# CHAPTER IV

# RESPONSE FUNCTION

When the system is stimulated by the force(or by the pressures, the displacements, the velocities, the accelerations, or a mixture of all of these), the system has a response. The stimulation(f(t)) is called the cause and the response(x(t)) is called the effect. This is shown[9] in Fig 4.1 .

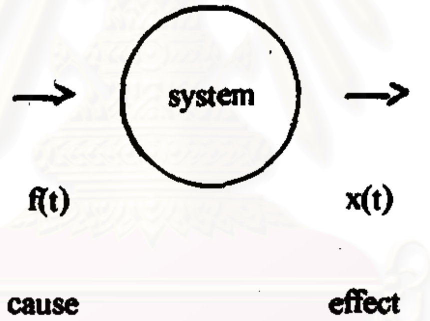


Fig 4.1 The Cause and the Effect Parameters.

The relationship between the physical cause and effect is replaced by the response function. Then, if there is no cause, the effect does not occur. Example, if we pull the spring, it is extended, or when we push the mass, it occurs an accleration. The cause and the effect do not occur in the same time, and the response function is meaningless if the effect occurs before the cause.

In this chapter, the problem of the semi-infinite monatomic chain solved by response function has been reviewed[1].

## The Relationship between the Cause and the Effect

Consider the time-dependent field(cause), f(t) and the change of physical magnitude (effect),x(t), at time t. The relationship between the cause and the effect can be written as*

$$x(t) = \int_{-\infty}^{\infty} \chi(t,t') f(t') dt' \quad , \qquad (4.1)$$

where $\chi(t,t')$ is the response function.

We can plot the graph in the three-dimensions between the response function $\chi(t,t')$ and t and t' for three cases by program mathcad(MCAD)**. Which are shown in Fig 4.2 , 4.3 and 4.4 .

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*See APPENDIX D.

**See APPENDIX F.

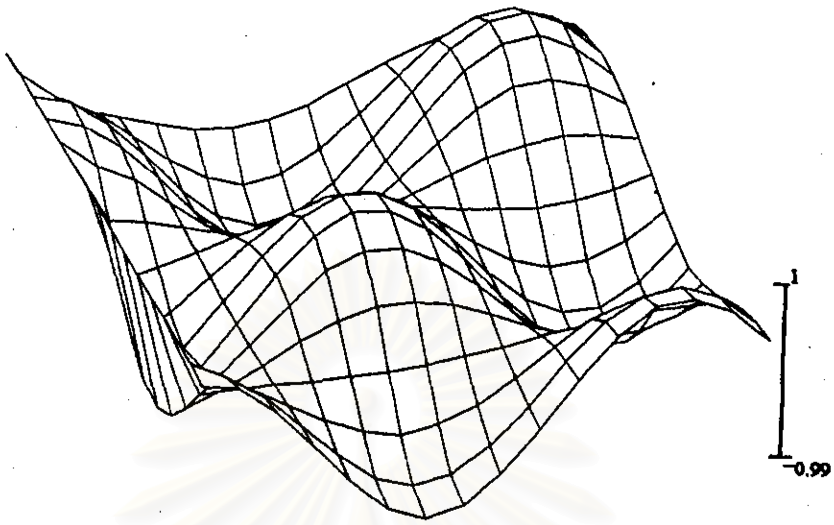


Fig 4.2 The General Response Function.

Fig 4.2 is shown in case $\chi$ (t,t')(general response function) depending on t and t', but t and t' are independent.

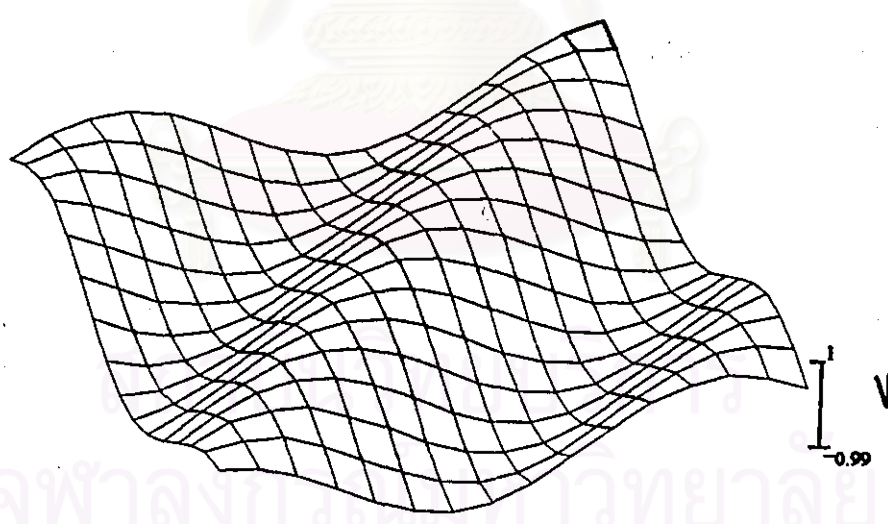


Fig 4.3 The Response Function depending on t-t'.

Fig 4.3 is shown in case $\chi(t,t')$ depending on the difference between the cause and the effect,t-t', i.e., $\chi(t,t') = \chi(t-t')$ .

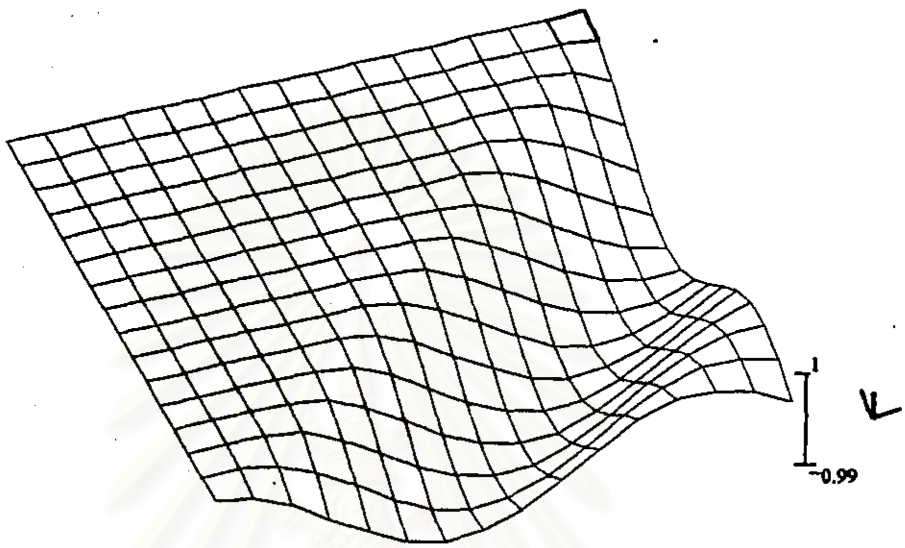


Fig 4.4 The Response Function depending on t-t' but t > t'.

Fig 4.4 is shown in case $\chi(t,t')$ depending on t-t', but t > t'. If t < t', the response function $\chi(t,t')$ is equal to zero.

From Eq.(4.1), when $f(t) = f_0 \exp(i\omega t)$, the relation between the cause and the effect is, thus

$$x(t) = \int_{-\infty}^{\infty} f_0 \exp(i\omega t')\chi(t-t')dt' \quad .$$

Therefore,

$$x(t) = f_0 \exp(i\omega t)\chi(\omega) \quad , \qquad (4.2)$$

where

$$\chi(\omega) = \int_{-\infty}^{\infty} \chi(t-t')\exp(-i\omega[t-t'])d(t-t') \quad .$$

**The Response Function of the Semi-Infinite Monatomic Chain**

Consider the particle,m, when the force f(t) is applied. Define f(t) as the cause , then we must find the effect. The position and the velocity are unsuitable because they increase without limits(their response functions are not integrable) but the acceleration is suitable as the effect. From the second law of the equation of the motion F = ma, that is to say the response function is

$$\chi(t) = \frac{\delta(t)}{m} \quad . \qquad (4.3)$$

The response function $\chi(t)$ in Eq.(4.3) is called the force-acceleration response function because it is due to the force(cause) and the accleration(effect)


Fig 4.5 The Force is applied to the Semi-Infinite Monatomic Chain.

In the semi-infinite monatomic chain case,Fig 4.5, we choose the sinusoidally varying force[10] acting on the first mass for the cause. Then,we choose the acceleration of the first mass for the effect and there is the same frequency dependence of the force. This means that, the displacement of the first mass will have the form $u_0(t) = u_0(\omega)\exp(i\omega t)$ ($u_0(\omega)$ is the Fourier Transform of $u_0(t)$), if the external force is $F(t) = F_0\exp(i\omega t)$.

From Eq.(4.2), we will get the relationship between the acceleration of the first mass and the force act on it from the left. We can rewrite the definition of the Fourier transform of the response function*, $\chi(\omega)$, thus

$$\ddot{u}_0(t) = \chi(\omega)F_0 \qquad . \qquad (4.4)$$

If we consider the semi-infinite chain in such a way that it is the infinite chain as seen only from the n th mass to the right(or the left) as if seen from the first mass. The acceleration of n th mass must then have the relationship with the force from the left and there is the same response function of the first mass. Thus

$$\ddot{u}_n(t) = \chi(\omega)C[u_{n-1}(\omega)-u_n(\omega)] \qquad , \qquad (4.5)$$

where C is the spring constant.



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*See APPENDIX E.

From the Harmonic motion, $\ddot{u}_n(\omega) = -\omega^2 u_n$ for all n. Inserting this into Eq.(4.5) yields

$$-\omega^2 u_n = \chi(\omega)C[u_{n-1}(\omega) - u_n(\omega)] \quad . \qquad (4.5.1)$$

Rewrite the Eq.(4.5.1) to obtain

$$\frac{u_n}{u_{n-1}} = \frac{\chi(\omega)C}{\chi(\omega)C - \omega^2} \quad . \qquad (4.6)$$

The Eq.(4.6) is called the recursion relation, that it means if we know $\chi(\omega)$, we can find the motion of all mass.

Consider the equation of motion for the first mass

$$m\ddot{u}_0 = C[u_1 - u_0] + F_0 \quad . \qquad (4.7)$$

Inserting Eq.(4.6)(write $u_1$ in terms of $u_0$) into Eq.(4.7) yields

$$m\ddot{u}_0 = C[u_0 \frac{\chi(\omega)C}{\chi(\omega)C - \omega^2} - u_0] + F_0 \quad ,$$

$$= C\left[\frac{\chi(\omega)C - \chi(\omega)C + \omega^2}{\chi(\omega)C - \omega^2}\right]u_0 + F_0 \quad ,$$

$$= C\left[\frac{\omega^2}{\chi(\omega)C - \omega^2}\right]u_0 + F_0 \quad . \qquad (4.8)$$

From, the harmonic motion. $\ddot{u}_0(\omega) = -\omega^2 u_0$ Eq.(4.8) is written as

$$m\ddot{u}_0 = -C\ddot{u}_0 \left[\frac{\omega^2}{\chi(\omega)C-\omega^2}\right]u_0 + F_0 \quad ,$$

$$\left[m + \frac{C}{\chi(\omega)C-\omega^2}\right]\ddot{u}_0 = F_0$$

or $$\ddot{u}_0 = \left[m + \frac{C}{\chi(\omega)C-\omega^2}\right]^{-1} F_0 \quad . \quad (4.9)$$

Compairing between Eq.(4.9) and Eq.(4.4) gives

$$\chi(\omega) = \left[m + \frac{C}{\chi(\omega)C-\omega^2}\right]^{-1} \quad ,$$

$$= \left[\frac{m\chi(\omega)C-m\omega^2+C}{\chi(\omega)C-\omega^2}\right]^{-1} \quad ,$$

$$\chi(\omega) = \left[\frac{\chi(\omega)C-\omega^2}{m\chi(\omega)C-m\omega^2+C}\right] \quad ,$$

or. $$mC\chi^2(\omega)-m\omega^2\chi(\omega)+C\chi(\omega) = C\chi(\omega)-\omega^2$$

Therefore, $$mC\chi^2(\omega)-m\omega^2\chi(\omega)+\omega^2 = 0 \quad , \quad (4.10)$$

which its solution is

$$\chi(\omega) = \frac{m\omega^2 \pm \sqrt{m^2\omega^4 - 4mC\omega^2}}{2mC} ,$$

$$\chi(\omega) = \frac{\omega^2}{2C} \pm \frac{1}{2C}\sqrt{\frac{m^2\omega^4 - 4mC\omega^2}{m^2}} ,$$

$$= \frac{\omega^2}{2C} \pm \frac{1}{2C}\sqrt{1 - \frac{4C}{m\omega^2}} ,$$

Let $\omega_0^2 = 4C/m$ ,

therefore,

$$\chi(\omega) = \frac{\omega^2}{2C}\left[1 \pm \sqrt{1 - \frac{\omega_0^2}{\omega^2}}\right] , \quad (4.11)$$

In here

$$\chi_+(\omega) = \frac{\omega^2}{2C}\left[1 + \sqrt{1 - \frac{\omega_0^2}{\omega^2}}\right] ,$$

(4.12)

$$\chi_-(\omega) = \frac{\omega^2}{2C}\left[1 - \sqrt{1 - \frac{\omega_0^2}{\omega^2}}\right] .$$

For the high frequency $\omega > \omega_0$, $\chi_+(\omega)$ is the divergent function, where as $\chi_-(\omega) \approx 1/m$