CHAPTER IV

POSITIVE TOTALLY ORDERED 0-SEMIFIELDS

Definition 4.1. Let R be a positive ordered semiring. R is called a positive totally ordered semiring iff for evrey $x, y \in R$, $x \leq y$ or $y \leq x$.

Remark 4.2. Let K be a positive ordered semifield. Then the following statements hold :

(1) K is a positive totally ordered iff $K = P \cup P^{-1} \cup \{0\}$.

(2) K is a positive totally ordered iff for all $x \in K$, $x \geq 1$ or $x \leq 1$.

(3) Suppose that K is a positive totally ordered and C an o-convex subgroup of K. Then C is a convex subgroup of K.

Proof (1) and (2) are obvious.

To prove (3), let C be an o-convex subgroup of K. Let $x \in C$ and $a, b \in K$ be such that $a + b = 1$.

Case 1. $x \leq 1$. Then $ax \leq a$, so $ax + b \leq a + b = 1$. Since $x \leq 1$, $x^{-1} \geq 1$. Then $bx^{-1} \geq b$, so $(ax + b)x^{-1} = a + bx^{-1} \geq a + b = 1$. Hence $ax + b \geq x$. By the o-convexity of C and $x \leq ax + b \leq 1$, $ax + b \in C$.

Case 2. $x \geq 1$. The proof is similar to the proof of Case 1.

Therefore C is a convex subgroup of K. #

We shall give some examples of positive totally ordered semifields.

Examples 4.3. (1) Q_0^+, $\mathbb{R}_0^+$ are positive totally ordered semifields.

(2) From the Example 1.24. (3), $\{2^n \mid n \in Z\} \cup \{0\}$ is a positive totally ordered semifield.

(3) From the Example 2.6. (5), $K^* \times L^* \cup \{(0, 0)\}$ with $\leq^*$ is a positive totally ordered semifield where K and L are positive totally ordered semifields.

Let K be a positive totally ordered semifield and C a convex subgroup of K. Then $K/_C$ is a positive totally ordered semifield. To prove this, clearly $K/_C$ is a positive ordered semifield. Let $\alpha \in K/_C$. Choose $x \in \alpha$.

Case 1. $x \leq 1$. Then $\alpha \leq C$.

Case 2. $x \geq 1$. Then $\alpha \geq C$. By Remark 4.2. (2), $K/_C$ is a positive totally ordered semifield. #

Theorem 4.4. Let S be a commutative positive totally ordered semiring with multiplicative zero 0 which is the M.C. property and satisfies the property that for every $x, y, z \in S$, $xz < yz$ implies that $x < y$. Then S can be embedded into a positive totally ordered semifield.

Proof By Theorem 2.7., we have that $K = S \times (S - \{0\})/_\sim$ is a positive ordered semifield. To show that K is a positive totally ordered semifield, let $\alpha \in K$. Fix $a \in S - \{0\}$. Choose $(x, y) \in \alpha$. Then $xa \leq ya$ or $ya \leq xa$ which implies that $\alpha = [(x, y)] \leq [(a, a)]$ or $[(a, a)] \leq [(x, y)] = \alpha$. So by Remark 4.2. (2), K is a positive totally ordered semifield. #

Proposition 4.5. ([3]) Let $\{K_i \mid i \in I\}$ be a family of positive totally ordered semifields. Then $\prod_{i \in I} K_i$ is a positive totally ordered semifield if and only if either $I = \{i\}$ and K_i is a positive totally ordered semifield or there exists

$i_0 \in I$ such that K_{i_0} is a positive totally ordered semifield and $|K_i| = 2$ for all $i \in I - \{i_0\}$.

Proof See [3], pp. 46.. #

Let K be a semifield and A a nonempty subset of K^*.

Let $\mathscr{C} = \{B \subseteq K \mid A \subseteq B$ and B has the property that

1) $1 \in B$,
2) $B^2 \subseteq B$,
3) $B + K \subseteq B$ and
4) B is an a-convex subset of K$\}$. $\mathscr{C} \neq \varnothing$ since $K^* \in \mathscr{C}$. Then

the smallest subset of K satisfying 1) - 4) exists.

Definition 4.6. Let K be a semifield A a nonempty subset of K^*. The hull of A, denoted by H(A), is the smallest subset of K satisfying 1) - 4).

And A has property (*) iff for every $x_1, x_2, ..., x_n \in K^*$ there exist $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n \in \{-1, 1\}$ such that $H(P \cup \{x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, ..., x_n^{\varepsilon_n}\})$ is conic. From now on we shall use $H(P, x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, ..., x_n^{\varepsilon_n})$ instead of $H(P \cup \{x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, ..., x_n^{\varepsilon_n}\})$.

Lemma 4.7. ([2]) Let K be a positive ordered semifield and P the positive cone of K. Suppose that P satisfies property (*). Then for every $x \in K^*$ either $H(P, x)$ or $H(P, x^{-1})$ satisfies (1) - (4) of Theorem 2.11. and also satisfies property (*).

Proof Let $x \in K^*$. Suppose that $H(P, x)$ and $H(P, x^{-1})$ do not satisfy the property (*). Then there exist $a_1, a_2, ..., a_n, b_1, b_2, ..., b_m \in K^*$ such that $H(H(P, x), a_1^{\varepsilon_1}, a_2^{\varepsilon_2}, \ldots, a_n^{\varepsilon_n})$ and $H(H(P, x^{-1}), b_1^{\delta_1}, b_2^{\delta_2}, \ldots, b_m^{\delta_m})$ are not conic for every choice $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, \delta_1, \delta_2, \ldots, \delta_m$.

Claim that $H(P, x, a_1^{\varepsilon_1}, a_2^{\varepsilon_2}, \ldots, a_n^{\varepsilon_n}) = H(H(P, x), a_1^{\varepsilon_1}, a_2^{\varepsilon_2}, \ldots, a_n^{\varepsilon_n})$ for every choice of $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$. Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \{-1, 1\}$. Clear that $H(P, x, a_1^{\varepsilon_1}, a_2^{\varepsilon_2}, \ldots, a_n^{\varepsilon_n}) \subseteq H(H(P, x), a_1^{\varepsilon_1}, a_2^{\varepsilon_2}, \ldots, a_n^{\varepsilon_n})$. Since $H(P, x) \subseteq H(P, x, a_1^{\varepsilon_1}, a_2^{\varepsilon_2}, \ldots, a_n^{\varepsilon_n})$ and $a_1, a_2, \ldots, a_n \in H(P, x, a_1^{\varepsilon_1}, a_2^{\varepsilon_2}, \ldots, a_n^{\varepsilon_n})$, $H(H(P, x), a_1^{\varepsilon_1}, a_2^{\varepsilon_2}, \ldots, a_n^{\varepsilon_n}) \subseteq H(P, x, a_1^{\varepsilon_1}, a_2^{\varepsilon_2}, \ldots, a_n^{\varepsilon_n})$. Therefore $H(H(P, x), a_1^{\varepsilon_1}, a_2^{\varepsilon_2}, \ldots, a_n^{\varepsilon_n}) = H(P, x, a_1^{\varepsilon_1}, a_2^{\varepsilon_2}, \ldots, a_n^{\varepsilon_n})$. So we have the claim.

Similarly, $H(H(P, x^{-1}), b_1^{\delta_1}, b_2^{\delta_2}, \ldots, b_m^{\delta_m}) = H(P, x^{-1}, b_1^{\delta_1}, b_2^{\delta_2}, \ldots, b_m^{\delta_m})$ for all choice $\delta_1, \delta_2, \ldots, \delta_m$. Then we get that $H(P, x, a_1^{\varepsilon_1}, a_2^{\varepsilon_2}, \ldots, a_n^{\varepsilon_n})$ and $H(P, x^{-1}, b_1^{\delta_1}, b_2^{\delta_2}, \ldots, b_m^{\delta_m})$ are not conic. We shall show that $H(P, x^{\varepsilon}, a_1^{\varepsilon_1}, a_2^{\varepsilon_2}, \ldots, a_n^{\varepsilon_n}, b_1^{\delta_1}, b_2^{\delta_2}, \ldots, b_m^{\delta_m})$ is not conic for every choice $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, \delta_1, \delta_2, \ldots, \delta_m$.

Case 1. $\varepsilon = 1$. Since $H(P, x, a_1^{\varepsilon_1}, a_2^{\varepsilon_2}, \ldots, a_n^{\varepsilon_n})$ is not conic, there exists $y \in H(P, x, a_1^{\varepsilon_1}, a_2^{\varepsilon_2}, \ldots, a_n^{\varepsilon_n}) \cap H(P, x, a_1^{\varepsilon_1}, a_2^{\varepsilon_2}, \ldots, a_n^{\varepsilon_n})^{-1}$ and $y \neq 1$. Hence $y \in H(P, x^{\varepsilon}, a_1^{\varepsilon_1}, a_2^{\varepsilon_2}, \ldots, a_n^{\varepsilon_n}, b_1^{\delta_1}, b_2^{\delta_2}, \ldots, b_m^{\delta_m}) \cap H(P, x^{\varepsilon}, a_1^{\varepsilon_1}, a_2^{\varepsilon_2}, \ldots, a_n^{\varepsilon_n}, b_1^{\delta_1}, b_2^{\delta_2}, \ldots, b_m^{\delta_m})^{-1}$, so $H(P, x^{\varepsilon}, a_1^{\varepsilon_1}, a_2^{\varepsilon_2}, \ldots, a_n^{\varepsilon_n}, b_1^{\delta_1}, b_2^{\delta_2}, \ldots, b_m^{\delta_m})$ is not conic.

Case 2. $\varepsilon = -1$. Similarly to case 1.

It contradicts to property (*) of P, hence $H(P, x)$ or $H(P, x^{-1})$ satisfies (*). If $H(P, x)$ satisfies (*) then $H(P, x) = H(H(P, x), 1)$ is conic. Therefore $H(P, x)$ is a positive cone of K. By Theorem 2.11., $H(P, x)$ defines a positive compatible partial order on K.

Similarly, if $H(P, x^{-1})$ satisfies (*). #

Theorem 4.8. Let K be a positive ordered semifield and P the positive cone of K. Then $\leq_P$ (from Theorem 2.11.) can be extended to a totally order on K iff P satisfies the property (*).

Proof Assume that P can be extended to totally order of K, say Q. Let $x_1, x_2, ..., x_n \in K^*$. Choose $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n \in \{-1, 1\}$ such that $x_i^{\varepsilon_i} \in Q$ for all i. To show that $H(P, x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, ..., x_n^{\varepsilon_n})$ is conic, let $x \in H(P, x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, ..., x_n^{\varepsilon_n}) \cap H(P, x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, ..., x_n^{\varepsilon_n})^{-1}$ and suppose that $x \neq 1$.

Case 1. $x < 1$. Then $x \notin H(P, x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, ..., x_n^{\varepsilon_n})$ since $H(P, x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, ..., x_n^{\varepsilon_n}) \subseteq Q$, a contradiction.

Case 2. $x > 1$. Then $x^{-1} < 1$ and $x^{-1} \notin H(P, x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, ..., x_n^{\varepsilon_n})$, a contradiction.

Therefore $x = 1$, so $H(P, x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, ..., x_n^{\varepsilon_n})$ is conic.

Conversely, let $\mathscr{C} = \{Q \mid Q$ is a positive cone of K containing P and satisfies (*)$\}$. Then $\varnothing \neq \mathscr{C}$ since $P \in \mathscr{C}$. Let $\{Q_i \mid i \in I\}$ be a chain in $\mathscr{C}$. Clear that $\bigcup_{i\in I} Q_i$ is a positive cone of K containing P which is an upper bound of $\{Q_i \mid i \in I\}$. Suppose that $\bigcup_{i\in I} Q_i$ does not satisfy (*), there exist $x_1, x_2, ..., x_n \in K^*$ such that $H(\bigcup_{i\in I} Q_i, x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, ..., x_n^{\varepsilon_n})$ is not conic for every choice $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n$. So there is an $x \in H(\bigcup_{i\in I} Q_i, x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, ..., x_n^{\varepsilon_n}) \cap H(\bigcup_{i\in I} Q_i, x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, ..., x_n^{\varepsilon_n})^{-1}$ such that $x \neq 1$. Then $x, x^{-1} \in H(\bigcup_{i\in I} Q_i, x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, ..., x_n^{\varepsilon_n})$. Choose k large enough so that $x, x^{-1} \in H(Q_k, x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, ..., x_n^{\varepsilon_n})$, a contradiction since Q_k satisfies (*) and $x \neq 1$. Hence $\bigcup_{i\in I} Q_i$ satisfies (*). By Zorn's Lemma, $\mathscr{C}$ has a maximal element, say Q. By Lemma 4.7., for $x \in K^*$, $H(Q, x)$ or $H(Q, x^{-1}) \in C$, so by the maximality of Q, either $x \in Q$ or $x^{-1} \in Q$. Therefore Q defines a total order on K. #

Theorem 4.9. Let K be a positive ordered semifield and P the positive cone of K. Then $\leq_P$ is the intersection of all total order which are extension of $\leq_P$ iff for $x \in K^*$ there exist $x_1, x_2, ..., x_n \in K$ such that $H(P, x^{-1}, x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, ..., x_n^{\varepsilon_n})$ is not conic for every choice $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n \in \{1, -1\}$ implies that $x \in P$.

Proof Assume that $P = \bigcap_{i\in I} Q_i$ where Q_i is a total order that is an extension of P for all $i \in I$. Let $x \in K^*$. Suppose that $x \notin P$. Then there exist $i_0 \in I$ such that $x \notin Q_{i_0}$, so $x^{-1} \in Q_{i_0}$. Let $y \in H(P, x^{-1}) \cap H(P, x^{-1})^{-1}$. Thus $y, y^{-1} \in H(P, x^{-1}) \subseteq Q_{i_0}$, so $y = 1$. Therefore $H(P, x^{-1})$ is a positive cone of K which can be extended to a total order Q_{i_0}. By Theorem 4.8., $H(P, x^{-1})$ satisfies (*).

Conversely, it is clear that $P \subseteq \bigcap_{i\in I} Q_i$. Let $x \notin P$. By assumption for every nonzero $x_1, x_2, ..., x_n \in K$, there exist $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n \in \{-1, 1\}$ such that $H(P, x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, ..., x_n^{\varepsilon_n})$ is conic. By Theorem 4.8., there is a total order Q_{i_0} which is an extension of P. Thus $x^{-1} \in Q_{i_0}$, so $x \notin Q_{i_0}$. This prove that $\bigcap_{i\in I} Q_i \subseteq P$. Hence $P = \bigcap_{i\in I} Q_i$. #

Definition 4.10. Let K be a semifield. K is said to be O^*-semifield iff there exists a positive compatible total order on K.

Proposition 4.11. If a semifield K is an O^*-semifield then for every $x_1, x_2, ..., x_n \in K^*$, there exist $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n \in \{-1, 1\}$ such that $H(x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, ..., x_n^{\varepsilon_n})$ is conic.

Proof Assume that K is an O^*-semifield. Then there exists the positive cone P of K such that $K = P \cup P^{-1} \cup \{0\}$. Let $x_1, x_2, ..., x_n \in K^*$.

Then choose $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n \in \{-1, 1\}$ such that $x_i^{\varepsilon_i} \in P$ for all $i \in \{1, 2, ..., n\}$. Thus $H(x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, ..., x_n^{\varepsilon_n}) \cap H(x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, ..., x_n^{\varepsilon_n})^{-1} \subseteq P \cap P^{-1} = \{1\}$, hence $H(x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, ..., x_n^{\varepsilon_n})$ is conic. #

<u>Definition 4.12.</u> Let K be a positive ordered semifield. K is called a <u>vector semifield</u> iff it is a subbdirect product of positive totally ordered semifields.

Let $\{K_i \mid i \in I\}$ be a family of positive ordered semifield. Let K be a subsemifield of $\prod_{i\in I} K_i$. Then the j^{th} projection map from K into K_j is isotone for all $j \in I$.

To prove this, let $(x_i)_{i \in I}, (y_i)_{i \in I} \in K$ be such that $(x_i)_{i \in I} \leq (y_i)_{i \in I}$. Then $x_i \leq y_i$ for all $i \in I$. Let $j \in I$. Then $\Pi_j((x_i)_{i \in I}) = x_j \leq y_j = \Pi_j((x_i)_{i \in I})$. Therefore Π_j is isotone.

<u>Theorem 4.13.</u> Let K be a positive ordered semifield. Then K is a vector semifield iff its positive cone can be represented as the intersection of T_i where for each $i \in I$,

(1) T_i is a convex multiplicative subsemigroup containing P,

(2) $x \in K^* - T_i$ implies that $x^{-1} \in T_i$ and

(3) $1 + K \subseteq T_i$.

<u>Proof</u> Assume that K is a vector semifield. Then K is a subsemifield of $\prod_{i\in I} K_i$ where K_i is a positive totally ordered semifield for all $i \in I$.

Let $T_i = \Pi_i^{-1}(P_i)$ where P_i is a positive cone of K_i for all $i \in I$. Let $i \in I$.

(1) Let $x, y \in T_i$. Then $\Pi_i(x), \Pi_i(y) \in P_i$. So $\Pi_i(xy) = \Pi_i(x)\Pi_i(y) \in P_i$, so $xy \in \Pi_i^{-1}(P_i) = T_i$. Let $a, b \in K$ be such that $a + b = 1$.Then $\Pi_i(a) + \Pi_i(b) = \Pi_i(a + b) = \Pi_i(1) = 1_i$. Then $\Pi_i(ax + by) = \Pi_i(a)\Pi_i(x) + \Pi_i(b)\Pi_i(y) \in P_i$ since P_i is an a-convex subset of K_i. Thus $ax + by \in \Pi_i^{-1}(P_i) = T_i$.

Let $z \in K$ ze such that $x \leq z \leq y$. Since Π_i is isotone, $\Pi_i(x) \leq \Pi_i(z) \leq \Pi_i(y)$. By the o-convexity of P_i, $\Pi_i(z) \in P_i$. Hence $z \in \Pi_i^{-1}(P_i) = T_i$.

Let $p \in P$. Then $\Pi_i(p) \in P_i$, so $p \in \Pi_i^{-1}(P_i) = T_i$. Therefore T_i is a convex multiplicative subsemigroup containing P.

(2) Let $x \in K^* - T_i$. Then $x \notin T_i$, so $\Pi_i(x) \notin P_i$. Since P_i is totally order, $\Pi_i(x) \in P_i^{-1}$. So $\Pi_i(x^{-1}) = (\Pi_i(x))^{-1} \in P_i$, hence $x^{-1} \in \Pi_i^{-1}(P_i) = T_i$.

(3) Let $x \in K$. Then $1 + x \in P_i$. Since $\Pi_i(P_i) \subseteq P_i$ for all $i \in I$, $1 + x \in = \Pi_i^{-1}(P_i) = T_i$ for all $i \in I$. Hence $P \subseteq \bigcap_{i \in I} T_i$. Let $y \in \bigcap_{i \in I} T_i$. Then $y \in T_i$ for all $i \in I$, so $\Pi_i(y) \in P_i$ for all $i \in I$ which implies that $y \in P$. Hence $\bigcap_{i \in I} T_i \subseteq P$. Therefore $P = \bigcap_{i \in I} T_i$.

Conversely, assume that $P = \bigcap_{i \in I} T_i$, let $N_i = T_i \cap T_i^{-1}$ for all $i \in I$. To show that N_i is convex subgroup of K, let $i \in I$. Let $x, y \in N_i$. Then $x, y, x^{-1}, y^{-1} \in T_i$.

Then $xy, (xy)^{-1} \in T_i$. Hence $xy \in T_i \cap T_i^{-1} = N_i$. Clearly that $x^{-1} \in N_i$. Let $a, b \in K$ be such that $a + b = 1$. Then $ax + by \in T_i$ and $(ax + by)^{-1} = (a + b)(ax + by)^{-1} = [ax\,(ax + by)^{-1}]x^{-1} + [by(ax + by)^{-1}]y^{-1} \in T_i$.
Hence $ax + by \in T_i \cap T_i^{-1} = N_i$.

Let $z \in K$ be such that $x \leq z \leq y$. Then $y^{-1} \leq z^{-1} \leq x^{-1}$, by the o-convexity of T_i, $z \in T_i \cap T_i^{-1} = N_i$.

Therefore N_i is a convex subgroup of K. Thus $\bigcap_{i \in I} N_i = \bigcap_{i \in I} (T_i \cap T_i^{-1}) = (\bigcap_{i \in I} T_i) \cap (\bigcap_{i \in I} T_i)^{-1} = P \cap P^{-1} = \{1\}$, so $\bigcap_{i \in I} N_i = \{1\}$. Let $K_i = K/_{N_i}$ for all $i \in I$. Then K_i is a positive ordered semifield. Define $f : K \rightarrow \prod_{i \in I} K_i$ by

$f(x) = (xN_i)_{i \in I}$ for all $x \in K$. Then we have that f is a monomorphism.

Next, let $P_i = \Pi_i \circ f(T_i)$ for all $i \in I$. To show that P_i is the positive cone of K_i for all $i \in I$, let $i \in I$.

(1) Let $\alpha, \beta \in P_i$. Then there are $a, b \in T_i$ such that $\Pi_i \circ f(a) = \alpha$ and $\Pi_i \circ f(b) = \beta$. Then $aN_i = \alpha$ and $bN_i = \beta$. Hence $\alpha\beta = (aN_i)(bN_i) = (ab)N_i$ $= \Pi_i \circ f(ab) \in \Pi_i \circ f(T_i)$ since $ab \in T_i$.

(2) Let $\alpha \in P_i \cap P_i^{-1}$. Then there are $a, b \in T_i$ such that $aN_i =$ $\Pi_i \circ f(a) = \alpha$ and $bN_i = \Pi_i \circ f(b) = \alpha^{-1}$. So $(ab)N_i = (aN_i)(bN_i) = \alpha\alpha^{-1} = N_i$, hence $ab \in N_i = T_i \cap T_i^{-1}$. Thus ab, $(ab)^{-1} \in T_i$. Since $a \in T_i$, $b^{-1} = a(ab)^{-1} \in T_i$. Therefore $b \in T_i \cap T_i^{-1} = N_i$. Since $b \in N_i$, $aN_i = abN_i$. Since $abN_i = N_i$, $\alpha =$ $aN_i = N_i$.

(3) Let $\alpha, \beta \in P_i$, then there are $a, b \in T_i$ such that $aN_i = \alpha$ and $bN_i = \beta$. Let $C, D \in K_i$ be such that $C + D = N_i$. Since $1 \in N_i$, there exist $c \in C$ and $d \in D$ such that $1 = c + d$. Since T_i is a-convex, $ca + db \in T_i$. Then $\Pi_i \circ f(ca + db) = (ca + db)N_i = (cN_i)(aN_i) + (dN_i)(bN_i) = C\alpha + D\beta$, $C\alpha + D\beta \in \Pi_i \circ f(T_i) = P_i$.

(4) Let $\alpha \in K_i$. choose $x \in \alpha$. by assumption, $x + 1 \in T_i$. Thus $\Pi_i \circ f(x + 1) = (x + 1)N_i = xN_i + N_i = \alpha + N_i$, so $\alpha + N_i \in \Pi_i \circ f(T_i) = P_i$. By Theorem 2.11., P_i is a possitive cone of K_i for all $i \in I$.

Finally, to show that P_i is totally order for all $i \in I$, let $i \in I$. Let $\alpha \in K_i^*$. Choose $x \in \alpha$. If $x \in T_i$ then $\alpha = xN_i = \Pi_i \circ f(x) \in \Pi_i \circ f(T_i) = P_i$. Suppose $x \notin T_i$. By assumption, $x^{-1} \in T_i$. So $(x^{-1})N_i = \Pi_i \circ f(x^{-1}) \in \Pi_i \circ f(T_i) =$ P_i. Therefore $\alpha = xN_i \in P_i^{-1}$. #

<u>Corollary 4.14.</u> Let K be a positive ordered semifield. If K is a vector semifield then its positive cone P satisfies the property that for every $x_1, x_2, ..., x_n \in K^*$, $\cap H(P, x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, ..., x_n^{\varepsilon_n}) = P$ for $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n \in \{-1, 1\}$.

Proof Assume that K is a vector semifield. By Theorem 4.13., $P = \bigcap_{i \in I} T_i$ where T_i satisfies (1) - (3) all $i \in I$. Let $x_1, x_2, ..., x_n \in K^*$. By property (2) of T_i, we can choose $\varepsilon_{1_i}, \varepsilon_{2_i}, ..., \varepsilon_{n_i} \in \{-1, 1\}$ such that $x_1^{\varepsilon_{1i}}$, $x_2^{\varepsilon_{2i}}, ..., x_n^{\varepsilon_{ni}} \in T_i$ for all $i \in I$. Therefore $H(P, x_1^{\varepsilon_{1i}}, x_2^{\varepsilon_{2i}}, ..., x_n^{\varepsilon_{ni}}) \subseteq T_i$ for all $i \in I$ since $P \subseteq T_i$.

Then $P \subseteq \cap H(P, x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, ..., x_n^{\varepsilon_n}) \subseteq \bigcap_{i \in I} H(P, x_1^{\varepsilon_{1i}}, x_2^{\varepsilon_{2i}}, ..., x_n^{\varepsilon_{ni}}) \subseteq \bigcap_{i \in I} T_i = P$, $\cap H(P, x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, ..., x_n^{\varepsilon_n}) = P$. #

Theorem 4.15. Let K be a positive lattice ordered semifield. Then K either a positive totally ordered semifield or K is subdirectly reducible.

Proof Let K be a positive lattice ordered semifield. Suppose that K is not a positive totally ordered semifield. Then there are $x, y \in K^*$ such that $x \| y$. Then $xy^{-1} \| 1$. Let $a = xy^{-1}$. Then $(a \vee 1) > 1$ and $(a \wedge 1)^{-1} > 1$, so $< a \vee 1 >_L$ and $< (a \wedge 1)^{-1} >_L \neq \{1\}$. Since $1 = (a \vee 1) \wedge (a \wedge 1)^{-1} = |a \vee 1| \wedge |(a \wedge 1)^{-1}|$, by Proposition 3.24., $< a \vee 1 >_L \cap < (a \wedge 1)^{-1} >_L = \{1\}$.

Let $\mathscr{L} = \{ I \mid I$ is an L-ideal of K except $\{1\}\}$. Suppose that $\mathscr{L}$ has the minimum element, say I_m. Since $< a \vee 1 >_L$, $< (a \wedge 1)^{-1} >_L \in \mathscr{L}$, $I_m \subseteq < a \vee 1 >_L$, $< (a \wedge 1)^{-1} >_L$. Thus $I_m \subseteq < a \vee 1 >_L \cap < (a \wedge 1)^{-1} >_L = \{1\}$, so $I_m = \{1\}$, a contradiction. Hence $\mathscr{L}$ has no a minimum element. By Proposition 1.62., K is subdirectly reducible. #

Theorem 4.16. Let K be a positive lattice ordered semifield. Then K is a subdirect product of positive totally ordered semifields.

Proof Let K be a positive lattice ordered semifield. By Theorem 1.66., K is a subdirect product of subdirectly irreducible positive lattice ordered semifield which are also positive totally ordered semifield, by Theorem 4.15.. #

Definition 4.17. Let K be a positive totally ordered semifield. K is said to be Archimedian iff for every $x, y \in K^*$, if $x < y$ then

(1) there exists an $n \in Z^+$ such that $y < nx$ and

(2) there exists an $n \in Z$ such that $x < y^n$ if $x \neq 1$.

Proposition 4.18. Let K be an Archimedian positive totally ordered semifield such that $1+1 \neq 1$ and $K_o \subseteq K$, the prime semifield of K is order isomorphic to Q_0^+. Then the following statements hold :

(1) $1 = \inf \{1 + n^{-1} \mid n \in Z^+\}$

(2) For every $x, y \in K$, $x < y$ implies that there exists an $n \in Z^+$ such that $nx + 1 < ny$.

(3) For every $x \in K$, there exists an $n \in Z^+$ such that $n - 1 \leq x < n$.

Proof (1) Let K_o be the prime semifield of K. Claim that for every $p \in K_o$, there exists an $N \in Z^+$ such that $p < N^2(2N + 1)^{-1}$.

Let $p \in K_o$. Then there exist $m, n \in Z_0^+$ such` that $p = mn^{-1}$ and $n \neq 0$. Let $N = 4(m + n)$. Then $N \in Z^+$ and $N = 4(m + n) > (m + n) + 2(m+ n) > (m + n)N^{-1} + 2N(m + n)N^{-1} = (m + n)(1 + 2N)N^{-1}$, so $N^2 > (m + n)(1 + 2N)$. Therefore $N^2(2N + 1)^{-1} > m + n > mn^{-1} = p$. So we have the claim.

Clear that 1 is a lower bound of $\{1 + n^{-1} \mid n \in Z^+\}$. Let $z \in K$ be such that $z < 1 + n^{-1}$ for all $n \in Z^+$. To show that $z \leq 1$, suppose that $z > 1$. Then $z < z^2$.

Case 1. Suppose that there exists an $M \in Z^+$ such that $1 + M^{-1} \le z^2$. Since $z < 1 + n^{-1}$ for all $n \in Z^+$, $z^2 < (1 + n^{-1})^2$ for all $n \in Z^+$............(*). By the claim, there exists an $N \in Z^+$ such that $M < N^2(2N + 1)^{-1}$. Thus $(2N + 1)N^{-2} < M^{-1}$, it follows that $1 + (2N + 1)N^{-2} < 1 + M^{-1}$. Therefore $(1 + N^{-1})^2 = (N^2 + 2N + 1)N^{-2} < 1 + M^{-1}$. Since $1 + M^{-1} \le z^2$, $(1 + N^{-1})^2 < z^2$ which is contradicts to (*).

Case 2. Suppose that $z^2 < 1 + n^{-1}$ for all $n \in Z^+$. Since $z < 2$ and K is Archimedian, there exists $m \in Z$ such that $2 < z^m$. Then $m \ge 3$.

Claim that $z^{2^k} < 1 + n^{-1} < z^m$ for all $n \in Z^+$, for all $k \in Z^+$ and $k \ge 2$.

For $k = 2$.

Let $n_o \in Z^+$. There exists an $N \in Z^+$ such that $n_o < N^2(2N + 1)^{-1}$. Thus $(1 + N^{-1})^2 < 1 + n_o^{-1}$. Since $z^2 < 1 + n^{-1}$ for all $n \in Z^+$, $z^4 < (1 + n^{-1})^2$ for all $n \in Z^+$. Thus $z^4 < (1 + N^{-1})^2 < 1 + n_o^{-1} < z^m$. Since n_o is arbitrary, $z^{2^2} = z^4 < 1 + n^{-1} < z^m$ for all $n \in Z^+$. Suppose that the claim is true for $k - 1 \ge 2$. Then $z^{2^{k-1}} < 1 + n^{-1} < z^m$ for all $n \in Z^+$. Hence $z^{2^k} = (z^{2^{k-1}})^2 < (1 + n^{-1})^2$ for all $n \in Z^+$.............(*).

Let $n_o \in Z^+$. There exists an $N \in Z^+$ such that $n_o < N^2(2N + 1)^{-1}$. Thus $(1 + N^{-1})^2 < 1 + n_o^{-1}$. Since $z^{2^k} < (1 + N^{-1})^2$, $z^{2^k} < 1 + n_o < z^m$. Since n_o is arbitrary, $z^{2^k} < 1 + n^{-1} < z^m$ for all $n \in Z^+$. So we have the claim.

Since $m \ge 3$, $z^2 \le z^m$. By K is an Archimedian, $z^m < (z^2)^p$ for some $p \in Z$. Then $p \ge 2$, which contradicts to the claim.

Therefore $z \le 1$. This shows that $1 = \inf\{1 + n^{-1} \mid n \in Z^+\}$.

(2) See the proof of [5], pp. 35.

(3) Clearly. #

Proposition 4.19. Let K be an Archimedian positive totally ordered semifield such that $1+1 \neq 1$ and $K_o \subseteq K$, the prime semifield of K is order isomorphic to Q_0^+. Then K_o is dense in K.

Proof Let $x, y \in K$ be such that $x < y$. We shall show that there exists $p \in K_o$ such that $x < p < y$. By Proposition 4.18. (2), there exists $n \in Z^+$ such that $nx + 1 < ny$. By Proposition 4.18. (3), there exists $m \in Z^+$ and $m - 1 \leq nx < m$. since $m - 1 \leq nx$, $m \leq nx + 1$. Since $nx + 1 < ny$ and $nx < m$, $nx < m < ny$. Thus $x < mn^{-1} < y$. #

Theorem 4.20. Let K be an Archimedian positive totally ordered semifield such that $1+1 \neq 1$ and $K_o \subseteq K$, the prime semifield of K is order isomorphic to Q_0^+. Then K can be embedded into a complete positive totally ordered semifield.

Proof Let K_o be the prime semifield of K which is $K_o \cong_o Q_0^+$. By Proposition 4.19., K_o is dense in K.

Let $\mathcal{K}$ be the set of all subsets D of K_o with the following properties that

(1) $D \cap K_o \neq \emptyset$,

(2) there exists a $p \in K_o$ such that $p \notin D$,

(3) for every $d \in D$ there exists a $p \in K_o$ such that $d < p$ and

(4) for every $p, q \in K_o$, $p < q$ and $q \in D$ imply that $p \in D$.

First to show that $I = \{x \in K_o \mid x < 1\} \in \mathcal{K}$

(1) Since $1 < 2$, $2^{-1} < 1$. So $2^{-1} \in I \cap K_o$.

(2) Since $1 < 2$ and $2 \in K_o$, $2 \notin I$.

(3) Let $r \in I$. Then $r < 1$. Since $1, r \in K_o$, $1 + r \in K_o$. So $(r + 1)2^{-1} \in K_o$. Since $r < 1$, $2r = r + r < r + 1 < 1 + 1 = 2$. Thus $r < (r + 1)2^{-1} < 1$, $(r + 1)\, 2^{-1} \in I$.

(4) Let $p, q \in K_o$ be such that $p < q$ and $q \in I$. Then $q < 1$, hence $p \in I$.

Therefore $I \in \mathscr{K} \neq \varnothing$. Let $\mathscr{K}_0 = \mathscr{K} \cup \{\{0\}\}$. Define binary + and . on $\mathscr{K}_0$ as follow : let $C, D \in \mathscr{K}_0$

$$C + D = \{\, c + d \mid c \in C \text{ and } d \in D\} \text{ and}$$

$$CD = \{\, cd \mid c \in C \text{ and } d \in D\}$$

<u>Step 1.</u> To show that + is well-defined, claim that for every $p, q \in K_o$, $p < q$ implies there exists $r \in K_o$ such that $p + r = q$. Let $p, q \in K_o$ be such that $p < q$. Since $K_o \cong_o Q_0^+$, $q - p \in K_o$. Let $r = q - p$. Then $p + r = p + (q - p) = q$. So we have the claim.

Let $C, D \in \mathscr{K}_0$. suppose that $C, D \neq \{0\}$.

(1) Since $C \cap K_o \neq \varnothing$ and $D \cap K_o \neq \varnothing$, $(C + D) \cap K_o \neq \varnothing$.

(2) Let $p \in K_o - C$ and $q \in K_o - D$. Then $x < p$ and $y < q$ for all $x \in C$ and $y \in D$, so $x + y < p + q$ for all $x \in C$ and $y \in D$.
Hence $p + q \notin C + D$.

(3) Let $x \in C + D$. Then $x = c_1 + d_1$ for some $c_1 \in C$ and $d_1 \in D$. Since C and D are in $\mathscr{K}$, there exist $c_2 \in C$ and $d_2 \in D$ such that $c_1 < c_2$ and $d_1 < d_2$. Thus $x = c_1 + d_1 < c_2 + d_2 \in C + D$.

(4) Let $p, q \in K_o$ be such that $p < q$ and $q \in C + D$. Then $q = c + d$ for some $c \in C$ and $d \in D$. Thus $p < q = c + d$.

<u>Case 1.</u> $p < d$. Then $p \in D$. $p = p + 0 \in C + D$.

<u>Case 2.</u> $p = d$. Then $p \in D$, so $p \in C + D$.

<u>Case 3.</u> $p > d$. Then there exists an $r \in P$ such that $p = d + r < c + d$, so

$r < c$. Hence $r \in C$, so $p = d + r \in C + D$. Therefore $+$ is well-defined.

Step 2. To show that . is well-defined, let $C, D \in \mathscr{K}_0$. Suppose that $C, D \neq \{0\}$.

(1) Let $c \in C \cap K_o$ and $d \in D \cap K_o$. $cd \in CD \cap K_o \neq \emptyset$.

(2) Let $c \in K_o - C$ and $d \in K_o - D$. then $cd \in K_o$. Then $x < c$ and $y < d$ for all $x \in C$ and $y \in D$, so $xy < cd$ for all $x \in C$ and $d \in D$. Hence $cd \notin CD$.

(3) Let $x \in CD$. Then $x = c_1d_1$ for some $c_1 \in C$ and $d_1 \in D$. Since C and D are in $\mathscr{K}$, there exist $c_2 \in C$ and $d_2 \in D$ such that $c_1 < c_2$ and $d_1 < d_2$. Therefore $x = c_1d_1 < c_2d_2 \in CD$.

(4) Let $p, q \in P$ be such that $p < q$ and $q \in CD$. Then there exist $c \in C$ and $d \in D$ such that $q = cd$. Since $p < q = cd$, $pc^{-1} < d$. Thus $pc^{-1} \in D$. Hence $p = c(pc^{-1}) \in CD$.

Thererfore . is well-defined.

Step 3. To prove that $DI = D$ for all $D \in \mathscr{K}_0$, let $D \in \mathscr{K}_0$. If $D = \{0\}$ then done. Suppose that $D \neq \{0\}$. Choose $d \in D$. Let $p \in I^*$. Then $p < 1$, so $pd < d$. Since $d \in D$, $pd \in D$. Hence $DI \subseteq D$. Let $d \in D$. Then there exists $p \in D$ such that $d < p$, so $dp^{-1} < 1$. It follows that $dp^{-1} \in I$. Since $d = p(dp^{-1})$, $d \in DI$. Thus $D \subseteq DI$. Therefore $DI = D$ for all $D \in \mathscr{K}_0$.

Step 4. Let $D \in \mathscr{K}$. Let $D^{-1} = \{ p \in K_o \mid$ there exists $q \in K_o - D$ such that $p < q^{-1}\}$. We shall show that $D^{-1} \in \mathscr{K}$.

(1) Let $p \notin D$ and $p \in K_o$. Let $x = 2p$. Then $x \in K_o$ and $x^{-1} \in K_o$. Since $p < 2p = x$, $x^{-1} < p^{-1}$. Thus $x^{-1} \in D^{-1}$.

(2) Let $d \in D$. Then $d < x$ for all $x \in K_o - D$, so $x^{-1} < d^{-1}$ for all $x \in K_o - D$. Hence $d^{-1} \notin D^{-1}$.

(3) Let $d \in D^{-1}$. Then there exists a $q \in K_o - D$ such that $d < q^{-1}$. Hence there exists a $p \in K_o$ such that $d < p < q^{-1}$, so $p \in D^{-1}$.

(4) Let $p, q \in K_o$ be such that $p < q$ and $q \in D^{-1}$. Then there exists an $x \in K_o - D$ such that $q < x^{-1}$, hence $p < x^{-1}$. Hence $p \in D^{-1}$.

Therefore $D^{-1} \in \mathscr{K}$.

Step 5. To show that $DD^{-1} = I$ for all $D \in \mathscr{K}$, let $D \in \mathscr{K}$. Claim that for every $x \in K^*$ there exists a $q \in K_o - D$ such that $q - x \in D$.

Let $x \in K^*$. Suppose that $nx \in D$ for $n \in Z^+$. Let $p \in K_o$. Since K is Archimedian, there exists an $N \in Z^+$ such that $p < Nx$. Since $Nx \in D$, $p \in D$. This prove that $K_o \subseteq D$, a contradiction. Therefore there exists a $n \in Z^+$ such that $nx \notin D$.

Let $n_o = \min\{ n \in Z^+ \mid nx \notin D\}$. Then $n_o > 1$ and $(n_o - 1)x \in D$. Take $q = n_o x$. So we have the claim.

Let $a \in D$ and $b \in D^{-1}$. Then there exists a $y \in P - D$ such that $b < y^{-1}$. Hence $a < y$, so $ay^{-1} < 1$. Hence $ab < ay^{-1} < 1$, $ab \in I$. Therefore $DD^{-1} \subseteq I$. Let $p \in K_o$. Then $p < 1$, so $1 - p > 0$. Choose $a \in D^*$, so $a > 0$. Thus $a(1 - p) > 0$. By the claim, there exists a $y \in K_o - D$ such that $y - a(1 - p) \in D$.

Since $y \notin D$, $0 < a < y$. Then $a(1 - p) < y(1 - p) = y - yp$, hence $yp < y - a(1 - p)$. Then $p(y + dp^{-1}) = yp + d = y - a(1 - p)$ for some $d \in D$, so $p = [y - a(1 - p)](ay + dp^{-1})^{-1}$. Since $0 < y < y + dp^{-1}$, $(y + dp^{-1})^{-1} < y^{-1}$. Hence $(y + dp^{-1})^{-1} \in D^{-1}$, so $p \in DD^{-1}$. Therefore $I \subseteq DD^{-1}$, thus $I = DD^{-1}$.

Clearly, the commutative, associative and distributive laws hold. Also $D + \{0\} = D$ and $D\{0\} = \{0\}$ for all $D \in \mathscr{K}_0$.

Therefore $(\mathscr{K}_0, +, .)$ is a semifield.

Step 6. Define a relation $\leq$ on $\mathscr{K}_0$ by $D \leq C$ iff $D \subseteq C$ for all $C, D \in \mathscr{K}_0$. Then $\leq$ is a compatible partial order on $\mathscr{K}_0$ and obviously, $\{0\} \leq D$ for all $D \in \mathscr{K}_0$.

Hence we have that $\mathscr{K}_0$ is a positive ordered semifield.

Step 7. We shall show that $\mathscr{K}_0$ is a positive totally ordered semifield. Let $C, D \in \mathscr{K}_0$. Suppose that $C \neq D$.

Case 1. There exists a $d \in D - C$. Thus $x < d$ for all $x \in C$, so $x \in D$ for all $x \in C$. Therefore $C \subseteq D$.

Case 2. There exists $c \in C - D$. Similarly to Case 1.

Case 3. There exists $c \in C - D$ and $d \in D - C$. Since $c \in C - D$, $d < c$. Since $c \in C$, $d \in C$ which is a contradiction.

Therfore $\mathscr{K}_0$ is a positive totally ordered semifield.

Step 8. To show that $\mathscr{K}_0$ is complete, let $\{D_i \mid i \in I\}$ be a family in $\mathscr{K}_0$ such that $D_i \leq C$ for all $i \in I$. Claim that $\bigcup_{i \in I} D_i$ is a least upper bound of $\{D_i \mid i \in I\}$. Let $D = \bigcup_{i \in I} D_i$.

(1) Since $D_i \cap P \neq \varnothing$, $P \cap D \neq \varnothing$.

(2) Since $C \in \mathscr{K}_0$, there exists a $c \in P - C$. Hence $c \notin D_i$ for all $i \in I$, so $c \notin D$.

(3) Let $d \in D$. Then there exists an $i_o \in I$ such that $d \in D_{i_o}$. Since $D_{i_o} \in \mathscr{K}_0$, there exists a $d_{i_o} \in D_{i_o} \subseteq D$ such that $d < d_{i_o}$.

(4) Let $p, q \in P$ be such that $p < q$ and $q \in D$. there exists an $i_o \in I$ such that $q \in D_{i_o}$. Hence $p \in D_{i_o} \subseteq D$. Therefore $D \in \mathscr{K}_0$.

Clearly, D is a least upper bound of $\{D_i \mid i \in I\}$. Therefore $\mathscr{K}_0$ is complete.

<u>Step 9.</u> Define $f : K \to \mathscr{K}_0$ by $f(x) = L_x$ where $L_x = \{p \in K_o \mid p < x\}$ and $f(0) = \{0\}$. To show that f is an order monomorphism, let $x, y \in K$ be such that $f(x) = f(y)$. Then $L_x = L_y$. Suppose that $x \neq y$. Without loss of generality, suppose that $x < y$. Then there exists an $r \in K_o$ such that $x < r < y$. Thus $r \in L_y - L_x$, a contradiction. Hence $x = y$, so f is 1-1.

Let $x, y \in K$. Claim that $L_xL_y = L_{xy}$ and $L_x + L_y = L_{x+y}$. Let $a \in L_x$ and $b \in L_y$. Then $a < x$ and $b < y$. So $a + b < x + y$, hence $L_x + L_y \subseteq L_{x+y}$. Let $c \in L_{x+y}$. Then $c < x + y$ and there exists $p \in P$ such that $c < p < x + y$. Thus $c = c(pp^{-1}) < (cp^{-1})(x + y) = cxp^{-1} + cyp^{-1}$. Since $c < p$, $cp^{-1} < 1$, so $cxp^{-1} < x$. Hence $cxp^{-1} < q_x < x$ for some $q_x \in K_o$. Similarly, there exists a $q_y \in P$ such that $cyp^{-1} < q_y < y$. Then $c = c(q_x + q_y)(q_x + q_y) = cq_x(q_x + q_y) + cq_y(q_x + q_y)$. Since $c < cxp^{-1} + cyp^{-1} < q_x + q_y$, $c(q_x + q_y)^{-1} < 1$. Thus $cq_x(q_x + q_y)^{-1} < q_x$ and $cq_y(q_x + q_y)^{-1} < q_y$, so $c = cq_x(q_x + q_y)^{-1} + cq_y(q_x + q_y)^{-1} \in L_x + L_y$ since $q_x < x$ and $q_y < y$. Therefore $c \in L_x + L_y$. Hence $L_{x+y} = L_x + L_y$.

Next, let $a \in L_x$ and $b \in L_y$. Then $a < x$ and $b < y$, so $ab < xy$. Thus $ab < xy$. Let $c \in L_{xy}$. Then $c < xy$, so $cx^{-1} < y$. Then there exists $p \in P$ such that $cx^{-1} < p < y$. Then $p \in L_y$. Since $cp^{-1} \in K_o$ and $cp^{-1} < x$, $cp^{-1} \in L_x$. Hence $c = (cp^{-1})p \in L_xL_y$. therefore $L_{xy} = L_xL_y$. So we have the claim. This show that f is a homomorphism.

Finally, to show that $f(P_K) = P_{f(K)}$, cleary that f is isotone. Thus $f(P_K) \subseteq P_{f(K)}$. Let $D \in P_{f(K)}$. Then $D \geq I$. Since $P_{f(K)} \subseteq f(K)$, $L_x = f(x) = D$ for some $x \in K$. If $x \geq 1$ then $D \in f(P_K)$. Suppose that $x < 1$. Thus $L_x = D \subset I$, a contradiction since $D \geq I$. Hence $P_{f(K)} \subseteq f(P_K)$, so $f(P_K) = P_{f(K)}$.

Therefore K is embedded into a complete positive totally ordered semifield. #