CHAPTER II

THE λ-CALCULUS WITH PATTERNS

The purpose of this chapter is to cover the basic definitions and relevant lemmas used in the λ-calculus with patterns. These concern terms, free variables, substitution, changes of bound variables, contractions and reductions.

2.1 Terms, Free Variables, and Substitution

This section contains the most basic definitions and results, namely, those concerning patterns, terms, free and bound variables, and substitution.

Definition 2.1.1. Assume there are given an infinite sequence of distinct symbols $v_1, v_2, v_3, \ldots$, called **variables**, and a set of symbols which are distinct from the variables, called **constants**. The set of expressions called **patterns** is defined as follows.

a. All variables and constants are patterns.

b. If P_1 is a pattern which is not a variable, P_2 is any pattern, and no variable occurs in both P_1 and P_2, then (P_1P_2) is a pattern.

The set of **terms** is divided into sets of atoms, applications and abstractions, and is defined as follows.

a. All variables and constants are terms. (These are the **atoms**.)

b. If P and Q are any terms, then (PQ) is a term. (These are the **applications**.)

c. If P is any pattern, Q is any term, and A is any abstraction, then $(\lambda P.Q)$ and $((\lambda P.Q) \mid A)$ are terms. (These are the **abstractions**.)

Notation 2.1.2.

a. Capital Roman letters will denote arbitrary terms.

b. Small Roman letters will denote variables.

c. Parentheses will be omitted by using association to the left.

d. $\lambda P.MN$ will abbreviate $(\lambda P.(MN))$.

e. Syntactic identity of terms will be denoted by $\equiv$. That is, $M \equiv N$ if and only if M is exactly the same term as N.

Notes 2.1.3.

a. Every pattern is a term, so that everything we define or prove for terms will also apply to patterns.

b. i. If $MN \equiv PQ$, then $M \equiv P$ and $N \equiv Q$.

ii. If $\lambda M.N \equiv \lambda P.Q$, then $M \equiv P$ and $N \equiv Q$.

iii. If $(\lambda M.N \mid A) \equiv (\lambda P.Q \mid B)$, then $\lambda M.N \equiv \lambda P.Q$, and $A \equiv B$.

c. An abstraction of the form $\lambda P.Q$ (respectively $(\lambda P.Q \mid A)$) is called a simple (respectively compound) abstraction.

d. The five classes of terms, namely, variables, constants, applications, simple abstractions, and compound abstractions, are mutually disjoint. (The combination of this note and (b) tells us that every term has a unique structure.)

Definition 2.1.4. The **length** of a term M, denoted by lgh(M), is the total number of occurrences of atoms in M. More precisely, it is defined by

a. $lgh(a) = 1$ for any atom a;

b. $lgh(PQ) = lgh(P) + lgh(Q)$;

c. $lgh(\lambda P.Q) = lgh(P) + lgh(Q)$;

d. $lgh((\lambda P.Q \mid A)) = lgh(\lambda P.Q) + lgh(A)$.

Note 2.1.5. References to induction on M will actually mean induction on the length of M.

Definition 2.1.6. An occurrence of a variable x in a term M is **bound** if it is in a subterm of M of the form $\lambda P.Q$ and it occurs in P; otherwise it is **free.** If x has at least one free occurrence in M, it is called a **free variable** of M; the set of all such variables is denoted by FV(M). A more precise definition of FV(M) is as follows.

a. $FV(a) = \begin{cases} \varnothing & \text{if a is a constant;} \\ \{a\} & \text{if a is a variable;} \end{cases}$

b. $FV(PQ) = FV(P) \cup FV(Q)$;

c. $FV(\lambda P.Q) = FV(Q) - FV(P)$;

d. $FV((\lambda P.Q \mid A)) = FV(\lambda P.Q) \cup FV(A)$.

Definition 2.1.7. Let M and $N_1,\ldots, N_k$, $k \geq 1$, be terms and $x_1,\ldots, x_k$ be distinct variables. The result of substituting N_i for all free occurrences of x_i, $i = 1, 2,\ldots, k$, in M, denoted by $[N_1/x_1,\ldots, N_k/x_k]M$, is defined as follows.

a. $[N_1/x_1,\ldots, N_k/x_k]x_i \equiv N_i$ for all $1 \leq i \leq k$;

b. $[N_1/x_1,\ldots, N_k/x_k]a \equiv a$ for all atoms a such that $a \notin \{x_1,\ldots, x_k\}$;

c. $[N_1/x_1,\ldots, N_k/x_k](PQ) \equiv [N_1/x_1,\ldots, N_k/x_k]P\ [N_1/x_1,\ldots, N_k/x_k]Q$;

d. $[N_1/x_1,\ldots, N_k/x_k](\lambda P.Q) \equiv \begin{cases} \lambda P.Q \text{ if } \{x_1,\ldots, x_k\} \cap FV(\lambda P.Q) = \varnothing; \\ [N_{i_1}/x_{i_1},\ldots, N_{i_m}/x_{i_m}](\lambda P.Q) \text{ if} \\ \quad \{x_1,\ldots, x_k\} \cap FV(\lambda P.Q) = \{x_{i_1},\ldots, x_{i_m}\}; \end{cases}$

e. $[N_1/x_1,\ldots, N_k/x_k](\lambda P.Q) \equiv \lambda P.[N_1/x_1,\ldots, N_k/x_k]Q$ if $\{x_1,\ldots, x_k\} \subseteq FV(\lambda P.Q)$ and $FV(P) \cap FV(N_1\ldots N_k) = \varnothing$;

f. $[N_1/x_1,\ldots, N_k/x_k](\lambda P.Q) \equiv [N_1/x_1,\ldots, N_k/x_k](\lambda[z/y]P.[z/y]Q)$ if $\{x_1,\ldots, x_k\} \subseteq FV(\lambda P.Q)$ and $FV(P) \cap FV(N_1\ldots N_k) \neq \varnothing$, where y is the first variable in $FV(P) \cap FV(N_1\ldots N_k)$ and z is chosen to be the first variable which is not in $FV(PQN_1\ldots N_k)$;

g. $[N_1/x_1,\ldots, N_k/x_k](\lambda P.Q \mid A) \equiv ([N_1/x_1,\ldots, N_k/x_k](\lambda P.Q) \mid [N_1/x_1,\ldots, N_k/x_k]A)$.

Note 2.1.8. Observe that in cases (d) and (f) of the above definition we do not say immediately how to reduce the original substitution to one into a term of shorter length. Instead, in case (d) we tell how to modify the substitution so that it will fall into one of the cases (e) or (f), and in case (f) we tell how to reduce the cardinality of $FV(P) \cap FV(N_1...N_k)$, so that after a finite number of applications of the rule in this case we will obtain a substitution that falls into case (e) of the definition. The final result of this process is described by Part (c) of Lemma 2.1.10.

Examples 2.1.9.

a. $[\lambda x.xy/x, \lambda x.xy/y]((\lambda x.xy)(\lambda y.xy))$

$\equiv [\lambda x.xy/x, \lambda x.xy/y](\lambda x.xy)\ [\lambda x.xy/x, \lambda x.xy/y](\lambda y.xy)$ (by (c))

$\equiv [\lambda x.xy/y](\lambda x.xy)\ [\lambda x.xy/x](\lambda y.xy)$ (by (d))

$\equiv (\lambda x.[\lambda x.xy/y](xy))\ [\lambda x.xy/x](\lambda[z/y]y.[z/y](xy))$, where z is the first variable which is not in $FV(xy(\lambda x.xy)) = \{x, y\}$ (by (e), (f))

$\equiv (\lambda x.x(\lambda x.xy))\ [\lambda x.xy/x](\lambda z.xz)$ (by (a), (b), (c))

$\equiv (\lambda x.x(\lambda x.xy))(\lambda z.[\lambda x.xy/x](xz))$ (by (e))

$\equiv (\lambda x.x(\lambda x.xy))(\lambda z.(\lambda x.xy)z)$. (by (a), (b), (c))

b. Let c be a constant.

$[v_3/v_1, v_4/v_2, v_2/v_4, v_1/v_5](\lambda(cv_1)v_2.v_1v_4v_5)$

$\equiv [v_2/v_4, v_1/v_5](\lambda(cv_1)v_2.v_1v_4v_5)$ (by (d))

$\equiv [v_2/v_4, v_1/v_5](\lambda[v_3/v_1]((cv_1)v_2).[v_3/v_1](v_1v_4v_5))$ (by (f))

$\equiv [v_2/v_4, v_1/v_5](\lambda(cv_3)v_2.v_3v_4v_5)$ (by (a), (b), (c))

$\equiv [v_2/v_4, v_1/v_5](\lambda[v_6/v_2]((cv_3)v_2).[v_6/v_2](v_3v_4v_5))$ (by (f))

$\equiv [v_2/v_4, v_1/v_5](\lambda(cv_3)v_6.v_3v_4v_5)$ (by (a), (b), (c))

$\equiv \lambda(cv_3)v_6.[v_2/v_4, v_1/v_5](v_3v_4v_5)$ (by (e))

$\equiv \lambda(cv_3)v_6.v_3v_2v_1$. (by (a), (b), (c))

Lemma 2.1.10.

a. Let P be a pattern, and $x_1,\ldots, x_k$, $k \geq 1$, be distinct variables. If $y_1,\ldots, y_k$ are distinct variables such that $\{y_1,\ldots, y_k\} \cap (FV(P) - \{x_1,\ldots, x_k\}) = \varnothing$, then $[y_1/x_1,\ldots, y_k/x_k]P$ is a pattern.

b. Let M and $N_1,\ldots, N_k$, $k \geq 1$, be terms, and $x_1,\ldots, x_k$ be distinct variables. Then $[N_1/x_1,\ldots, N_k/x_k]M$ is a term, and if M is not a variable then $[N_1/x_1,\ldots, N_k/x_k]M$ is of the same form as M.

c. Let $\lambda P.Q$ be a simple abstraction, $x_1,\ldots, x_k$, $k \geq 1$, be distinct variables, and $N_1,\ldots, N_k$ be terms such that $\{x_1,\ldots, x_k\} \cap FV(\lambda P.Q) = \{x_{i_1},\ldots, x_{i_m}\}$, and $FV(P) \cap FV(N_{i_1}\ldots N_{i_m}) = \{y_1,\ldots, y_n\}$, where for each $1 \leq j \leq n$, y_j is the j^{th} variable in $FV(P) \cap FV(N_{i_1}\ldots N_{i_m})$. Then

$[N_1/x_1,..., N_k/x_k](\lambda P.Q) \equiv \lambda[z_n/y_n]\ldots[z_1/y_1]P.[N_{i_1}/x_{i_1},\ldots, N_{i_m}/x_{i_m}][z_n/y_n]\ldots[z_1/y_1]Q$,

where z_1 is the first variable which is not in $FV(N_{i_1}\ldots N_{i_m}PQ)$ and for each $1 < j \leq n$, z_j is the first variable which is not in $FV(N_{i_1}\ldots N_{i_m}[z_{j-1}/y_{j-1}]\ldots[z_1/y_1](PQ))$.

Proof of (a). Assume $y_1,\ldots, y_k$ are distinct variables such that $\{y_1,\ldots, y_k\} \cap (FV(P) - \{x_1,\ldots, x_k\}) = \varnothing$ and induct on P.

i. P is an atom.

$$\text{Then } [y_1/x_1,\ldots, y_k/x_k]P \equiv \begin{cases} y_t & \text{if } P \equiv x_t \text{ for some } 1 \leq t \leq k. \\ P & \text{otherwise.} \end{cases}$$

Hence $[y_1/x_1,\ldots, y_k/x_k]P$ is a pattern.

ii. $P \equiv P_1P_2$ for some patterns P_1 and P_2 such that P_1 is not a variable and no variable occurs in both P_1 and P_2. Since $\{y_1,\ldots, y_k\} \cap (FV(P) - \{x_1,\ldots, x_k\}) = \varnothing$, $\{y_1,\ldots, y_k\} \cap (FV(P_i) - \{x_1,\ldots, x_k\}) = \varnothing$, i = 1, 2. Hence, by induction $[y_1/x_1,\ldots, y_k/x_k]P_i$, i = 1, 2, is a pattern.

Claim. $[y_1/x_1,\ldots, y_k/x_k]P_1$ is not a variable.

Since P_1 is not a variable, P_1 falls into one of the following two cases.

Case 1. P_1 is a constant.

Then $[y_1/x_1,\ldots, y_k/x_k]P_1 \equiv P_1$ which is not a variable.

Case 2. $P_1 \equiv Q_1Q_2$ for some patterns Q_1 and Q_2.

Similar to the above, by induction $[y_1/x_1,\ldots, y_k/x_k]Q_i$, i = 1, 2, is a pattern. Since $[y_1/x_1,\ldots, y_k/x_k]P_1 \equiv [y_1/x_1,\ldots, y_k/x_k]Q_1[y_1/x_1,\ldots, y_k/x_k]Q_2$, $[y_1/x_1,\ldots, y_k/x_k]P_1$ is not a variable.

So we have the claim.

Since $y_1,\ldots, y_k$ are distinct variables such that $\{y_1,\ldots, y_k\} \cap (FV(P_1P_2) - \{x_1,\ldots, x_k\}) = \varnothing$ and no variable occurs in both P_1 and P_2, no variable occurs in both $[y_1/x_1,\ldots, y_k/x_k]P_1$ and $[y_1/x_1,\ldots, y_k/x_k]P_2$. Hence, by the claim $[y_1/x_1,\ldots, y_k/x_k]P_1[y_1/x_1,\ldots, y_k/x_k]P_2$ is a pattern. Since $[y_1/x_1,\ldots, y_k/x_k]P \equiv [y_1/x_1,\ldots, y_k/x_k]P_1[y_1/x_1,\ldots, y_k/x_k]P_2$, $[y_1/x_1,\ldots, y_k/x_k]P$ is a pattern.

□

Proof of (b). Induct on M.

i. M is an atom.

$$\text{Then}\quad [N_1/x_1,\dots, N_k/x_k]M \;\equiv\; \begin{cases} N_t & \text{if } M \equiv x_t \text{ for some } 1 \le t \le k. \\ M & \text{otherwise.} \end{cases}$$

Hence $[N_1/x_1,\dots, N_k/x_k]M$ is a term. In particular, if M is not a variable then $[N_1/x_1,\dots, N_k/x_k]M$ is M, so it is of the same form as M.

ii. $M \equiv M_1M_2$.

By induction, $[N_1/x_1,\dots, N_k/x_k]M_i$, i = 1, 2, is a term. Since $[N_1/x_1,\dots, N_k/x_k]M \equiv [N_1/x_1,\dots, N_k/x_k]M_1\ [N_1/x_1,\dots, N_k/x_k]M_2$, $[N_1/x_1,\dots, N_k/x_k]M$ is an application.

iii. $M \equiv \lambda P.Q$.

Case 1. $\{x_1,\dots, x_k\} \cap FV(M) = \varnothing$.

Then $[N_1/x_1,\dots, N_k/x_k]M \equiv M$. Hence $[N_1/x_1,\dots, N_k/x_k]M$ is a simple abstraction.

Case 2. $\{x_1,\dots, x_k\} \cap FV(M) \neq \varnothing$.

By Definition 2.1.7(d), we may assume that $\{x_1,\dots, x_k\} \subseteq FV(M)$.

Let $m = |FV(P) \cap FV(N_1\dots N_k)|$ and induct on m.

For the case m = 0, we have $[N_1/x_1,\dots, N_k/x_k]M \equiv \lambda P.[N_1/x_1,\dots, N_k/x_k]Q$. By induction on M, $[N_1/x_1,\dots, N_k/x_k]Q$ is a term. Hence $[N_1/x_1,\dots, N_k/x_k]M$ is a simple abstraction.

Now assume m > 0. Let u be the first variable in $FV(P) \cap FV(N_1\dots N_k)$ and z be the first variable which is not in $FV(PQN_1\dots N_k)$.

Then $[N_1/x_1,\dots, N_k/x_k]M \equiv [N_1/x_1,\dots, N_k/x_k](\lambda[z/u]P.[z/u]Q)$. By (a), $[z/u]P$ is a pattern and, by induction (on M) $[z/u]Q$ is a term. Hence $\lambda[z/u]P.[z/u]Q$ is a simple abstraction. Since $|FV([z/u]P) \cap FV(N_1\dots N_k)| = m - 1$, by induction $[N_1/x_1,\dots, N_k/x_k](\lambda[z/u]P.[z/u]Q)$ is a simple abstraction. That is, $[N_1/x_1,\dots, N_k/x_k]M$ is a simple abstraction.

iv. $M \equiv (\lambda P.Q \mid A)$.

By induction, $[N_1/x_1,\dots, N_k/x_k](\lambda P.Q)$ is a simple abstraction and $[N_1/x_1,\dots, N_k/x_k]A$ is an abstraction. Since

$[N_1/x_1,..., N_k/x_k]M \equiv ([N_1/x_1,..., N_k/x_k](\lambda P.Q) \mid [N_1/x_1,..., N_k/x_k]A)$, $[N_1/x_1,..., N_k/x_k]M$ is a compound abstraction. □

Proof of (c). Let z_1 be the first variable which is not in $FV(N_{i_1}\ldots N_{i_m}PQ)$. Induct on n.

If n = 1, then $FV(P) \cap FV(N_{i_1}\ldots N_{i_m}) = \{y_1\}$, so it follows directly from the definition that

$$\begin{aligned}[N_1/x_1,..., N_k/x_k](\lambda P.Q) &\equiv [N_{i_1}/x_{i_1},..., N_{i_m}/x_{i_m}](\lambda P.Q)\\ &\equiv [N_{i_1}/x_{i_1},..., N_{i_m}/x_{i_m}](\lambda[z_1/y_1]P.[z_1/y_1]Q)\\ &\equiv \lambda[z_1/y_1]P.[N_{i_1}/x_{i_1},..., N_{i_m}/x_{i_m}][z_1/y_1]Q.\end{aligned}$$

Now assume n > 1. Then we can use induction to obtain

$$\begin{aligned}&[N_1/x_1,..., N_k/x_k](\lambda P.Q)\\ &\quad\equiv [N_{i_1}/x_{i_1},..., N_{i_m}/x_{i_m}](\lambda P.Q)\\ &\quad\equiv [N_{i_1}/x_{i_1},..., N_{i_m}/x_{i_m}](\lambda[z_1/y_1]P.[z_1/y_1]Q)\\ &\quad\equiv \lambda[z_n/y_n]\ldots[z_2/y_2][z_1/y_1]P.[N_{i_1}/x_{i_1},..., N_{i_m}/x_{i_m}][z_n/y_n]\ldots[z_2/y_2][z_1/y_1]Q,\end{aligned}$$

where z_j is the first variable which is not in $FV(N_{i_1}...N_{i_m}[z_{j-1}/y_{j-1}]\ldots[z_1/y_1](PQ))$ for all $1 < j \leq n$. □

Lemma 2.1.11. Let $x_1,..., x_k$, $k \geq 1$, be distinct variables, and M, $N_1,\ldots, N_k$ be terms. Then for each $1 \leq i \leq k$,

a. if $x_i \notin FV(M)$, then

$$[N_1/x_1,..., N_k/x_k]M \equiv [N_1/x_1,..., N_{i-1}/x_{i-1}, N_{i+1}/x_{i+1},\ldots, N_k/x_k]M;$$

b. $[N_1/x_1,..., N_{i-1}/x_{i-1}, x_i/x_i, N_{i+1}/x_{i+1},\ldots, N_k/x_k]M$

$\equiv [N_1/x_1,\ldots, N_{i-1}/x_{i-1}, N_{i+1}/x_{i+1},\ldots, N_k/x_k]M.$

Proof. Fix $1 \leq i \leq k$.

Proof of (a). Assume $x_i \notin FV(M)$, and induct on M.

i. M is an atom.

Case 1. $M \equiv x_t$ for some $1 \leq t \leq k$.

Since $x_i \notin FV(M)$, $i \neq t$. Hence

$[N_1/x_1,..., N_k/x_k]M \equiv N_t \equiv [N_1/x_1,..., N_{i-1}/x_{i-1}, N_{i+1}/x_{i+1},\ldots, N_k/x_k]M.$

Case 2. $M \not\equiv x_j$ for all $1 \le j \le k$.

Then $[N_1/x_1,..., N_k/x_k]M \equiv M \equiv [N_1/x_1,..., N_{i-1}/x_{i-1}, N_{i+1}/x_{i+1},\ldots, N_k/x_k]M$.

ii. $M \equiv M_1M_2$.

Since $x_i \notin FV(M)$, $x_i \notin FV(M_1)$ and $x_i \notin FV(M_2)$. So this case follows straightforwardly by induction.

iii. $M \equiv \lambda P.Q$.

Case 1. $\{x_1,..., x_k\} \cap FV(M) = \varnothing$.

Then $[N_1/x_1,..., N_k/x_k]M \equiv M \equiv [N_1/x_1,..., N_{i-1}/x_{i-1}, N_{i+1}/x_{i+1},\ldots, N_k/x_k]M$.

Case 2. $\{x_1,..., x_k\} \cap FV(M) = \{x_{j_1},..., x_{j_n}\}$.

Since $x_i \notin FV(M)$, $(\{x_1,..., x_k\} - \{x_i\}) \cap FV(M) = \{x_{j_1},..., x_{j_n}\}$. Hence

$[N_1/x_1,..., N_k/x_k]M \equiv [N_{j_1}/x_{j_1},\ldots, N_{j_n}/x_{j_n}]M$

$\equiv [N_1/x_1,..., N_{i-1}/x_{i-1}, N_{i+1}/x_{i+1},\ldots, N_k/x_k]M$.

iv. $M \equiv (\lambda P.Q \mid A)$.

This is similar to (ii). □

Proof of (b). If $x_i \notin FV(M)$, then (b) follows from (a).

Suppose $x_i \in FV(M)$ and induct on M.

i. $M \equiv x_i$.

$[N_1/x_1,..., N_{i-1}/x_{i-1}, x_i/x_i, N_{i+1}/x_{i+1},\ldots, N_k/x_k]M$

$\equiv x_i \equiv [N_1/x_1,..., N_{i-1}/x_{i-1}, N_{i+1}/x_{i+1},\ldots, N_k/x_k]M$.

ii. $M \equiv M_1M_2$.

This case follows straightforwardly by induction.

iii. $M \equiv \lambda P.Q$.

Since $x_i \in FV(M)$, $\{x_1,..., x_k\} \cap FV(M) \neq \varnothing$. By Definition 2.1.7(d), we may assume that $\{x_1,..., x_k\} \subseteq FV(M)$.

Let $m = |FV(P) \cap FV(N_1\ldots N_{i-1}\, x_i\, N_{i+1}\ldots N_k)|$ and induct on m.

If $m = 0$, then

$[N_1/x_1,..., N_{i-1}/x_{i-1}, x_i/x_i, N_{i+1}/x_{i+1},\ldots, N_k/x_k]M$

$\equiv \lambda P.[N_1/x_1,..., N_{i-1}/x_{i-1}, x_i/x_i, N_{i+1}/x_{i+1},\ldots, N_k/x_k]Q$

$\equiv \lambda P.[N_1/x_1,..., N_{i-1}/x_{i-1}, N_{i+1}/x_{i+1},\ldots, N_k/x_k]Q$ (by induction on M)

$\equiv [N_1/x_1,..., N_{i-1}/x_{i-1}, N_{i+1}/x_{i+1},\ldots, N_k/x_k]M$.

Now assume $m > 0$. Let u be the first variable in $FV(P) \cap FV(N_1\ldots N_{i-1}x_iN_{i+1}\ldots N_k)$ and let z be the first variable which is not in $FV(PQN_1\ldots N_{i-1}x_iN_{i+1}\ldots N_k)$. Since $x_i \in FV(M)$, $x_i \notin FV(P)$ and $x_i \in FV(Q)$. Hence $FV(P) \cap FV(N_1\ldots N_{i-1}x_iN_{i+1}\ldots N_k) = FV(P) \cap FV(N_1\ldots N_{i-1}N_{i+1}\ldots N_k)$ and $FV(PQN_1\ldots N_{i-1}x_iN_{i+1}\ldots N_k) = FV(PQN_1\ldots N_{i-1}N_{i+1}\ldots N_k)$. Thus u is the first variable in $FV(P) \cap FV(N_1\ldots N_{i-1}N_{i+1}\ldots N_k)$ and z is the first variable which is not in $FV(PQN_1\ldots N_{i-1}N_{i+1}\ldots N_k)$. Since $|FV([z/u]P) \cap FV(N_1\ldots N_{i-1}x_iN_{i+1}\ldots N_k)| = m - 1$, by induction $[N_1/x_1,..., N_{i-1}/x_{i-1}, x_i/x_i, N_{i+1}/x_{i+1},\ldots, N_k/x_k]M$

$$\equiv [N_1/x_1,..., N_{i-1}/x_{i-1}, x_i/x_i, N_{i+1}/x_{i+1},\ldots, N_k/x_k](\lambda[z/u]P.[z/u]Q)$$

$$\equiv [N_1/x_1,..., N_{i-1}/x_{i-1}, N_{i+1}/x_{i+1},\ldots, N_k/x_k](\lambda[z/u]P.[z/u]Q)$$

$$\equiv [N_1/x_1,..., N_{i-1}/x_{i-1}, N_{i+1}/x_{i+1},\ldots, N_k/x_k](\lambda P.Q)$$

$$\equiv [N_1/x_1,..., N_{i-1}/x_{i-1}, N_{i+1}/x_{i+1},\ldots, N_k/x_k]M.$$

iv. $M \equiv (\lambda P.Q \mid A)$.

This is similar to (ii). □

Corollary 2.1.12. Let $x_1,..., x_k$, $k \geq 1$, be distinct variables, $M, N_1,\ldots, N_k$ be terms, and $\lambda P.Q$ be a simple abstraction.

a. If $\{x_1,..., x_k\} \cap FV(M) = \varnothing$, then $[N_1/x_1,..., N_k/x_k]M \equiv M$.

b. If $\{x_1,..., x_k\} \cap FV(M) = \{x_{i_1},..., x_{i_m}\}$, then $[N_1/x_1,..., N_k/x_k]M \equiv [N_{i_1}/x_{i_1},..., N_{i_m}/x_{i_m}]M$.

c. $[x_1/x_1,..., x_k/x_k]M \equiv M$.

d. If $FV(P) \cap FV(x_1\ldots x_kN_1\ldots N_k) = \varnothing$, then $[N_1/x_1,..., N_k/x_k](\lambda P.Q) \equiv \lambda P.[N_1/x_1,..., N_k/x_k]Q$.

Proof. Parts (a) and (b) follow from Lemma 2.1.11(a), while Part (c) follows from Lemma 2.1.11(b).

Proof of (d). Assume $FV(P) \cap FV(x_1...x_kN_1...N_k) = \varnothing$.

Since $FV(\lambda P.Q) = FV(Q) - FV(P)$ and $\{x_1,..., x_k\} \cap FV(P) = \varnothing$,

$\{x_1,\ldots, x_k\} \cap FV(\lambda P.Q) = \{x_1,\ldots, x_k\} \cap FV(Q)$.

Case 1. $\{x_1,\ldots, x_k\} \cap FV(\lambda P.Q) = \varnothing$.

Then $\{x_1,\ldots, x_k\} \cap FV(Q) = \varnothing$. Hence

$$[N_1/x_1,\ldots, N_k/x_k](\lambda P.Q) \equiv \lambda P.Q \qquad \text{(by Definition 2.1.7(d))}$$

$$\equiv \lambda P.[N_1/x_1,\ldots, N_k/x_k]Q. \qquad \text{(by (a))}$$

Case 2. $\{x_1,\ldots, x_k\} \cap FV(\lambda P.Q) = \{x_{i_1},\ldots, x_{i_m}\}$.

Then $\{x_1,\ldots, x_k\} \cap FV(Q) = \{x_{i_1},\ldots, x_{i_m}\}$. Hence

$$[N_1/x_1,\ldots, N_k/x_k](\lambda P.Q) \equiv [N_{i_1}/x_{i_1},\ldots, N_{i_m}/x_{i_m}](\lambda P.Q) \qquad \text{(by Definition 2.1.7(d))}$$

$$\equiv \lambda P.[N_{i_1}/x_{i_1},\ldots, N_{i_m}/x_{i_m}]Q \qquad \text{(by Definition 2.1.7(e))}$$

$$\equiv \lambda P.[N_1/x_1,\ldots, N_k/x_k]Q. \qquad \text{(by (b))} \quad \square$$

Lemma 2.1.13. Let $x_1,\ldots, x_k$, $k \geq 1$, be distinct variables, and $M, N_1,\ldots, N_k$ be terms. If $\{x_1,\ldots, x_k\} \subseteq FV(M)$, then

$FV([N_1/x_1,\ldots, N_k/x_k]M) = FV(N_1\ldots N_k) \cup (FV(M) - \{x_1,\ldots, x_k\})$.

Proof. Assume $\{x_1,\ldots, x_k\} \subseteq FV(M)$. Induct on M.

i. $M \equiv x_1$.

Then $FV([N_1/x_1]M) = FV(N_1) = FV(N_1) \cup (FV(M) - \{x_1\})$.

ii. $M \equiv M_1M_2$.

Since $\{x_1,\ldots, x_k\} \subseteq FV(M)$, $\{x_1,\ldots, x_k\} \cap FV(M_1) \neq \varnothing$ or $\{x_1,\ldots, x_k\} \cap FV(M_2) \neq \varnothing$. Without loss of generality, assume $\{x_1,\ldots, x_k\} \cap FV(M_1) \neq \varnothing$, the case $\{x_1,\ldots, x_k\} \cap FV(M_2) \neq \varnothing$ being similar.

Case 1. $\{x_1,\ldots, x_k\} \cap FV(M_2) = \varnothing$.

Then $\{x_1,\ldots, x_k\} \subseteq FV(M_1)$. Hence

$$FV([N_1/x_1,\ldots, N_k/x_k]M) = FV([N_1/x_1,\ldots, N_k/x_k]M_1\ [N_1/x_1,\ldots, N_k/x_k]M_2)$$

$$= FV(([N_1/x_1,\ldots, N_k/x_k]M_1)M_2) \qquad \text{(by Corollary 2.1.12(a))}$$

$$= FV([N_1/x_1,\ldots, N_k/x_k]M_1) \cup FV(M_2)$$

$$= FV(N_1\ldots N_k) \cup (FV(M_1) - \{x_1,\ldots, x_k\}) \cup FV(M_2)$$

(by induction)

$$= FV(N_1\ldots N_k) \cup ((FV(M_1) \cup FV(M_2)) - \{x_1,\ldots, x_k\})$$

$= FV(N_1 \ldots N_k) \cup (FV(M) - \{x_1, \ldots, x_k\})$.

Case 2. $\{x_1, \ldots, x_k\} \cap FV(M_2) = \{x_{j_1}, \ldots, x_{j_n}\}$.

Let $\{x_1, \ldots, x_k\} \cap FV(M_1) = \{x_{i_1}, \ldots, x_{i_m}\}$. Since $\{x_1, \ldots, x_k\} \subseteq FV(M)$, $\{i_1, \ldots, i_m\} \cup \{j_1, \ldots, j_n\} = \{1, \ldots, k\}$. So we have

$$
\begin{aligned}
FV([N_1/x_1, \ldots, N_k/x_k]M) \\
&= FV([N_{i_1}/x_{i_1}, \ldots, N_{i_m}/x_{i_m}]M_1[N_{j_1}/x_{j_1}, \ldots, N_{j_n}/x_{j_n}]M_2) \\
&= FV([N_{i_1}/x_{i_1}, \ldots, N_{i_m}/x_{i_m}]M_1) \cup FV([N_{j_1}/x_{j_1}, \ldots, N_{j_n}/x_{j_n}]M_2) \\
&= FV(N_{i_1} \ldots N_{i_m}) \cup (FV(M_1) - \{x_{i_1}, \ldots, x_{i_m}\}) \cup FV(N_{j_1} \ldots N_{j_n}) \\
&\quad \cup (FV(M_2) - \{x_{j_1}, \ldots, x_{j_n}\}) \\
&= FV(N_1 \ldots N_k) \cup (FV(M_1) - \{x_1, \ldots, x_k\}) \cup (FV(M_2) - \{x_1, \ldots, x_k\}) \\
&= FV(N_1 \ldots N_k) \cup ((FV(M_1) \cup FV(M_2)) - \{x_1, \ldots, x_k\}) \\
&= FV(N_1 \ldots N_k) \cup (FV(M) - \{x_1, \ldots, x_k\}).
\end{aligned}
$$

iii. $M \equiv \lambda P.Q$.

Since $\{x_1, \ldots, x_k\} \subseteq FV(M)$, $\{x_1, \ldots, x_k\} \subseteq FV(Q)$ and $\{x_1, \ldots, x_k\} \cap FV(P) = \varnothing$.

Let $m = |FV(P) \cap FV(N_1 \ldots N_k)|$ and induct on m.

If $m = 0$, then $[N_1/x_1, \ldots, N_k/x_k]M \equiv \lambda P.[N_1/x_1, \ldots, N_k/x_k]Q$, so

$$
\begin{aligned}
FV([N_1/x_1, \ldots, N_k/x_k]M) &= FV(\lambda P.[N_1/x_1, \ldots, N_k/x_k]Q) \\
&= FV([N_1/x_1, \ldots, N_k/x_k]Q) - FV(P) \\
&= (FV(N_1 \ldots N_k) \cup (FV(Q) - \{x_1, \ldots, x_k\})) - FV(P) \\
&= FV(N_1 \ldots N_k) \cup ((FV(Q) - FV(P)) - \{x_1, \ldots, x_k\}) \\
&= FV(N_1 \ldots N_k) \cup (FV(M) - \{x_1, \ldots, x_k\}).
\end{aligned}
$$

Now assume $m > 0$. Let u be the first variable in $FV(P) \cap FV(N_1 \ldots N_k)$ and z be the first variable which is not in $FV(PQN_1 \ldots N_k)$.Then, by induction on m we have

$$
\begin{aligned}
FV([N_1/x_1, \ldots, N_k/x_k]M) &= FV([N_1/x_1, \ldots, N_k/x_k](\lambda[z/u]P.[z/u]Q)) \\
&= FV(N_1 \ldots N_k) \cup (FV(\lambda[z/u]P.[z/u]Q) - \{x_1, \ldots, x_k\}) \\
&= FV(N_1 \ldots N_k) \cup ((FV([z/u]Q) - FV([z/u]P)) - \{x_1, \ldots, x_k\}).
\end{aligned}
$$

Let $A = FV(N_1 \ldots N_k) \cup ((FV([z/u]Q) - FV([z/u]P)) - \{x_1, \ldots, x_k\})$.

Case 1. $u \in FV(Q)$.

Then

$A = FV(N_1 \ldots N_k) \cup (((\{z\} \cup (FV(Q) - \{u\})) - (\{z\} \cup (FV(P) - \{u\}))) - \{x_1, \ldots, x_k\})$

$= FV(N_1 \ldots N_k) \cup ((FV(Q) - FV(P)) - \{x_1, \ldots, x_k\})$

$= FV(N_1 \ldots N_k) \cup (FV(M) - \{x_1, \ldots, x_k\})$.

Case 2. $u \notin FV(Q)$.

Then $A = FV(N_1 \ldots N_k) \cup ((FV(Q) - (\{z\} \cup (FV(P) - \{u\}))) - \{x_1, \ldots, x_k\})$

$= FV(N_1 \ldots N_k) \cup ((FV(Q) - FV(P)) - \{x_1, \ldots, x_k\})$

$= FV(N_1 \ldots N_k) \cup (FV(M) - \{x_1, \ldots, x_k\})$.

iv. $M \equiv (\lambda P.Q \mid A)$.

This is similar to (ii). □

Corollary 2.1.14. Let $x_1, \ldots, x_k$, $k \geq 1$, be distinct variables, and $M, N_1, \ldots, N_k$ be terms. Then $FV([N_1/x_1, \ldots, N_k/x_k]M) \subseteq FV(N_1 \ldots N_k) \cup (FV(M) - \{x_1, \ldots, x_k\})$.

Proof. This follows from Corollary 2.1.12(a), (b) and Lemma 2.1.13. □

Lemma 2.1.15. Let $x_1, \ldots, x_k$, $k \geq 1$, be distinct variables, $y_1, \ldots, y_k$ be variables, and M be a term. Then $lgh([y_1/x_1, \ldots, y_k/x_k]M) = lgh(M)$.

Proof. By Corollary 2.1.12(a), (b), we may assume that $\{x_1, \ldots, x_k\} \subseteq FV(M)$. Induct on M.

i. $M \equiv x_1$.

Then $[y_1/x_1]M \equiv y_1$. Hence $lgh([y_1/x_1]M) = 1 = lgh(M)$.

ii. $M \equiv M_1M_2$.

This case follows straightforwardly by induction.

iii. $M \equiv \lambda P.Q$.

Let $m = |FV(P) \cap \{y_1, \ldots, y_k\}|$ and induct on m.

If $m = 0$, then $[y_1/x_1, \ldots, y_k/x_k]M \equiv \lambda P.[y_1/x_1, \ldots, y_k/x_k]Q$, so we have

$lgh([y_1/x_1, \ldots, y_k/x_k]M) = lgh(\lambda P.[y_1/x_1, \ldots, y_k/x_k]Q)$

$= lgh(P) + lgh([y_1/x_1, \ldots, y_k/x_k]Q)$

$= lgh(P) + lgh(Q)$ (by induction on M)

$= lgh(\lambda P.Q)$

$= lgh(M)$.

Now assume $m > 0$. Let u be the first variable in $FV(P) \cap \{y_1,\ldots, y_k\}$ and z be the first variable which is not in $FV(y_1\ldots y_kPQ)$. Hence $|FV([z/u]P) \cap \{y_1,\ldots, y_k\}| = m - 1$ and so

$$
\begin{aligned}
lgh([y_1/x_1,\ldots, y_k/x_k]M) &= lgh([y_1/x_1,\ldots, y_k/x_k](\lambda[z/u]P.[z/u]Q)) \\
&= lgh(\lambda[z/u]P.[z/u]Q) && \text{(by induction on m)} \\
&= lgh([z/u]P) + lgh([z/u]Q) \\
&= lgh(P) + lgh(Q) && \text{(by induction on M)} \\
&= lgh(\lambda P.Q) \\
&= lgh(M).
\end{aligned}
$$

iv. $M \equiv (\lambda P.Q \mid A)$.

This is similar to (ii). □

Lemma 2.1.16. Let $x_1,\ldots, x_m, y_1,\ldots, y_n$, $m \geq 1$, $n \geq 1$, be distinct variables, $U_1,\ldots, U_m, V_1,\ldots,V_n$ be terms and M be a term such that no variable bound in M is free in $x_1\ldots x_m y_1\ldots y_n U_1\ldots U_m V_1\ldots V_n$.

a. For each $1 \leq i \leq m$, $1 \leq j \leq n$,

$[U_1/x_1,\ldots, U_m/x_m][V_1/y_1,\ldots,V_{j-1}/y_{j-1},V_j/x_i,V_{j+1}/y_{j+1},\ldots,V_n/y_n]M$

$\equiv [[U_1/x_1,\ldots, U_m/x_m]V_1/y_1,\ldots,[U_1/x_1,\ldots, U_m/x_m]V_{j-1}/y_{j-1},[U_1/x_1,\ldots, U_m/x_m]V_j/ x_i,$
$[U_1/x_1,\ldots, U_m/x_m]V_{j+1}/y_{j+1},\ldots,[U_1/x_1,\ldots, U_m/x_m]V_n/y_n, U_1/x_1,\ldots, U_{i-1}/x_{i-1}, U_{i+1}/x_{i+1},\ldots,$
$U_m/x_m]M$.

b. $[U_1/x_1,\ldots, U_m/x_m][V_1/y_1,\ldots,V_n/y_n]M$

$\equiv [[U_1/x_1,\ldots, U_m/x_m]V_1/y_1,\ldots,[U_1/x_1,\ldots, U_m/x_m]V_n/y_n, U_1/x_1,\ldots, U_m/x_m]M$.

Proof of (a). Let $1 \leq i \leq m$ and $1 \leq j \leq n$ and induct on M.

i. M is an atom.

Case 1. $M \equiv x_i$ or $M \equiv y_t$ for some $1 \leq t \leq n$, $t \neq j$. Let

$$
s = \begin{cases} j & \text{if } M \equiv x_i. \\ t & \text{if } M \equiv y_t. \end{cases}
$$

Then $[[U_1x_1,\ldots, U_m/x_m]V_1/y_1,\ldots,[U_1/x_1,\ldots, U_m/x_m]V_{j-1}/y_{j-1}, [U_1/x_1,\ldots, U_m/x_m]V_j/x_i,$ $[U_1/x_1,\ldots, U_m/x_m]V_{j+1}/y_{j+1},\ldots,[U_1/x_1,\ldots, U_m/x_m]V_n/y_n, U_1/x_1,\ldots, U_{i-1}/x_{i-1}, U_{i+1}/x_{i+1},\ldots,$ $U_m/x_m]M$

$\equiv [U_1/x_1,\ldots, U_m/x_m]V_s$

$\equiv [U_1/x_1,\ldots, U_m/x_m][V_1/y_1,\ldots,V_{j-1}/y_{j-1},V_j/x_i,V_{j+1}/y_{j+1},\ldots,V_n/y_n]M.$

Case 2. $M \not\equiv x_i$ and $M \not\equiv y_r$ for all $1 \le r \le n$, $r \ne j$.

Then

$[[U_1/x_1,\ldots, U_m/x_m]V_1/y_1,\ldots,[U_1/x_1,\ldots, U_m/x_m]V_{j-1}/y_{j-1},[U_1/x_1,\ldots, U_m/x_m]V_j/x_i,$ $[U_1/x_1,\ldots, U_m/x_m]V_{j+1}/y_{j+1},\ldots,[U_1/x_1,\ldots, U_m/x_m]V_n/y_n, U_1/x_1,\ldots, U_{i-1}/x_{i-1}, U_{i+1}/x_{i+1},\ldots,$ $U_m/x_m]M$

$\equiv [U_1/x_1,\ldots, U_{i-1}/x_{i-1}, U_{i+1}/x_{i+1},\ldots, U_m/x_m]M$ (by Lemma 2.1.11(a))

$\equiv [U_1/x_1,\ldots, U_m/x_m]M$ (by Lemma 2.1.11(a))

$\equiv [U_1/x_1,\ldots, U_m/x_m][V_1/y_1,\ldots,V_{j-1}/y_{j-1},V_j/x_i,V_{j+1}/y_{j+1},\ldots,V_n/y_n]M.$

(by Corollary 2.1.12(a))

ii. $M \equiv M_1M_2$.

This follows easily by induction.

iii. $M \equiv \lambda P.Q$.

Since no variable bound in M is free in $x_1\ldots x_m y_1\ldots y_n U_1\ldots U_m V_1\ldots V_n$, $FV(x_1\ldots x_m y_1\ldots y_n U_1\ldots U_m V_1\ldots V_n) \cap FV(P) = \varnothing$. Hence, by Corollary 2.1.12(d)

$[U_1/x_1,\ldots, U_m/x_m][V_1/y_1,\ldots, V_{j-1}/y_{j-1},V_j/x_i,V_{j+1}/y_{j+1},\ldots,V_n/y_n]M$

$\equiv \lambda P.[U_1/x_1,\ldots, U_m/x_m][V_1/y_1,\ldots, V_{j-1}/y_{j-1},V_j/x_i,V_{j+1}/y_{j+1},\ldots,V_n/y_n]Q$

$\equiv \lambda P.[[U_1/x_1,\ldots, U_m/x_m]V_1/y_1,\ldots, [U_1/x_1,\ldots, U_m/x_m]V_{j-1}/y_{j-1},$
$[U_1/x_1,\ldots, U_m/x_m]V_j/x_i,[U_1/x_1,\ldots, U_m/x_m]V_{j+1}/y_{j+1},\ldots,$
$[U_1/x_1,\ldots, U_m/x_m]V_n/y_n, U_1/x_1,\ldots, U_{i-1}/x_{i-1}, U_{i+1}/x_{i+1},\ldots, U_m/x_m]Q$

(by induction)

$\equiv [[U_1/x_1,\ldots, U_m/x_m]V_1/y_1,\ldots,[U_1/x_1,\ldots, U_m/x_m]V_{j-1}/y_{j-1},[U_1/x_1,\ldots, U_m/x_m]V_j/x_i,$
$[U_1/x_1,\ldots, U_m/x_m]V_{j+1}/y_{j+1},\ldots, [U_1/x_1,\ldots, U_m/x_m]V_n/y_n, U_1/x_1,\ldots, U_{i-1}/x_{i-1},$
$U_{i+1}/x_{i+1},\ldots, U_m/x_m]M.$ (by Corollaries 2.1.14 and 2.1.12(d))

iv. $M \equiv (\lambda P.Q \mid A)$.

This is similar to (ii).

Part (b) can be proved similarly. □

Corollary 2.1.17. Let $x_1,..., x_m$, $m \geq 1$, be distinct variables, $y_1,..., y_n$, $n \geq 1$, be distinct variables, $U_1,\ldots, U_m$, $V_1,\ldots,V_n$ be terms, and M be a term such that no bound variable in M is free in $x_1\ldots x_m y_1\ldots y_n U_1\ldots U_m V_1\ldots V_n$.

a. If $FV(M) \cap (\{x_1,\ldots, x_m\} - \{y_1,\ldots, y_n\}) = \varnothing$, then

$$[U_1/x_1,\ldots, U_m/x_n][V_1/y_1,\ldots,V_n/y_n]M$$
$$\equiv [[U_1/x_1,\ldots, U_m/x_m]V_1/y_1,\ldots,[U_1/x_1,\ldots, U_m/x_m]V_n/y_n]M.$$

b. If $FV(M) \cap (\{x_1,\ldots, x_m\} - \{y_1,\ldots, y_n\}) = \{x_{i_1},\ldots, x_{i_k}\}$, then

$$[U_1/x_1,\ldots, U_m/x_m][V_1/y_1,\ldots,V_n/y_n]\, M$$
$$\equiv [[U_1/x_1,\ldots, U_m/x_m]V_1/y_1,\ldots,[U_1/x_1,\ldots, U_m/x_m]V_n/y_n, U_{i_1}/x_{i_1},\ldots, U_{i_k}/x_{i_k}]M.$$

c. If $\{x_1,\ldots, x_m\} \cap FV\,(y_1\ldots y_n V_1\ldots V_n) = \varnothing$, then

$$[U_1/x_1,\ldots, U_m/x_m][V_1/y_1,\ldots,V_n/y_n]M \equiv [U_1/x_1,\ldots, U_m/x_m, V_1/y_1,\ldots,V_n/y_n]M.$$

Proof. All parts follow from Lemma 2.1.16 and Corollary 2.1.12(a), (b). □

2.2 Changes of Bound Variables

As mentioned in Chapter I, congruent terms of the original λ-calculus have identical interpretations. This also holds for the new λ-calculus. However, the definition of changes of bound variables has to be adjusted slightly since we have patterns in the new λ-calculus.

Definition 2.2.1. Let A be an occurrence of a simple abstraction λP.Q in a term M. Let $x \in FV(P)$ and $y \notin FV(PQ)$. The act of replacing A by $\lambda[y/x]P.[y/x]Q$ is called a **change of bound variable** or an **α-step** in M. We say M is **congruent** to a term N, or M **α-converts** to N, denoted by $M \equiv_\alpha N$, if N is obtained from M by a finite sequence of changes of bound variables.

Lemma 2.2.2. Let M be a term and N be an expression. If $M \equiv_\alpha N$, then N is a term and $lgh(M) = lgh(N)$.

Proof. Assume $M \equiv_\alpha N$. Then N is obtained from M by a finite sequence of changes of bound variables. Therefore to prove this lemma, it is enough to prove the result for a single change of bound variable. So suppose N is obtained from M by a single change of bound variable. Let A be the occurrence of a simple abstraction $\lambda P.Q$ in M such that A changes to $\lambda[y/x]P.[y/x]Q$, where $x \in FV(P)$ and $y \notin FV(PQ)$, in N. Since the only part of M that changes is A, it is enough to prove that $\lambda[y/x]P.[y/x]Q$ is a simple abstraction and $lgh(\lambda P.Q) = lgh(\lambda[y/x]P.[y/x]Q)$.

By Lemma 2.1.10(a), (b), $[y/x]P$ is a pattern and $[y/x]Q$ is a term. Hence $\lambda[y/x]P.[y/x]Q$ is a simple abstraction.

By Lemma 2.1.15, $lgh(P) = lgh([y/x]P)$ and $lgh(Q) = lgh([y/x]Q)$. Hence $lgh(\lambda P.Q) = lgh(P) + lgh(Q) = lgh([y/x]P) + lgh([y/x]Q) = lgh(\lambda[y/x]P.[y/x]Q)$. □

Lemma 2.2.3. The relation $\equiv_\alpha$ is transitive, reflexive and symmetric.

Proof. Transitivity and reflexivity are obvious. For symmetry, it is enough to consider a single change of bound variable. That is, it is enough to prove that $\lambda[y/x]P.[y/x]Q \equiv_\alpha \lambda P.Q$, where P is a pattern, Q is a term, $x \in FV(P)$, and $y \notin FV(PQ)$. Since $y \in FV([y/x]P)$ and $x \notin FV([y/x](PQ))$, we have $\lambda[y/x]P.[y/x]Q \equiv_\alpha \lambda[x/y][y/x]P.[x/y][y/x]Q \equiv \lambda P.[x/y][y/x]Q$. Thus, it is enough to prove that $[x/y][y/x]Q \equiv_\alpha Q$ wherever $y \notin FV(Q)$.

If $x \notin FV(Q)$, then $[x/y][y/x]Q \equiv [x/y]Q \equiv Q$. Therefore suppose $x \in FV(Q)$, and induct on Q.

i. $Q \equiv x$.

Then $[x/y][y/x]Q \equiv [x/y]y \equiv x \equiv Q$.

ii. $Q \equiv Q_1Q_2$.

This case follows straightforwardly by induction.

iii. $Q \equiv \lambda U.V$.

Since $x \in FV(Q)$, $x \notin FV(U)$ and $x \in FV(V)$.

Case 1. $y \notin FV(U)$.

Since $y \notin FV(Q)$, $y \notin FV(V)$. Hence, by induction

$[x/y][y/x]Q \equiv [x/y](\lambda U.[y/x]V) \equiv \lambda U.[x/y][y/x]V \equiv_\alpha \lambda U.V \equiv Q$.

Case 2. $y \in FV(U)$.

Let z be the first variable which is not in $FV(yUV)$. Since $x \in FV(V)$, $x \not\equiv z$ and since $x \not\equiv y$, $x \in FV(\lambda[z/y]U.[z/y]V)$ and so $y \in FV(\lambda[z/y]U.[y/x][z/y]V)$. Note that y is the first variable in $FV(U) \cap \{y\}$. Hence

$$
\begin{aligned}
[x/y][y/x]Q &\equiv [x/y][y/x](\lambda[z/y]U.[z/y]V) \\
&\equiv [x/y](\lambda[z/y]U.[y/x][z/y]V) \\
&\equiv \lambda[z/y]U.[x/y][y/x][z/y]V \\
&\equiv_\alpha \lambda[z/y]U.[z/y]V && \text{(by induction)} \\
&\equiv_\alpha \lambda[y/z][z/y]U.[y/z][z/y]V \\
&\equiv \lambda U.[y/z][z/y]V && \text{(by Lemma 2.1.16 and Corollary 2.1.12(c))} \\
&\equiv_\alpha \lambda U.V \equiv Q. && \text{(by induction)}
\end{aligned}
$$

iv. $Q \equiv (\lambda U.V \mid A)$.

This is similar to (ii). □

Lemma 2.2.4. Let M and N be terms such that $M \equiv_\alpha N$.

a. If $M \equiv M_1M_2$, then $N \equiv N_1N_2$ for some terms N_1 and N_2, where $M_i \equiv_\alpha N_i$, $i = 1, 2$.

b. If $M \equiv \lambda P.Q$ and no variable in P has been changed, then $N \equiv \lambda P.Q'$ for some term Q', where $Q \equiv_\alpha Q'$.

c. If $M \equiv (\lambda P.Q \mid A)$, then $N \equiv (\lambda P'.Q' \mid A')$ for some abstractions $\lambda P'.Q'$ and A', where $\lambda P.Q \equiv_\alpha \lambda P'.Q'$ and $A \equiv_\alpha A'$.

Proof. Since $M \equiv_\alpha N$, there exists a sequence of terms $M \equiv L_1, L_2, \ldots, L_n \equiv N$, $n \geq 1$, such that for each $1 \leq i < n$, L_{i+1} is obtained from L_i by a single change of bound variable.

Proof of (a). Assume $M \equiv M_1M_2$, and induct on n.

If n = 1, then $M \equiv N$ and there is nothing to prove.

Now assume n > 1. Since $M \equiv_\alpha L_{n-1}$, by induction $L_{n-1} \equiv T_1T_2$ for some terms T_1 and T_2, where $M_i \equiv_\alpha T_i$, i = 1, 2. Note that N is obtained from L_{n-1} by a single change of bound variable. Let A be the occurrence of a simple abstraction $\lambda P.Q$ in L_{n-1} such that A changes to $A' \equiv \lambda[y/x]P.[y/x]Q$, where $x \in FV(P)$ and $y \notin FV(PQ)$, in N. Since A is an abstraction, A is either in T_1 or in T_2. Assume A is in T_1. When A changes to A', suppose T_1 changes to N_1. So $N \equiv N_1T_2$, where $T_1 \equiv_\alpha N_1$. The case A is in T_2 can be proved similarly. So we have $N \equiv N_1N_2$ for some terms N_1 and N_2, where $T_i \equiv_\alpha N_i$, i = 1, 2. By the transitivity of the relation $\equiv_\alpha$, $M_i \equiv_\alpha N_i$, i = 1, 2.

Parts (b) and (c) can be proved similarly. □

Lemma 2.2.5.

a. For any terms M and N, if $M \equiv_\alpha N$, then FV(M) = FV(N).

b. For any term M, any variables $x_1, \ldots, x_n$, $n \geq 1$, there exists a term M' such that $M \equiv_\alpha M'$ and none of $x_1, \ldots, x_n$ is bound in M'.

Proof of (a). Let M and N be terms such that $M \equiv_\alpha N$. Then N is obtained from M by a finite sequence of changes of bound variables. Therefore to prove this lemma, it is enough to prove the result for a single change of bound variable. So suppose N is obtained from M by a single change of bound variable. Let A be the occurrence of a simple abstraction $\lambda P.Q$ in M such that A changes to $\lambda[y/x]P.[y/x]Q$, where $x \in FV(P)$ and $y \notin FV(PQ)$, in N. Since the only part of M that changes is A, it is enough to show that $FV(\lambda P.Q) = FV(\lambda[y/x]P.[y/x]Q)$.

Case 1. $x \notin FV(Q)$.

$$\begin{aligned} FV(\lambda[y/x]P.[y/x]Q) &= FV(\lambda[y/x]P.Q) \\ &= FV(Q) - FV([y/x]P) \\ &= FV(Q) - (\{y\} \cup (FV(P) - \{x\})) \quad \text{(by Lemma 2.1.13)} \\ &= FV(Q) - FV(P) \\ &= FV(\lambda P.Q). \end{aligned}$$

Case 2. $x \in FV(Q)$.

$$
\begin{aligned}
FV(\lambda[y/x]P.[y/x]Q) &= FV([y/x]Q) - FV([y/x]P) \\
&= (\{y\} \cup (FV(Q) - \{x\})) - (\{y\} \cup (FV(P) - \{x\})) \\
&= FV(Q) - (\{x\} \cup (FV(P) - \{x\})) \\
&= FV(Q) - FV(P) \\
&= FV(\lambda P.Q).
\end{aligned}
$$

□

Proof of (b). Let M be a term and $x_1,\dots, x_n$, $n \geq 1$, be variables and induct on M.

i. M is an atom.

The result is obvious since M contains no bound variables.

ii. $M \equiv M_1M_2$.

By induction, there exist terms M_1' and M_2' such that $M_1 \equiv_\alpha M_1'$, $M_2 \equiv_\alpha M_2'$ and none of $x_1,\dots, x_n$ is bound in M_1' or M_2'. Let $M' \equiv M_1'M_2'$. Then none of $x_1,\dots, x_n$ is bound in M' and $M \equiv M_1M_2 \equiv_\alpha M_1'M_2' \equiv M'$.

iii. $M \equiv \lambda P.Q$.

Let $m = |FV(P) \cap \{x_1,\dots, x_n\}|$ and induct on m.

Suppose $m = 0$, so that $FV(P) \cap \{x_1,\dots, x_n\} = \varnothing$. By induction, there exists a term Q' such that $Q \equiv_\alpha Q'$ and none of $x_1,\dots, x_n$ is bound in Q'. Let $M' \equiv \lambda P.Q'$. Then none of $x_1,\dots, x_n$ is bound in M' and $M \equiv \lambda P.Q \equiv_\alpha \lambda P.Q' \equiv M'$.

Now assume $m > 0$. Let $x_t \in FV(P) \cap \{x_1,\dots, x_n\}$ and $y \notin FV(x_1\dots x_nPQ)$.

Let $M_0 \equiv \lambda[y/x_t]P.[y/x_t]Q$. By induction on m, there exists a term M' such that $M_0 \equiv_\alpha M'$ and none of $x_1,\dots, x_n$ is bound in M'. Since $M \equiv_\alpha M_0 \equiv_\alpha M'$ and the relation $\equiv_\alpha$ is transitive, $M \equiv_\alpha M'$.

iv. $M \equiv (\lambda P.Q \mid A)$.

This is similar to (ii). □

Lemma 2.2.6. Let $x, v, y_1,\dots, y_n$, $n \geq 1$, be distinct variables, and $V, U_1,\dots, U_n$ be terms.

a. For any term M, if $v \notin FV(M)$, then $[V/v][v/x]M \equiv_\alpha [V/x]M$.

b. For any term M, if $x \notin FV(U_1 \ldots U_n)$, then

$[U_1/y_1,\ldots, U_n/y_n][V/x]M \equiv_\alpha [[U_1/y_1,\ldots, U_n/y_n]V/x][U_1/y_1,\ldots, U_n/y_n]M$.

Lemma 2.2.7. Let $x_1,\ldots, x_k$, $k \geq 1$, be distinct variables, and $N_1,\ldots, N_k, N_1',\ldots, N_k'$ be terms such that $N_i \equiv_\alpha N_i'$ for all $1 \leq i \leq k$. Then, for any terms M and M′, if $M \equiv_\alpha M'$, then $[N_1/x_1,\ldots, N_k/x_k]M \equiv_\alpha [N_1'/x_1,\ldots, N_k'/x_k]M'$.

Proof of Lemmas 2.2.6 and 2.2.7.

Let M and M′ be terms.

For Lemma 2.2.6(a), assume $v \notin FV(M)$, for Lemma 2.2.6(b), assume $x \notin FV(U_1 \ldots U_n)$, and for Lemma 2.2.7, assume $M \equiv_\alpha M'$.

For Lemma 2.2.6, suppose $x \notin FV(M)$. Then,

for (a) we have $[V/v][v/x]M \equiv [V/v]M \equiv M \equiv [V/x]M$ and

for (b), since $x \notin FV(U_1 \ldots U_n)$ and $x \notin FV(M)$, by Corollary 2.1.14 $x \notin FV([U_1/y_1,\ldots, U_n/y_n]M)$. Hence, by Corollary 2.1.12(a)

$$\begin{aligned}[][[U_1/y_1,\ldots, U_n/y_n]V/x][U_1/y_1,\ldots, U_n/y_n]M &\equiv [U_1/y_1,\ldots, U_n/y_n]M \\ &\equiv [U_1/y_1,\ldots, U_n/y_n][V/x]M.\end{aligned}$$

So from this point on we may suppose $x \in FV(M)$.

For Lemma 2.2.7, by Corollary 2.1.12(a), (b) and Lemma 2.2.5(a) we may assume that $\{x_1,\ldots, x_k\} \subseteq FV(M)$.

We will now prove Lemmas 2.2.6 and 2.2.7 simultaneously, by induction on M.

i. M is an atom.

For Lemma 2.2.6, $M \equiv x$. Thus, for (a) we have

$[V/v][v/x]M \equiv [V/v]v \equiv V \equiv [V/x]M$ and for (b) we have

$$\begin{aligned}[][U_1/y_1,\ldots, U_n/y_n][V/x]M &\equiv [U_1/y_1,\ldots, U_n/y_n]V \\ &\equiv [[U_1/y_1,\ldots, U_n/y_n]V/x]x \\ &\equiv [[U_1/y_1,\ldots, U_n/y_n]V/x][U_1/y_1,\ldots, U_n/y_n]x \\ &\equiv [[U_1/y_1,\ldots, U_n/y_n]V/x][U_1/y_1,\ldots, U_n/y_n]M.\end{aligned}$$

For Lemma 2.2.7, $M \equiv x_1 \equiv M'$. Hence $[N_1/x_1]M \equiv N_1 \equiv_\alpha N_1' \equiv [N_1'/x_1]M'$.

ii. $M \equiv M_1M_2$.

This case follows straightforwardly by induction, using the facts that $v \notin FV(M)$ implies $v \notin FV(M_1)$ and $v \notin FV(M_2)$, and $M \equiv_\alpha M'$ implies $M' \equiv M_1'M_2'$, where $M_1 \equiv_\alpha M_1'$ and $M_2 \equiv_\alpha M_2'$.

iii. $M \equiv \lambda P.Q$.

We will prove Lemma 2.2.7 first.

Since $M \equiv_\alpha M'$, we have a sequence of α-congruences in which each congruence is of the form $\lambda P'.Q' \equiv_\alpha \lambda P'.Q''$, where $Q' \equiv_\alpha Q''$, or of the form $\lambda P'.Q' \equiv_\alpha \lambda[w/u]P'.[w/u]Q'$, where $u \in FV(P')$ and $w \notin FV(P'Q')$. By the transitivity of the relation $\equiv_\alpha$, this says we only need to consider the cases $M' \equiv \lambda P.Q'$, where $Q' \equiv_\alpha Q$, and $M' \equiv \lambda[w/u]P.[w/u]Q$, where $u \in FV(P)$ and $w \notin FV(PQ)$.

Case I. $M' \equiv \lambda P.Q'$, where $Q' \equiv_\alpha Q$.

Let $m = |FV(P) \cap FV(N_1 \ldots N_K)|$ and induct on m. Note that $FV(N_1 \ldots N_k) = FV(N_1' \ldots N_k')$ and $FV(Q) = FV(Q')$.

If $m = 0$, then
$$\begin{aligned}[N_1/x_1,\ldots, N_k/x_k]M &\equiv \lambda P.[N_1/x_1,\ldots, N_k/x_k]Q.\\ &\equiv_\alpha \lambda P.[N_1'/x_1,\ldots, N_k'/x_k]Q' \quad \text{(by induction on M)}\\ &\equiv [N_1'/x_1,\ldots, N_k'/x_k]M'.\end{aligned}$$

Now assume $m > 0$. Let u be the first variable in $FV(P) \cap FV(N_1 \ldots N_k)$ and z be the first variable which is not in $FV(PQN_1 \ldots N_k)$. By induction on M, $[z/u]Q \equiv_\alpha [z/u]Q'$, so $\lambda[z/u]P.[z/u]Q \equiv_\alpha \lambda[z/u]P.[z/u]Q'$. Hence
$$\begin{aligned}[N_1/x_1,\ldots, N_k/x_k]M &\equiv [N_1/x_1,\ldots, N_k/x_k](\lambda[z/u]P.[z/u]Q)\\ &\equiv_\alpha [N_1'/x_1,\ldots, N_k'/x_k](\lambda[z/u]P.[z/u]Q') \quad \text{(by induction on m)}\\ &\equiv [N_1'/x_1,\ldots, N_k'/x_k](\lambda P.Q')\\ &\equiv [N_1'/x_1,\ldots, N_k'/x_k]M'.\end{aligned}$$

Case II. $M' \equiv \lambda[w/u]P.[w/u]Q$, where $u \in FV(P)$, $w \notin FV(PQ)$.

Since $\{x_1,\ldots, x_k\} \subseteq FV(M) = FV(Q) - FV(P)$, $x_i \not\equiv u$ and $x_i \not\equiv w$ for all

$1 \leq i \leq k$. Let $m = |FV([w/u]P) \cap FV(N_1' \ldots N_k')|$ and induct on m.

For the case m = 0, we have

$$[N_1'/x_1,\ldots, N_k'/x_k]M' \equiv \lambda[w/u]P.[N_1'/x_1,\ldots, N_k'/x_k][w/u]Q$$
$$\equiv_\alpha \lambda[w/u]P.[N_1/x_1,\ldots, N_k/x_k][w/u]Q. \qquad \text{(by induction on M)}$$

Let $T \equiv \lambda[w/u]P.[N_1/x_1,\ldots, N_k/x_k][w/u]Q$.

Case 1. $u \notin FV(N_1\ldots N_k)$, so $FV(P) \cap FV(N_1\ldots N_k) = \varnothing$. Then

$$T \equiv_\alpha \lambda[u/w][w/u]P.[u/w][N_1/x_1,\ldots, N_k/x_k][w/u]Q$$
$$\equiv_\alpha \lambda P.[N_1/x_1,\ldots, N_k/x_k][u/w][w/u]Q$$
$$\text{(by 2.1.17, 2.1.12(c) and induction (2.2.6(b)))}$$
$$\equiv_\alpha \lambda P.[N_1/x_1,\ldots, N_k/x_k]Q \qquad \text{(by induction (2.2.6(a), 2.2.7) and 2.1.12(c)))}$$
$$\equiv [N_1/x_1,\ldots, N_k/x_k](\lambda P.Q)$$
$$\equiv [N_1/x_1,\ldots, N_k/x_k]M.$$

Case 2. $u \in FV(N_1\ldots N_k)$, so $FV(P) \cap FV(N_1\ldots N_k) = \{u\}$.

Let z be the first variable which is not in $FV(PQN_1\ldots N_k)$. Then

$$T \equiv_\alpha \lambda[z/w][w/u]P.[z/w][N_1/x_1,\ldots, N_k/x_k][w/u]Q$$
$$\equiv_\alpha \lambda[z/u]P.[N_1/x_1,\ldots, N_k/x_k][z/w][w/u]Q \quad \text{(by 2.1.16 and induction (2.2.6(b)))}$$
$$\equiv_\alpha \lambda[z/u]P.[N_1/x_1,\ldots, N_k/x_k][z/u]Q \qquad \text{(by induction (2.2.6(a), 2.2.7))}$$
$$\equiv [N_1/x_1,\ldots, N_k/x_k](\lambda[z/u]P.[z/u]Q)$$
$$\equiv [N_1/x_1,\ldots, N_k/x_k](\lambda P.Q)$$
$$\equiv [N_1/x_1,\ldots, N_k/x_k]M.$$

Now assume m > 0. Let v' be the first variable in $FV([w/u]P) \cap FV(N_1'\ldots N_k')$ and z be the first variable which is not in $FV(N_1'\ldots N_k'[w/u](PQ))$. Then

$$[N_1'/x_1,\ldots, N_k'/x_k]M' \equiv [N_1'/x_1,\ldots, N_k'/x_k](\lambda[w/u]P.[w/u]Q)$$
$$\equiv [N_1'/x_1,\ldots, N_k'/x_k](\lambda[z/v'][w/u]P.[z/v'][w/u]Q).$$

Let $T \equiv [N_1'/x_1,\ldots, N_k'/x_k](\lambda[z/v'][w/u]P.[z/v'][w/u]Q)$. There are cases and subcases as follows.

(1) u is the first variable in $FV(P) \cap FV(N_1\ldots N_k)$, so $u \not\equiv z$.

(1.1) $v' \equiv w$, so $w \in FV(N_1\ldots N_k)$ and z is the first variable which is not in $FV(PQN_1\ldots N_k)$.

$T \equiv [N_1'/x_1,\ldots, N_k'/x_k](\lambda[z/u]P.[z/v'][w/u]Q)$ (by 2.1.17)

$\equiv_\alpha [N_1/x_1,\ldots, N_k/x_k](\lambda[z/u]P.[z/u]Q)$ (by induction (2.2.6(a)) and Case I)

$\equiv [N_1/x_1,\ldots, N_k/x_k](\lambda P.Q)$

$\equiv [N_1/x_1,\ldots, N_k/x_k]M.$

(1.2) $v' \not\equiv w$, so $v' \in FV(P) \cap FV(N_1\ldots N_k)$.

(1.2.1) $w \in FV(N_1\ldots N_k)$, so $|FV(P) \cap FV(N_1\ldots N_k)| = m$.

$T \equiv_\alpha [N_1/x_1,\ldots, N_k/x_k](\lambda[w/u][z/v']P.[w/u][z/v']Q)$

(by 2.1.17, induction(2.2.6(b)) and Case I)

$\equiv_\alpha [N_1/x_1,\ldots, N_k/x_k](\lambda[z/v']P.[z/v']Q)$ (by induction on m)

$\equiv_\alpha [N_1/x_1,\ldots, N_k/x_k](\lambda P.Q)$ (by induction on m)

$\equiv [N_1/x_1,\ldots, N_k/x_k]M.$

(1.2.2) $w \notin FV(N_1\ldots N_k)$, so $|FV(P) \cap FV(N_1\ldots N_k)| = m + 1$.

(1.2.2.1) w is the first variable which is not in $FV(PQN_1\ldots N_k)$.

$T \equiv_\alpha [N_1/x_1,\ldots, N_k/x_k](\lambda[w/u]P.[w/u]Q)$ (by induction on m)

$\equiv [N_1/x_1,\ldots, N_k/x_k](\lambda P.Q)$

$\equiv [N_1/x_1,\ldots, N_k/x_k]M.$

(1.2.2.2) w is not the first variable which is not in $FV(PQN_1\ldots N_k)$.

Thus z is the first variable which is not in $FV(PQN_1\ldots N_k)$ since $z \not\equiv u$.

Let $z' \notin FV(zwPQN_1\ldots N_k)$.

$T \equiv_\alpha [N_1/x_1,\ldots, N_k/x_k](\lambda[z'/z][z/v'][w/u]P.[z'/z][z/v'][w/u]Q)$

$\equiv_\alpha [N_1/x_1,\ldots, N_k/x_k](\lambda[z'/v'][w/u]P.[z'/v'][w/u]Q)$

$\equiv_\alpha [N_1/x_1,\ldots, N_k/x_k](\lambda[z/w][z'/v'][w/u]P.[z/w][z'/v'][w/u]Q)$

$\equiv_\alpha [N_1/x_1,\ldots, N_k/x_k](\lambda[z'/v'][z/w][w/u]P.[z'/v'][z/w][w/u]Q)$

$\equiv_\alpha [N_1/x_1,\ldots, N_k/x_k](\lambda[z'/v'][z/u]P.[z'/v'][z/u]Q)$

$\equiv_\alpha [N_1/x_1,\ldots, N_k/x_k](\lambda[z/u]P.[z/u]Q)$

$\equiv [N_1/x_1,\ldots, N_k/x_k](\lambda P.Q)$

$\equiv [N_1/x_1,\ldots, N_k/x_k]M.$

(2) u is not the first variable in $FV(P) \cap FV(N_1\ldots N_k)$.

(2.1) $v' \equiv w$, so $w \in FV(N_1 \ldots N_k)$ and $|FV(P) \cap FV(N_1 \ldots N_k)| \leq m$, since $w \notin FV(P)$.

$T \equiv_\alpha [N_1/x_1, \ldots, N_k/x_k](\lambda[z/u]P.[z/v'][w/u]Q)$

$\equiv_\alpha [N_1/x_1, \ldots, N_k/x_k](\lambda[z/u]P.[z/u]Q)$

$\equiv_\alpha [N_1/x_1, \ldots, N_k/x_k](\lambda P.Q)$

$\equiv [N_1/x_1, \ldots, N_k/x_k]M.$

(2.2) $v' \not\equiv w$, so v' is the first variable in $FV(P) \cap FV(N_1 \ldots N_k)$.

(2.2.1) $w \in FV(N_1 \ldots N_k)$, so $|FV(P) \cap FV(N_1 \ldots N_k)| \leq m$.

(2.2.1.1) $z \not\equiv u$.

$T \equiv_\alpha [N_1/x_1, \ldots, N_k/x_k](\lambda[w/u][z/v']P.[w/u][z/v']Q)$

$\equiv_\alpha [N_1/x_1, \ldots, N_k/x_k](\lambda[z/v']P.[z/v']Q)$

$\equiv_\alpha [N_1/x_1, \ldots, N_k/x_k](\lambda P.Q)$

$\equiv [N_1/x_1, \ldots, N_k/x_k]M.$

(2.2.1.2) $z \equiv u$, so $FV(P) \cap FV(N_1 \ldots N_k) = m - 1$.

$T \equiv_\alpha [N_1/x_1, \ldots, N_k/x_k](\lambda[w/u]P.[w/u]Q)$ (by induction on m)

$\equiv_\alpha [N_1/x_1, \ldots, N_k/x_k](\lambda[u/w][w/u]P.[u/w][w/u]Q)$ (by induction on m)

$\equiv [N_1/x_1, \ldots, N_k/x_k](\lambda P.[u/w][w/u]Q)$ (by 2.1.17 and 2.1.12(c))

$\equiv_\alpha [N_1/x_1, \ldots, N_k/x_k](\lambda P.Q)$ (by Case I, induction (2.2.6(a)) and 2.1.12(c))

$\equiv [N_1/x_1, \ldots, N_k/x_k]M.$

(2.2.2) $w \notin FV(N_1 \ldots N_k)$, so $|FV(P) \cap FV(N_1 \ldots N_k)| \leq m + 1$.

(2.2.2.1) z is the first variable which is not in $FV(PQN_1 \ldots N_k)$, so $z \not\equiv u$.

$T \equiv_\alpha [N_1/x_1, \ldots, N_k/x_k](\lambda[w/u][z/v']P.[w/u][z/v']Q)$

$\equiv_\alpha [N_1/x_1, \ldots, N_k/x_k](\lambda[z/v']P.[z/v']Q)$

$\equiv [N_1/x_1, \ldots, N_k/x_k](\lambda P.Q)$

$\equiv [N_1/x_1, \ldots, N_k/x_k]M.$

(2.2.2.2) z is not the first variable which is not in $FV(PQN_1 \ldots N_k)$.

Let z'' be the first variable which is not in $FV(PQN_1 \ldots N_k)$ and choose $z' \notin FV(zz''wPQN_1 \ldots N_k)$.

$T \equiv_\alpha [N_1/x_1,\ldots, N_k/x_k](\lambda[z'/w][z/v'][w/u]P.[z'/w][z/v'][w/u]Q)$

$\equiv_\alpha [N_1/x_1,\ldots, N_k/x_k](\lambda[z/v'][z'/w][w/u]P.[z/v'][z'/w][w/u]Q)$

$\equiv_\alpha [N_1/x_1,\ldots, N_k/x_k](\lambda[z/v'][z'/u]P.[z/v'][z'/u]Q)$

$\equiv_\alpha [N_1/x_1,\ldots, N_k/x_k](\lambda[z''/z][z/v'][z'/u]P.[z''/z][z/v'][z'/u]Q)$

$\equiv_\alpha [N_1/x_1,\ldots, N_k/x_k](\lambda[z''/v'][z'/u]P.[z''/v'][z'/u]Q)$

$\equiv_\alpha [N_1/x_1,\ldots, N_k/x_k](\lambda[z'/u][z''/v']P.[z'/u][z''/v']Q)$

$\equiv_\alpha [N_1/x_1,\ldots, N_k/x_k](\lambda[z''/v']P.[z''/v']Q)$

$\equiv [N_1/x_1,\ldots, N_k/x_k](\lambda P.Q)$

$\equiv [N_1/x_1,\ldots, N_k/x_k]M.$

This finishes the proof of Lemma 2.2.7 for $M \equiv \lambda P.Q$.

Next we will prove Lemma 2.2.6. By Lemma 2.2.5(b), there exists a term M_1 such that $M_1 \equiv_\alpha M$ and no variable bound in M_1 is free in $xvU_1\ldots U_kV$, so $FV(M_1) = FV(M)$ (by Lemma 2.2.5(a)). Hence, by Corollary 2.1.17 and Lemma 2.2.7 for (a) we have $[V/v][v/x]M \equiv_\alpha [V/v][v/x]M_1 \equiv [[V/v]v/x, V/v]M_1$

$\equiv [V/x]M_1 \equiv_\alpha [V/x]M$ and

for (b) we have

$[U_1/y_1,\ldots, U_n/y_n][V/x]M \equiv_\alpha [U_1/y_1,\ldots, U_n/y_n][V/x]M_1$

$\equiv [[U_1/y_1,\ldots, U_n/y_n]V/x, U_1/y_1,\ldots, U_n/y_n]M_1$

$\equiv [[U_1/y_1,\ldots, U_n/y_n]V/x][U_1/y_1,\ldots, U_n/y_n]M_1$

$\equiv_\alpha [[U_1/y_1,\ldots, U_n/y_n]V/x][U_1/y_1,\ldots, U_n/y_n]M.$

iv. $M \equiv (\lambda P.Q \mid A)$.

This is similar to (ii). □

Corollary 2.2.8. If we remove the condition on variables bound in M from Lemma 2.1.16 and Corollary 2.1.17 and replace $\equiv$ by $\equiv_\alpha$, then those two results remain true.

Proof. This follows from Lemmas 2.2.5(b), 2.2.7 and 2.1.16, and Corollary 2.1.17. □

2.3 Contractions and Reductions

As with the original λ-calculus, we think of abstractions as representing functions and applications of abstractions to terms as representing functions applied to arguments. We also follow the original λ-calculus in calculating the results of such applications by performing contractions. However, since our terms are more complicated, our definitions of contraction and reduction are more complicated as well. In particular, we find it necessary to define three types of contraction – β-contractions, γ-contractions, and δ-contractions – and corresponding types of reduction – β-reduction, βγ-reduction, βδ-reduction. Ultimately, we are only interested in βδ-reduction, but we need the other two types of reduction to define it properly.

Definition 2.3.1. For any abstraction A, and any term N, AN is called a **potential redex**, and if A is a simple (respectively compound) abstraction, then we call AN a **simple** (respectively **compound**) **potential redex**.

Definition 2.3.2. For any pattern P with $FV(P) = \{x_1,\ldots, x_k\}$, $k \geq 1$ (respectively $FV(P) = \varnothing$), and any term N, if there exist terms $N_1,\ldots, N_k$ such that $[N_1/x_1,\ldots, N_k/x_k]P \equiv N$ (respectively $P \equiv N$), then for any term Q, $(\lambda P.Q)N$ is called a **β-redex** and the corresponding term $[N_1/x_1,\ldots, N_k/x_k]Q$ (respectively Q) is called its **β-contractum**.

Let R be an occurrence of a β-redex in a term M. If we replace R by its β-contractum, and the result is the expression M', then we say M **β-contracts** to M', which we denote by $M \rhd_{1\beta} M'$.

Definition 2.3.3. For any terms M and M', we say M **β-reduces** to M', denoted by $M \rhd_{\beta} M'$, if there exists a sequence of terms $M \equiv M_1, M_2,\ldots, M_n \equiv M'$, $n \geq 1$, such that for each $1 \leq i < n$, $M_i \rhd_{1\beta} M_{i+1}$ or $M_i \equiv_{\alpha} M_{i+1}$.

Definition 2.3.4. Let $(\lambda P.Q \mid A)$ be a compound abstraction and N be a term and let $FV(N) = \{y_1,\ldots, y_m\}$, $m \geq 1$ (respectively $FV(N) = \varnothing$). We will call $(\lambda P.Q \mid A)N$ a **γ-redex** with **γ-contractum** S if one of the following two conditions holds:

a. $(\lambda P.Q)N$ is a β-redex, in which case $S \equiv (\lambda P.Q)N$; or

b. for all terms $U_1,\ldots, U_m$ and all terms N' such that $[U_1/y_1,\ldots, U_m/y_m]N \rhd_\beta N$ (respectively $N \rhd_\beta N'$), $(\lambda P.Q)N'$ is not a β-redex, in which case $S \equiv AN$.

Let R be an occurrence of a γ-redex in a term M. If we replace R by its γ-contractum, and the result is the expression M', then we say M **γ-contracts** to M', which we denote by $M \rhd_{1\gamma} M'$.

Definition 2.3.5. For any terms M and M', we say M **βγ-reduces** to M', denoted by $M \rhd_{\beta\gamma} M'$, if there exists a sequence of terms $M \equiv M_1, M_2,\ldots, M_n \equiv M'$, $n \geq 1$, such that for each $1 \leq i < n$, $M_i \rhd_{1\beta} M_{i+1}$, $M_i \rhd_{1\gamma} M_{i+1}$, or $M_i \equiv_\alpha M_{i+1}$.

Definition 2.3.6. Let $(\lambda P.Q \mid A)$ be a compound abstraction and N be a term and let $FV(N) = \{y_1,\ldots, y_m\}$, $m \geq 1$ (respectively $FV(N) = \varnothing$). We will call $(\lambda P.Q \mid A)N$ a **δ-redex** with **δ-contractum** S if one of the following two conditions holds:

a. $(\lambda P.Q)N$ is a β-redex, in which case $S \equiv (\lambda P.Q)N$; or

b. for all terms $U_1,\ldots, U_m$ and all terms N' such that $[U_1/y_1,\ldots, U_m/y_m]N \rhd_{\beta\gamma} N'$ (respectively $N \rhd_{\beta\gamma} N'$), $(\lambda P.Q)N'$ is not a β-redex, in which case $S \equiv AN$.

Let R be an occurrence of a δ-redex in a term M. If we replace R by its δ-contractum, and the result is the expression M', then we say M **δ-contracts** to M', which we denote by $M \rhd_{1\delta} M'$.

Definition 2.3.7. For any terms M and M', we say M **βδ-reduces** to M', denoted by $M \rhd_{\beta\delta} M'$, if there exists a sequence of terms $M \equiv M_1, M_2,\ldots, M_n \equiv M'$, $n \geq 1$, such that for each $1 \leq i < n$, $M_i \triangleright_{1\beta} M_{i+1}$, $M_i \triangleright_{1\delta} M_{i+1}$, or $M_i \equiv_\alpha M_{i+1}$.

Notes 2.3.8.

a. If $M \triangleright_{1\delta} N$, then $M \triangleright_{1\gamma} N$.

b. If $M \triangleright_{\beta\delta} N$, then $M \triangleright_{\beta\gamma} N$.

c. The relations $\triangleright_\beta$, $\triangleright_{\beta\gamma}$, and $\triangleright_{\beta\delta}$ are transitive and reflexive but not symmetric.

Notation 2.3.9. The expression $M \triangleright_{1\beta,1\delta} N$ will mean "$M \triangleright_{1\beta} N$ or $M \triangleright_{1\delta} N$".

Definition 2.3.10. For any potential redex R, R is called a **contractible redex** if R is either a β-redex or a δ-redex.

Definition 2.3.11. A term M which contains no contractible redexes is called a **βδ-normal form** (or a term in βδ-normal form). The class of all βδ-normal forms is called **βδ-nf**. If a term M βδ-reduces to a term N in βδ-nf, then N is called a **βδ-normal form** of M.

Examples 2.3.12.

a. Let c be a constant.

i. The term $(\lambda cx.xc)(xc)$ is not a β-redex since $[N/x](cx) \not\equiv xc$ for all terms N.

ii. $(\lambda cx.xc)(c(\lambda c.cx)) \triangleright_{1\beta} (\lambda c.cx)c \triangleright_{1\beta} cx$.

iii. The term $(\lambda cx.xc \mid \lambda x.cx)(xc)$ is not a δ-redex since $(\lambda cx.xc)(xc)$ is not a β-redex (from (i)), and furthermore $[c/x](xc) \triangleright_{\beta\gamma} cc$ and $(\lambda cx.xc)(cc)$ is a β-redex.

iv. $(\lambda cx.xc \mid \lambda x.cx)((\lambda cx.xy)(cc)) \triangleright_{1\beta} (\lambda cx.xc \mid \lambda x.cx)(cy)$

$\triangleright_{1\delta} (\lambda cx.xc)(cy) \triangleright_{1\beta} yc$.

v. $(\lambda cx.xc \mid \lambda x.x(cx))(\lambda cx.x) \triangleright_{1\delta} (\lambda x.x(cx))(\lambda cx.x)$

$\triangleright_{1\beta} (\lambda cx.x)(c(\lambda cx.x)) \triangleright_{1\beta} \lambda cx.x$.

b. Let c be a constant.

Let $A \equiv (\lambda c(cx).c \mid \lambda cx.x)$,

$M \equiv (\lambda x.y(xx))(\lambda x.y(xx))$, and

$M' \equiv (\lambda x.c(xx))(\lambda x.c(xx))$.

Since the only reductions for M and M' are

$M \rhd_{1\beta} yM \rhd_{1\beta} y(yM) \rhd_{1\beta} \ldots\ldots \rhd_{1\beta} y(y(\ldots(yM)\ldots)) \rhd_{1\beta} \ldots$, and

$M' \rhd_{1\beta} cM' \rhd_{1\beta} c(cM') \rhd_{1\beta} \ldots\ldots \rhd_{1\beta} c(c(\ldots(cM')\ldots)) \rhd_{1\beta} \ldots$, respectively,

M and M' have no $\beta\delta$-normal forms.

The term AM' can be reduced as follows.

i. $AM' \rhd_{1\beta} A(cM') \rhd_{1\beta} A(c(cM')) \rhd_{1\beta} \ldots\ldots \rhd_{1\beta} A(c(c(\ldots(cM')\ldots))) \rhd_{1\beta} \ldots$,

ii. $AM' \rhd_{1\beta} A(cM') \rhd_{1\beta} A(c(cM')) \rhd_{1\delta} (\lambda c(cx).c)(c(cM')) \rhd_{1\beta} c$.

Hence AM' has a $\beta\delta$-normal form c, but also has an infinite reduction.

The term AM is not a δ-redex since $(\lambda c(cx).c)M$ is not a β-redex (because $[N/x](c(cx)) \not\equiv M$ for all terms N) and furthermore $[c/y]M \equiv M' \rhd_{\beta\gamma} c(cM')$ and $(\lambda c(cx).c)(c(cM'))$ is a β-redex. Since the only possible reduction for AM is

$AM \rhd_{1\beta} A(yM) \rhd_{1\beta} A(y(yM)) \rhd_{1\beta} \ldots\ldots \rhd_{1\beta} A(y(y(\ldots(yM)\ldots))) \rhd_{1\beta} \ldots$,

AM has no $\beta\delta$-normal forms.

c. Assume $\{0,S\}$ is a set of constants, and let $P \equiv (\lambda 0.0 \mid \lambda Sx.x)$.

Then $P0 \equiv (\lambda 0.0 \mid \lambda Sx.x)0$

$\rhd_{1\delta} (\lambda 0.0)0 \rhd_{1\beta} 0$,

$P(S0) \equiv (\lambda 0.0 \mid \lambda Sx.x)(S0)$

$\rhd_{1\delta} (\lambda Sx.x)(S0) \rhd_{1\beta} 0$,

$P(S(S0)) \equiv (\lambda 0.0 \mid \lambda Sx.x)(S(S0))$

$\rhd_{1\delta} (\lambda Sx.x)(S(S0)) \rhd_{1\beta} S0$,

⋮

$$P(\underbrace{S(S(\ldots(S}_{\text{n copies}}0)\ldots))) \equiv (\lambda 0.0 \mid \lambda Sx.x)(\underbrace{S(S(\ldots(S}_{\text{n copies}}0)\ldots)))$$

$$\rhd_{1\delta} (\lambda Sx.x)(\underbrace{S(S(\ldots(S0)\ldots))}_{\text{n copies}})$$

$$\rhd_{1\beta} \underbrace{S(\ldots(S0)\ldots)}_{\text{n-1 copies}}.$$

If we think of S as representing the successor function, then P represents the predecessor function.

Lemma 2.3.13. Let M be a term and N be an expression. If $M \rhd_{1\beta,1\delta} N$, then N is a term, and if M is not the potential redex which is contracted when $M \rhd_{1\beta,1\delta} N$ then N is of the same form as M.

Proof. Assume that $M \rhd_{1\beta,1\delta} N$ and R is the occurrence of a potential redex in M which is contracted. Induct on M, and note that since R is in M, M is not an atom.

i. $M \equiv M_1M_2$.

Case 1. $M \not\equiv R$.

Then R is either in M_1or in M_2. Without loss of generality, assume R is in M_1. When R is contracted, suppose M_1 changes to M_1'. Then $M_1 \rhd_{1\beta,1\delta} M_1'$ and hence $N \equiv M_1'M_2$. By induction, $M_1'$is a term. Hence N is an application.

Case 2. $M \equiv R$.

Subcase 2.1. $M_1 \equiv \lambda P.Q$.

Then $M \equiv (\lambda P.Q)M_2$, which is a β-redex. Since $M \rhd_{1\beta} N$, N is either Q or some substitution of Q. Hence, by Lemma 2.1.10(b) N is a term.

Subcase 2.2. $M_1 \equiv (\lambda P.Q \mid A)$.

Then $M \equiv (\lambda P.Q \mid A)M_2$, which is a δ-redex. Since $M \rhd_{1\delta} N$, $N \equiv (\lambda P.Q)M_2$ or $N \equiv AM_2$. In either case N is a term.

ii. $M \equiv \lambda P.Q$.

Then R is in Q. When R is contracted, suppose Q changes to Q'. So $Q \rhd_{1\beta,1\delta} Q'$ and $N \equiv \lambda P.Q'$. By induction, Q' is a term. Hence N is a simple abstraction.

iii. $M \equiv (\lambda P.Q \mid A)$.

Then R is either in Q or in A. Similar to the case where M is a simple abstraction, $N \equiv (\lambda P.Q' \mid A')$ for some Q' and A' such that either $Q \rhd_{1\beta,1\delta} Q'$ and $A \equiv A'$ or $Q \equiv Q'$ and $A \rhd_{1\beta,1\delta} A'$. By induction, Q' is a term and A' is an abstraction. Hence N is a compound abstraction. □

Note 2.3.14. From the proof of the above lemma, we have that if M and N are terms such that $M \rhd_{1\beta,1\delta} N$ and R is the occurrence of a potential redex which is contracted when $M \rhd_{1\beta,1\delta} N$, then

a. if $M \equiv M_1M_2$ and $M \not\equiv R$ then $N \equiv N_1N_2$ for some terms N_1 and N_2 such that either $M_1 \rhd_{1\beta,1\delta} N_1$ and $M_2 \equiv N_2$ or $M_1 \equiv N_1$ and $M_2 \rhd_{1\beta,1\delta} N_2$;

b. if $M \equiv \lambda P.Q$ then $N \equiv \lambda P.Q'$ for some term Q' such that $Q \rhd_{1\beta,1\delta} Q'$;

c. if $M \equiv (\lambda P.Q \mid A)$ then $N \equiv (\lambda P.Q' \mid A')$ for some term Q', and some abstraction A' such that either $Q \rhd_{1\beta,1\delta} Q'$ and $A \equiv A'$ or $Q \equiv Q'$ and $A \rhd_{1\beta,1\delta} A'$.

Corollary 2.3.15. For any term M, if $M \rhd_{\beta\delta} N$, then N is a term and

a. if $M \equiv M_1M_2$ and $M \rhd_{\beta\delta} N$ by a sequence of terms $M \equiv M_1, M_2, \ldots, M_n \equiv N$, $n \geq 1$, such that for each $1 \leq i < n$, M_i is not the potential redex which is contracted then $N \equiv N_1N_2$ for some terms N_1 and N_2 such that $M_i \rhd_{\beta\delta} N_i$, $i = 1, 2$;

b. if $M \equiv \lambda P.Q$, and no variable in P has been changed when $M \rhd_{\beta\delta} N$ then $N \equiv \lambda P.Q'$ for some term Q' such that $Q \rhd_{\beta\delta} Q'$;

c. if $M \equiv (\lambda P.Q \mid A)$ then $N \equiv (\lambda P'.Q' \mid A')$ for some abstractions $\lambda P'.Q'$ and A' such that $\lambda P.Q \rhd_{\beta\delta} \lambda P'.Q'$ and $A \rhd_{\beta\delta} A'$.

Proof. This follows from Note 2.3.14 and Lemma 2.2.4. □

Lemma 2.3.16.

a. For any terms M and N, if $M \rhd_{\beta\delta} N$, then $FV(N) \subseteq FV(M)$.

b. Let $x_1,\ldots, x_k$, $k \geq 1$, be distinct variables and $M, U_1,\ldots, U_k, V_1,\ldots, V_k$ be terms. If $U_i \rhd_{\beta\delta} V_i$ for all $1 \leq i \leq k$, then $[U_1/x_1,\ldots, U_k/x_k]M \rhd_{\beta\delta} [V_1/x_1,\ldots, V_k/x_k]M$.

Proof of (a). Let M and N be terms such that $M \rhd_{\beta\delta} N$. By the definition of $\beta\delta$-reduction and Lemma 2.2.5(a), we may assume $M \rhd_{1\beta,1\delta} N$. Furthermore, since the only part of M that changes is the redex which is contracted, we may assume that M is a contractible redex and N is its contractum.

i. $M \equiv (\lambda P.Q)L$.

Case 1. $FV(P) = \varnothing$.

Then $P \equiv L$ and $M \rhd_{1\beta} Q$ so that $N \equiv Q$. Hence $FV(L) = FV(P) = \varnothing$. Thus $FV(N) = (FV(Q) - FV(P)) \cup FV(L) = FV(\lambda P.Q) \cup FV(L) = FV((\lambda P.Q)L) = FV(M)$.

Case 2. $FV(P) = \{x_1,\ldots, x_k\}$.

Then there exist terms $N_1,\ldots, N_k$ such that $[N_1/x_1,\ldots, N_k/x_k]P \equiv L$ and $M \rhd_{1\beta} [N_1/x_1,\ldots, N_k/x_k]Q$ so that $N \equiv [N_1/x_1,\ldots, N_k/x_k]Q$. So we have

$$\begin{aligned}
FV(N) &= FV([N_1/x_1,\ldots, N_k/x_k]Q) \\
&\subseteq FV(N_1\ldots N_k) \cup (FV(Q) - \{x_1,\ldots, x_k\}) && \text{(by Corollary 2.1.14)} \\
&= FV(N_1\ldots N_k) \cup (FV(Q) - FV(P)) \\
&= (FV(N_1\ldots N_k) \cup (FV(P) - \{x_1,\ldots, x_k\})) \cup FV(\lambda P.Q) \\
&= FV([N_1/x_1,\ldots, N_k/x_k]P) \cup FV(\lambda P.Q) && \text{(by Lemma 2.1.13)} \\
&= FV(L) \cup FV(\lambda P.Q) \\
&= FV((\lambda P.Q)L) \\
&= FV(M).
\end{aligned}$$

ii. $M \equiv (\lambda P.Q \mid A)L$.

Case 1. $M \rhd_{1\delta} (\lambda P.Q)L$, so that $N \equiv (\lambda P.Q)L$.

$$\begin{aligned}
\text{Then } FV(N) &= FV(\lambda P.Q) \cup FV(L) \\
&\subseteq FV(\lambda P.Q) \cup FV(L) \cup FV(A) \\
&= FV((\lambda P.Q \mid A)L) \\
&= FV(M).
\end{aligned}$$

Case 2. $M \rhd_{1\delta} AL$.

This is similar to Case 1. □

Proof of (b). Assume $U_i \rhd_{\beta\delta} V_i$ for all $1 \leq i \leq k$. By Corollary 2.1.12(a), (b), we may assume that $\{x_1,\ldots, x_k\} \subseteq FV(M)$ and induct on M.

i. $M \equiv x_1$.

Then $[U_1/x_1]M \equiv U_1 \rhd_{\beta\delta} V_1 \equiv [V_1/x_1]M$.

ii. $M \equiv M_1M_2$.

This case follows straightforwardly by induction.

iii. $M \equiv \lambda P.Q$.

Let $m = |FV(P) \cap FV(U_1\ldots U_k)|$ and induct on m.

For the case $m = 0$, we have $FV(P) \cap FV(U_1\ldots U_k) = \varnothing$. Since $U_i \rhd_{\beta\delta} V_i$ for all $1 \leq i \leq k$, by (a), $FV(V_1\ldots V_k) \subseteq FV(U_1\ldots U_k)$. Hence $FV(P) \cap FV(V_1\ldots V_k) = \varnothing$.

Thus $[U_1/x_1,\ldots, U_k/x_k]M \equiv \lambda P.[U_1/x_1,\ldots, U_k/x_k]Q$

$\rhd_{\beta\delta} \lambda P.[V_1/x_1,\ldots, V_k/x_k]Q$ (by induction on M)

$\equiv [V_1/x_1,\ldots, V_k/x_k]M$.

Now assume $m > 0$. Let u be the first variable in $FV(P) \cap FV(U_1\ldots U_k)$ and z be the first variable which is not in $FV(PQU_1\ldots U_k)$. Then

$[U_1/x_1,\ldots, U_k/x_k]M \equiv [U_1/x_1,\ldots, U_k/x_k](\lambda[z/u]P.[z/u]Q)$

$\rhd_{\beta\delta} [V_1/x_1,\ldots, V_k/x_k](\lambda[z/u]P.[z/u]Q)$ (by induction on m)

$\equiv_\alpha [V_1/x_1,\ldots, V_k/x_k](\lambda P.Q)$ (by Lemma 2.2.7)

$\equiv [V_1/x_1,\ldots, V_k/x_k]M$.

Hence $[U_1/x_1,\ldots, U_k/x_k]M \rhd_{\beta\delta} [V_1/x_1,\ldots, V_k/x_k]M$.

iv. $M \equiv (\lambda P.Q \mid A)$.

This is similar to (ii). □

Lemma 2.3.17. The class $\beta\delta$-nf is the smallest class such that

a. all atoms are in the class;

b. if $M_0, M_1,\ldots, M_n$, $n \geq 1$, are in the class and M_0M_1 is not a contractible redex, then $M_0M_1\ldots M_n$ is in the class;

c. if Q is in the class, then for any pattern P, $\lambda P.Q$ is in the class; and

d. if $\lambda P.Q$ and an abstraction A are in the class, then $(\lambda P.Q \mid A)$ is in the class.

Proof. Let $\mathscr{B}$ be the intersection of all classes satisfying properties (a) – (d).

Claim 1. $\mathscr{B}$ satisfies properties (a) – (d).

Proof of Claim 1.

i. Since all atoms are in $\mathscr{A}$ for all classes $\mathscr{A}$ which satisfy properties (a) – (d), all atoms are in $\mathscr{B}$. Hence $\mathscr{B}$ satisfies property (a).

ii. Assume $M_0, M_1, \ldots, M_n \in \mathscr{B}, n \geq 1$, and M_0M_1 is not a contractible redex. Then $M_0, M_1, \ldots, M_n \in \mathscr{A}$ for all classes $\mathscr{A}$ which satisfy properties (a) – (d). Since M_0M_1 is not a contractible redex, $M_0M_1 \ldots M_n \in \mathscr{A}$ for all classes $\mathscr{A}$ which satisfy properties (a) – (d). Thus $M_0M_1 \ldots M_n \in \mathscr{B}$. Hence $\mathscr{B}$ satisfies property (b).

iii. Assume P is a pattern and $Q \in \mathscr{B}$. Then $Q \in \mathscr{A}$ for all classes $\mathscr{A}$ which satisfy properties (a) – (d). Thus $\lambda P.Q \in \mathscr{A}$ for all classes $\mathscr{A}$ which satisfy properties (a) – (d). Hence $\lambda P.Q \in \mathscr{B}$, so $\mathscr{B}$ satisfies property (c).

iv. Assume abstractions $\lambda P.Q$ and A are in $\mathscr{B}$. Then $\lambda P.Q, A \in \mathscr{A}$ for all classes $\mathscr{A}$ which satisfy properties (a) – (d). Hence $(\lambda P.Q \mid A) \in \mathscr{A}$ for all classes $\mathscr{A}$ which satisfy properties (a) – (d). Thus $(\lambda P.Q \mid A) \in \mathscr{B}$, so $\mathscr{B}$ satisfies property (d).

Thus we have Claim 1.

Since $\mathscr{B} \subseteq \mathscr{A}$ for all classes $\mathscr{A}$ which satisfy properties (a) – (d), by Claim 1 $\mathscr{B}$ is the smallest class which satisfies properties (a) – (d).

Claim 2. $\beta\delta$-nf $= \mathscr{B}$.

Proof of Claim 2.

To show that $\beta\delta\text{-nf} \subseteq \mathscr{B}$, let $M \in \beta\delta$-nf. We will show that $M \in \mathscr{B}$ by induction on M.

i. M is an atom.

Since $\mathscr{B}$ satisfies property (a), $M \in \mathscr{B}$.

ii. $M \equiv M_1M_2$.

Since $M \in \beta\delta$-nf, M contains no contractible redexes, and so neither do M_1 and M_2. Hence $M_1, M_2 \in \mathcal{B}$-nf. By induction, $M_1, M_2 \in \mathcal{B}$. Since M contains no contractible redexes and $M \equiv M_1M_2$, M_1M_2 is not a contractible redex. Since $\mathcal{B}$ satisfies property (b), $M_1M_2 \in \mathcal{B}$.

iii. $M \equiv \lambda P.Q$.

Since $M \in \beta\delta$-nf, M contains no contractible redexes, and so neither does Q. Then $Q \in \beta\delta$-nf. By induction, $Q \in \mathcal{B}$. Since $\mathcal{B}$ satisfies (c), $\lambda P.Q \in \mathcal{B}$.

iv. $M \equiv (\lambda P.Q \mid A)$.

By an argument similar to the one above, by induction $\lambda P.Q, A \in \mathcal{B}$. Since $\mathcal{B}$ satisfies property (d), $(\lambda P.Q \mid A) \in \mathcal{B}$.

Next we will prove that $\mathcal{B} \subseteq \beta\delta$-nf. Let $M \in \mathcal{B}$. We will show that $M \in \beta\delta$-nf by induction on M. Since $\mathcal{B}$ satisfies properties (a) – (d), M must fall into one of the following categories.

i. M is an atom.

Then M contains no contractible redexes. Hence $M \in \beta\delta$-nf.

ii. $M \equiv M_0M_1\ldots M_n$ for some $M_0, M_1,\ldots, M_n \in \mathcal{B}$, $n \geq 1$, such that M_0M_1 is not a contractible redex. By induction, $M_0, M_1,\ldots, M_n \in \beta\delta$-nf, so M_i contains no contractible redexes for all $1 \leq i \leq n$. We will shows that $M \in \beta\delta$-nf by induction on n.

Suppose $n = 1$, so that $M \equiv M_0M_1$. Since M_0M_1 is not a contractible redex and both M_0 and M_1 contain no contractible redexes, M_0M_1 contains no contractible redexes. Hence $M \in \beta\delta$-nf.

Now assume $n > 1$. By induction, $M_0M_1\ldots M_{n-1} \in \beta\delta$-nf, so it contains no contractible redexes. Since $M_0M_1\ldots M_{n-1}$ is an application, it is not an abstraction. Hence $M_0M_1\ldots M_n$ is not a potential redex, and so it is not a contractible redex. Hence $M_0M_1\ldots M_n$ contains no contractible redexes. Thus $M \in \beta\delta$-nf.

iii. $M \equiv \lambda P.Q$ for some $Q \in \mathcal{B}$.

By induction, $Q \in \beta\delta$-nf, so Q contains no contractible redexes, and so neither does $\lambda P.Q$. Hence $M \in \beta\delta$-nf.

iv. $M \equiv (\lambda P.Q \mid A)$ for some $\lambda P.Q, A \in \mathcal{B}$.

By induction, $\lambda P.Q$, $A \in \beta\delta$-nf, so they contain no contractible redexes, and so neither does $(\lambda P.Q \mid A)$. Hence $M \in \beta\delta$-nf.

Thus we have Claim 2.

Since $\mathscr{B}$ is the smallest class satisfying properties (a) – (d), by Claim 2 $\beta\delta$-nf is the smallest class satisfying properties (a) – (d). □