CHAPTER III

POSITIVE LATTICE ORDERED 0-SEMIFIELDS

Definition 3.1. Let R be a positive ordered semiring. R is called a positive lattice ordered semiring iff the partial order of R is a lattice, that is for every $x, y \in R$, $x \vee y$ and $x \wedge y$ exist.

Examples 3.2. (1) Q_0^+, $\mathbb{R}_0^+$ are positive lattice ordered semifields.

(2) From Example 2.6. (2), we have that K is a positive lattoce ordered semifield. To prove this, let $x, y \in K$. Then $x \leq x + y$, $y \leq x + y$. Thus $x + y$ is an upper bound of x and y. Let $w \in K$ be such that $x \leq w$ and $y \leq w$. Then $x + w = w$ and $y + w = w$, $(x + y) + w = x + (y + w) = x + w = w$. Thus $x + y \leq w$, so $x \vee y = x + y$.

If $x = 0$ or $y = 0$ then $x \wedge y$ exists. Suppose that x, y are nonzero. So $x + y \neq 0$. Since $x + xy(x + y)^{-1} = [x(x + y) + xy](x + y)^{-1} = (x^2 + xy + xy)(x + y)^{-1} = (x^2 + xy)(x + y)^{-1} = [x(x + y)](x + y)^{-1} = x$, $xy(x + y)^{-1} \leq x$.

Similarly, $xy(x + y)^{-1} \leq y$. Then $xy(x + y)^{-1}$ is a lower bound of x and y. Let $w \in K$ be such that $x \geq w$ and $y \geq w$. Then $x + w = x$ and $y + w = w$. Therefore $w + xy(x + y)^{-1} = [w(x + y) + xy](x + y)^{-1} = (wx + wy + xy)(x + y)^{-1} = [wx + (w + x)y](x + y)^{-1} = (wx + xy)(x + y)^{-1} = [x(w + y)x](x + y)^{-1} = xy(x + y)^{-1}$, so $w \leq xy(x + y)^{-1}$. Hence $x \wedge y = xy(x + y)^{-1}$.

(3) Let $(G, . , \leq)$ be a lattice ordered group. Let $K = G \cup \{a\}$ where a is a new symbol such that $a \notin G$. Define + on K by $x + y = x \vee y$ and $a + x = x + a = x$ for all $x, y \in K$ and define $a.x = x.a = a$ and $a \leq x$ for all $x \in K$.

Then we have that K is a positive lattice ordered semifield where

$x + x = x$ for all $x \in K$.

(4) Let K be a positive lattice ordered semifield. Then $(K, +^*, \cdot, \leq)$ is a positive lattice ordered semifield such that $x +^* x = x$ for all $x \in K$ where $x +^* y = x \vee y$ for all $x, y \in K$.

Remark 3.3. Let K be a positive ordered semifield. Then the following statements hold :

(1) For every $x, y \in K$, if $x \vee y$ exists then $xw \vee yw$ exists for all $w \in K$. Moreover, $(x \vee y)w = xw \vee yw$.

(2) For every $x, y \in K$, if $x \wedge y$ exists then $xw \wedge yw$ exists for all $w \in K$. Moreover, $(x \wedge y)w = xw \wedge yw$.

Proof (1) Let $x, y \in K$ be such that $x \vee y$ exists. Let $w \in K$. If $w = 0$ then done. Suppose that $w \neq 0$. Since $x \leq x \vee y$ and $y \leq x \vee y$, $xw \leq (x \vee y)w$ and $yw \leq (x \vee y)w$. Hence $(x \vee y)w$ is an upper bound of xw and yw. Let $z \in K$ be such that $xw \leq z$ and $yw \leq z$. Then $x \leq zw^{-1}$ and

$y \leq zw^{-1}$, so $x \vee y \leq zw^{-1}$. Hence $(x \vee y)w \leq z$. Therefore $(x \vee y)w = wx \vee wy$.

(2) Dually (1). #

Theorem 3.4. ([1]) Every positive lattice ordered group G is a distributive lattice, that is for any $x, y, z \in G$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Proof See [1], pp. 294. #

Proposition 3.5. Let K be a positive ordered semifield. Then the following statements are equivalent :

(1) K is a lattice.

(2) for every $x \in K$, $x \vee 1$ exists.

(3) for every $x \in K$, $x \wedge 1$ exists.

(4) P is a lattice where P is the positive cone of K.

Proof Obviously, (1) $\rightarrow$ (2) and (1) $\rightarrow$ (4).

To show that (2) $\rightarrow$ (3), assume (2). Let $x \in K$. If $x = 0$ then $0 = x \wedge 1$, so done. Suppose that $x \neq 0$. By assumption, $x^{-1} \vee 1$ exists. Let $y = x^{-1} \vee 1$. Claim that $x \wedge 1 = y^{-1}$. Since $x^{-1} \leq y$ and $1 \leq y$, $y^{-1} \leq 1$ and $y^{-1} \leq x$. Thus y^{-1} is a lower bound of x and 1. Let $w \in K$ be such that $w \leq 1$ and $w \leq x$. Assume that $w \neq 0$. Then $1 \leq w^{-1}$ and $x^{-1} \leq w^{-1}$. Since $y = x^{-1} \vee 1$, $y \leq w^{-1}$. So $w \leq y^{-1}$, hence $x \wedge 1 = y^{-1}$. So we have the claim.

To prove (3) $\rightarrow$ (1), assume (3). Let $w, y \in K$. If $x = 0$ then $0 = 0 \wedge y = x \wedge y$, so done. Assume that $x \neq 0$. By assumption, $yx^{-1} \wedge 1 = z$ for some $z \in K$. By Remark 3.3., $x \wedge y = x[(yx^{-1}) \wedge 1] = xz$. Thus $x \wedge y$ exists. Next, to show that $x \vee y$ exists for all $x, y \in K$, let $x, y \in K$. If $x = 0$ or $y = 0$ then done. Suppose that x, y are nonzero. By above $x^{-1} \wedge y^{-1} = w$ for some $w \in K$. Since $(x + y)^{-1} \leq x^{-1}$ and $y^{-1} \leq w$, $0 \neq (x + y)^{-1} \leq w$, so $w \neq 0$. Since $w \leq x^{-1}$ and $w \leq y^{-1}$, $x \leq w^{-1}$ and $y \leq w^{-1}$. Thus w^{-1} is an upper bound of x and y. Let $z \in K$ be such that $x \leq z$ and $y \leq z$. If $z = 0$ then done. Suppose that $z \neq 0$. Since $z^{-1} \leq x^{-1}$ and $z^{-1} \leq y^{-1}$ and $x^{-1} \wedge y^{-1} = w$, $z^{-1} \leq w$. Thus $w^{-1} \leq z$, so $x \vee y = w^{-1}$. Hence (1) holds.

To prove (4) $\rightarrow$ (3), assume (4). Let $x \in K$. If $x = 0$ then $x \wedge 1$ exists. Suppose that $x \neq 0$. By Remark 2.10. (6), $x = pq^{-1}$ for some $p, q \in P$. By assumption, $p \wedge q = z$ for some $z \in P$. Hence $x \wedge 1 = pq^{-1} \wedge 1 = q^{-1}(p \wedge q) = zq^{-1}$, so we have (3). #

Proposition 3.6. Let K be a positive lattice ordered semifield. Then the following statements hold :

(1) For every nonzero element $x \in K$, $x = pq^{-1}$ for some $p, q \in P$ such that $p \wedge q = 1$.

(2) For every $x, y \in K$, $(x \vee y)(x \wedge y) = xy$.

(3) For every $x \in K$, $x = (x \vee 1)(x \wedge 1)$.

(4) For every $x, y \in K^*$, $(x \vee y)^{-1} = x^{-1} \wedge y^{-1}$ and $(x \wedge y)^{-1} = x^{-1} \vee y^{-1}$.

Proof (1) Let $x \in K$ be such that $x \neq 0$. By Remark 2.10. (6), $x = pq^{-1}$ for some $p, q \in P$. Let $a = p(p \wedge q)^{-1}$ and $b = q(p \wedge q)^{-1}$. So $a, b \in P$, $a \wedge b = [p(p \wedge q)^{-1}] \wedge [q(p \wedge q)^{-1}] = 1$ and $x = pq^{-1} = [p(p \wedge q)^{-1}][(p \wedge q)q^{-1}] = ab^{-1}$.

(2) Let $x, y \in K$. If $x = 0$ or $y = 0$ then done. Suppose that x, y are nonzero. Then $(x \vee y) \neq 0$. Since $x \leq (x \vee y)$ and $y \leq (x \vee y)$, $(x \vee y)^{-1} \leq x^{-1}$ and $(x \vee y)^{-1} \leq y^{-1}$. So $xy(x \vee y)^{-1} \leq x$ and $xy(x \vee y)^{-1} \leq y$. Hence $xy(x \vee y)^{-1}$ is a lower bound of x and y. Let $z \in K$ be such that $z \leq x$ and $z \leq y$. Thus $zx \leq xy$ and $zy \leq xy$, so $z(x \vee y) = zx \vee zy \leq xy$. Thus $z \leq xy(x \vee y)^{-1}$. Therefore $x \wedge y = xy(x \vee y)^{-1}$, so $(x \vee y)(x \wedge y) = xy$.

(3) Follows directly from (2).

(4) Let $x, y \in K^*$. Since $x \leq x \vee y$ and $y \leq x \vee y$, $(x \vee y)^{-1} \leq x^{-1}$ and $(x \vee y)^{-1} \leq y^{-1}$. Then $(x \vee y)^{-1}$ is a lower bound of x^{-1} and y^{-1}. Let $w \in K$ be such that $w \leq x^{-1}$ and $w \leq y^{-1}$. Hence $xw \leq 1$ and $yw \leq 1$, so $(x \vee y)w = xw \vee yw \leq 1$. Then $w \leq (x \vee y)^{-1}$, so $(x \vee y)^{-1} = x^{-1} \wedge y^{-1}$. Next, to show that $(x \wedge y)^{-1} = x^{-1} \vee y^{-1}$, since $x, y \neq 0$, xy, $x \vee y$ are nonzero. Then $x \wedge y = xy(x \vee y)^{-1} \neq 0$. Therefore $(x \wedge y)^{-1} = (x \vee y)(xy)^{-1} = x^{-1} \vee y^{-1}$. #

Note that for nonzero elements x, y in a positive lattice ordered semifield K, $x \vee y$ and $x \wedge y$ are nonzero.

Proposition 3.7. Let K be a positive lattice ordered semifield. Then the following statements hold : for every $x, y, z \in K$

(1) if $x \leq y$ then $x \vee z \leq y \vee z$ and $x \wedge z \leq y \wedge z$,

(2) $x + (y \wedge z) \leq (x + y) \wedge (x + z)$ and $x + (y \vee z) \geq (x + z) \vee (x + z)$ and

(3) $(x + y) \wedge z \leq (x \wedge z) + (y \wedge z)$ and $(x + y) \vee z \leq (x \vee z) + (y \vee z)$

Proof Let $x, y, z \in K$.

(1) Assume that $x \leq y$. Since $y \leq y \vee z$ and $z \leq y \vee z$, $x \vee z \leq y \vee z$. Since $x \wedge z \leq x \leq y$ and $x \wedge z \leq z$, $x \wedge z \leq y \wedge z$.

(2) Since $y \wedge z \leq y$ and z, $x + (y \wedge z) \leq x + y$ and $x + z$. Hence $x + (y \wedge z) \leq (x + y) \wedge (x + z)$. Dually, $x + (y \vee z) \geq (x + y) \vee (x + z)$.

(3) Suppose that $x, y, z \neq 0$. Since $x \leq x + y$, by (1) $x \wedge z \leq (x + y) \wedge z$. Thus $[(x + y) \wedge z]^{-1} \leq (x \wedge z)^{-1}$, so $xz[(x + z) \wedge z]^{-1} \leq xz(x \wedge y)^{-1}$. Similarly, $yz[(x + y) \wedge z]^{-1} \leq yz(y \wedge z)^{-1}$. Therefore $(x + y) \wedge z =$ $z(x + y)[(x + y) \vee z]^{-1} = zx[(x + y) \vee z]^{-1} + zy[(x + y) \vee z]^{-1} \leq xz(x \vee z) + yz(y \vee z)$ $= (x \wedge z) + (y \wedge z)$. Since $x \leq x \vee z$ and $y \leq y \vee z$, $(x + y) \leq (x \vee z) + (y \wedge z)$. Clear that $z \leq (x \vee z) + (y \vee z)$. Therefore $(x + y) \vee z \leq (x \vee z) + (y \vee z)$. #

Theorem 3.8. ([2]) Let K be a positive lattice ordered semifield and $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n \in P$ such that $a_1a_2 \ldots a_m = b_1b_2 \ldots b_n$. Then there exist elements $c_{ij} \in P$ for all $i \in \{1,2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$ satisfying

(1) $a_i = c_{i1}c_{i2} \ldots c_{in}$ $i \in \{1,2, \ldots, m\}$,

(2) $b_j = c_{1j}c_{2j} \ldots c_{mj}$ $i \in \{1,2, \ldots, n\}$,

(3) $c_{i+1,j}c_{i+2,j} \ldots c_{mj} \wedge c_{i,j+1}c_{i,j+2} \ldots c_{in} = 1$ for all $i < m$, $j < n$.

Proof See [2], pp. 68. #

Corollary 3.9. If $a, b_1, b_2, ..., b_n$ are in the positive cone of a positive lattice ordered semifiled K such that $a \le b_1 b_2 \dots b_n$ then there exist $a_1, a_2, ..., a_n \in P$ satisfying $a = a_1 a_2 \dots a_m$ with $a_i \le b_i$ for all $i \in \{1, 2, ..., n\}$.

Proof It follows from Theorem 3.8.. #

Proposition 3.10. Let K be a positive lattice ordered semifield. Then the following statements hold :

(1) For every $x, y \in K$, $(x \vee y)^n = x^n \vee x^{n-1}y \vee x^{n-2}y^2 \vee \dots \vee xy^{n-1} \vee y^n$ and $(x \wedge y)^n = x^n \wedge x^{n-1}y \wedge x^{n-2}y^2 \wedge \dots \wedge xy^{n-1} \wedge y^n$.

(2) If $x \in K$ and $x^n \ge 1$ for some $n \in Z^+$ then $x \ge 1$.

(3) For every $x \in K$, $(x \vee 1)^n = x^n \vee 1$ and $(x \wedge 1)^n = x^n \wedge 1$ for all $n \in Z^+$.

Proof (1) We shall prove by induction on n. Let $x, y \in K$, for $n = 1$, then done. Assume that (1) is true for $n - 1 \ge 1$.

Then $(x \vee y)^n = (x \vee y)^{n-1}(x \vee y)$

$= (x^{n-1} \vee x^{n-2}y \vee \dots \vee xy^{n-2} \vee y^{n-1})(x \vee y)$

$= (x^{n-1} \vee x^{n-2}y \vee \dots \vee xy^{n-2} \vee y^{n-1})x \vee (x^{n-1} \vee x^{n-2}y \vee \dots \vee xy^{n-2} \vee y^{n-1})y$

$= x^n \vee x^{n-1}y \vee x^{n-2}y^2 \vee \dots \vee x^2y^{n-2} \vee y^{n-1}x \vee x^{n-1}y \vee x^{n-2}y^2 \vee \dots \vee xy^{n-1} \vee y^n$

$= x^n \vee x^{n-1}y \vee x^{n-2}y^2 \vee \dots \vee xy^{n-1} \vee y^n$.

Dually, $(x \wedge y)^n = x^n \wedge x^{n-1}y \wedge x^{n-2}y^2 \wedge \dots \wedge xy^{n-1} \wedge y^n$.

(2) Let $x \in K$ and $x^n \ge 1$ for some $n \in Z^+$. If $n = 1$ then done. Suppose that $n \ge 2$. Then by (1), $(x \vee 1)^n = x^n \vee x^{n-1} \vee \dots \vee x \vee 1 = (x \vee y)^n$ $= x^n \vee x^{n-1} \vee \dots \vee x$ (since $x^n \ge 1$) $= x(x^{n-1} \vee \dots \vee 1) = x(x \vee 1)^{n-1}$. Therefore $x \vee 1 = x$, so $1 \le x \vee 1 = x$.

(3) Let $x \in K$ and $n \in Z^+$. If $x = 0$ then done. Suppose that $x \ne 0$. For $0 \le k \le n$, $(x^{n-k} \vee x^{-k})^n = x^{(n-k)n} \vee \dots \vee x^{(n-k)k}(x^{-k})^{(n-k)} \vee \dots \vee (x^{-k})^n$. By (2), $(x^{n-k} \vee x^{-k}) \ge 1$, so $x^n \vee 1 \ge x^k$ for all $0 \le k \le n$.

Hence $(x \vee 1)^n = x^n \vee x^{n-1} \vee \dots \vee x \vee 1 = x^n \vee 1$.

Next, $(x \wedge 1)^n = [x(x \vee 1)^{-1}]^n = x^n(x \vee 1)^{-n} = x^n(x^n \vee 1)^{-1} = x^n \wedge 1$. #

Theorem 3.11. Let S be a positive lattice ordered commutative semiring with multiplicative zero 0 which is M.C.. If S satisfies the property that : for every $x, y, z \in S$, $xz < yz$ implies that $x < y$ then S can be embeded into a positive lattice ordered semifield.

Proof By Theorem 2.7., we have that $K = S \times (S - \{0\})/_\sim$ is the positive ordered semifield. To show that K is a lattice, let $\alpha \in K$. Fix $a \in S - \{0\}$. If $\alpha = 0$ then done. Suppose that $a \neq 0$. Choose $(x , y) \in \alpha$. Then $x \vee a$ and $y \vee a$ exist in S. Claim that $\alpha \vee [(a , a)] = [(xa \vee ya , ya)]$. Clearly, $[(xa \vee ya , ya)]$ is upper bound of α and $[(a , a)]$. Let $\beta \in K$ be such that α and $[(a , a)] \leq \beta$. Then there exist (b_1 , b_2), $(b_3 , b_4) \in \beta$, $(z , w) \in \alpha$ and $c \in S - \{0\}$ such that $zb_2 \leq wb_1$ and $cb_4 \leq cb_3$. Since $c \neq 0$, $b_4 \leq b_3$. Then $yab_4 \leq yab_3$. Since $zb_2 \leq wb_1$ and $b_1b_4 = b_2b_3$, $zb_2b_4 \leq wb_1b_4 = wb_2b_3$. Since $b_2 \neq 0$, $zb_4 \leq wb_3$, so $zyb_4 \leq wyb_3$. Since $zy = xw$, $xwb_4 \leq wyb_4$, so $xb_4 \leq yb_4$. Hence $xab_4 \leq yab_3$. This show that $(xa \vee ya)b_4 = xab_4 \vee yab_4 \leq yab_3$, so $[(xa \vee ya , ya)] \leq [(b_3 , b_4)] = \beta$. Therefore $\alpha \vee [(a , a)] = [(xa \vee ya , ya)]$. So we have the claim. By Proposition 3.5. (2), K is a positive lattice ordered semifield. #

Definition 3.12. Let K be a positive lattice ordered semifield and $x \in K^*$. The absolute value of x, denoted by $|x|$, is defined to be $x \vee x^{-1}$.

In [2], pp. 76 we have the following elementary properties : for every $x, y \in K^*$

(1) $|x| \geq 1$ and $|x| = |x^{-1}|$.

(2) $|x| = 1$ iff $x = 1$.

(3) $|xy^{-1}| = (x \vee y)(x \wedge y)^{-1}$.

(4) $|x| = (x \vee 1)(x \wedge 1)^{-1}$

(5) $|x^n| = |x|^n$ for all $n \in Z^+$.

(6) $|xy| \leq |x|\,|y|$

Propositon 3.13. Let K be a positive lattice ordered semfield and $x, y, z \in K^*$.

(1) $|(x \vee z)(y \vee z)^{-1}|\,|(x \wedge z)(y \wedge z)^{-1}| = |xy^{-1}|$.

(2) $|(x \vee z)(y \vee z)^{-1}| \leq |xy^{-1}|$ and $|(x \wedge z)(y \wedge z)^{-1}| \leq |xy^{-1}|$.

(3) $|x + y| \leq |x| + |y|$.

(4) $|(x + z)(y + z)^{-1}| \leq |xy^{-1}|$.

Proof Let $x, y, z \in K^*$.

(1) $|(x \vee z)(y \vee z)^{-1}|\,|(x \wedge z)(y \wedge z)^{-1}|$

$= [(x \vee z) \vee (y \vee z)][(x \vee z) \wedge (y \vee z)]^{-1}[(x \wedge z) \vee (y \wedge z)][(x \wedge z) \wedge (y \wedge z)]^{-1}$

$= [(x \vee y) \vee z][(x \wedge y) \vee z]^{-1}[(x \vee y) \wedge z][(x \wedge y) \wedge z]^{-1}$

$= [(x \vee y) \vee z][(x \wedge y) \vee z]^{-1}[(x \vee y)z][(x \vee y) \vee z]^{-1}[(x \wedge y) \vee z][(x \wedge y)z]^{-1}$

$= (x \vee y)(x \wedge y)^{-1} = |xy^{-1}|$.

(2) Since $1 \leq |(x \vee z)(y \vee z)^{-1}|$, $|(x \wedge z)(y \wedge z)^{-1}| \leq |(x \wedge z)(y \wedge z)^{-1}|$ $|(x \vee z)(y \vee z)^{-1}| = |xy^{-1}|$. Thus $|(x \wedge z)(y \wedge z)^{-1}| \leq |xy^{-1}|$. Similarly, $|(x \vee z)(y \vee z)^{-1}| \leq |xy^{-1}|$.

(3) Since $x \leq |x|$ and $y \leq |y|$, $x + y \leq |x| + |y|$. Since $x \leq x + y$ and $y \leq x + y$, $(x + y)^{-1} \leq x^{-1}$ and $(x + y)^{-1} \leq y^{-1}$. Then $(x + y)^{-1} \leq (x + y)^{-1} + (x + y)^{-1}$ $\leq x^{-1} + y^{-1} \leq |x| + |y|$. This prove that $|x| + |y|$ is an upper bound of $x + y$ and $(x + y)^{-1}$. Hence $|x + y| = (x + y) \vee (x + y)^{-1} \leq |x| + |y|$.

(4) $|(x + z)(y + z)^{-1}| = [(x + z) \vee (y + z)][(x + z) \wedge (y + z)]^{-1}$

$\leq [(x \vee y) + z][(x + z) \wedge (y + z)]^{-1}$

$$\leq [(x \vee y) + z][(x \wedge y) + z]^{-1}.$$

Claim that $[(x \vee y) + z][(x \wedge y) + z]^{-1} \leq (x \vee y)(x \wedge y)^{-1}$.

Since $x \wedge y \leq x \vee y$, $z(x \wedge y) \leq z(x \vee y)$. Thus

$$\begin{aligned}(x \wedge y)[(x \vee y) + z] &= (x \wedge y)(x \vee y) + (x \wedge y)z \\ &\leq (x \wedge y)(x \vee y) + (x \vee y)z \\ &= (x \vee y)[(x \wedge z) + z].\end{aligned}$$

Hence $[(x \vee y) + z][(x \wedge y) + z]^{-1} \leq (x \vee y)(x \wedge y)^{-1}$, so we have the claim. Since $|(x + z)(y + z)^{-1}| \leq [(x \vee y) + z][(x \wedge y) + z]^{-1}$, by the claim $|(x + z)(y + z)^{-1}| \leq (x \vee y)(x \wedge y)^{-1} = |xy^{-1}|$. Hence $|(x + z)(y + z)^{-1}| \leq |xy^{-1}|$. #

Definition 3.14. Let K be a positive lattice ordered semifield. Let $x, y \in K$, x and y are said to be orthogonal iff $x \wedge y = 1$, denoted by $x \perp y$.

Let $\perp_x$ be the set of all elements $y \in K$ such that $x \perp y$, that is $\perp_x = \{y \in K \mid x \wedge y = 1\}$.

Remark 3.15. ([2]) Let K be a positive lattice ordered semifield. the following statements hold : for every $x, y, z \in K$

(1) $x \perp y$ and $x \perp z$ imply that $x \perp yz$.

(2) $x \perp y$ and $z \geq 1$ imply that $x \wedge yz = x \wedge z$.

(3) $x \perp y$ and $x \perp z$ imply that $x \perp (y \vee z)$ and $x \perp (y \wedge z)$.

(4) $x \perp y$ implies that $x \vee y = xy$.

(5) $x \perp y$ implies that $|x| \perp |y|$.

Definition 3.16. Let K be a positive lattice ordered semifield. Let I be a convex subgroup of K. I is said to be an L-ideal iff for every $x, y \in I$, $x \vee y, x \wedge y \in I$.

Remark 3.17. Let K be a positive lattice ordered semifield. Then the following statements clearly hold :

(1) $\{1\}$ and K^* are trivial L-ideals of K.

(2) The intersection of a family of L-ideal of K is an L-ideal of K. Also the union of an increasing chain of L-ideals of K is an L-ideal.

(3) Let I be a convex subgroup of K. I is an L-ideal of K iff $x \vee 1 \in I$ for all $x \in I$.

Proposition 3.18. Let K be a positive lattice ordered semifield and $I \subseteq K$. Then I is an L-ideal of K iff it is an a-convex subgroup of K such that for every $a \in I$, $x \in K$ if $|x| \leq |a|$ then $x \in I$.

Proof Assume that I is an L-ideal of K. Let $a \in I$, $x \in K$ be such that $|x| \leq |a|$. then $x, x^{-1} \leq |a|$ and $|a| \in I$. By the o-convexity of I and $|a|^{-1} \leq x \leq |a|$, $x \in I$.

Conversely, we must show that I is o-convex and $x \vee 1 \in I$ for all $x \in I$. Let $x, y \in I$, $z \in K$ be such that $x \leq z \leq y$. Then $1 \leq zx^{-1} \leq yx^{-1}$. Since $|zx^{-1}| = zx^{-1} \leq yx^{-1} \leq |yx^{-1}|$ and $yx^{-1} \in I$, by assumption $zx^{-1} \in I$. Hence $z \in I$. Let $x \in I$. Since $1 \leq |x \vee 1| = x \vee 1 \leq |x|$, $x \vee 1 \in I$. Therefore I is an L-ideal of K. #

Proposition 3.19. Let A and B be L-ideals of a positive lattice ordered semifield K. Then AB is an L-ideal of K which is the smallest L-ideal containing A and B.

Proof By Proposition 1.34., we have that AB is an a-convex subgroup of K containing A and B. Let $x \in A$, $y \in B$ and $c \in K$ be such that

$|c| \leq |xy|$. Since $|xy| \leq |x|\,|y|$, $|c| \leq |x|\,|y|$. By Corollary 3.9., the exist $a, b \in P$, such that $a \leq |x|$ and $b \leq |y|$ and $|c| = ab$. Since $a, b \in P$, $a \in A$ and $b \in B$. Hence $|c| \in AB$. Because $1 \leq |c \vee 1| = c \vee 1 \leq |c|$, we can prove in a manner similar to the above that $c \vee 1 \in AB$. Since $|c| = (c \vee 1)(c \wedge 1)^{-1}$ and $|c|, c \vee 1 \in AB$, $c \wedge 1 \in AB$. Since $c = (c \vee 1)(c \wedge 1)$, $c \in AB$. Thus by Proposition 3.18, AB is an L-ideal.

Next, let I be an L-ideal of K such that $A, B \subseteq I$. Let $a \in A$ and $b \in B$. Then $a, b \in I$, so that $ab \in I$. Hence $AB \subseteq I$. #

Corollary 3.20. Let K be a positive lattice ordered semifield and I an L-ideal of K. For every $x, y \in I$ and $z \in K^*$, $(x \vee z)(y \vee z)^{-1}$, $(x \wedge z)(y \wedge z)^{-1} \in I$.

Proof Let $x, y \in I$ and $z \in K - \{0\}$. By Proposition 3.13. (2), $|(x \vee z)(x \vee z)^{-1}|$, $|(x \wedge z)(x \wedge z)^{-1}| \leq |xy^{-1}|$. By Proposition 3.18., $(x \vee z)(y \vee z)^{-1}$, $(x \wedge z)(x \wedge z)^{-1} \in I$. #

Proposition 3.21. Let K be a positive lattice ordered semifield and $a \in K^*$. Then $A = \{x \in K \mid |x| \perp |a|\}$ is an L-ideal of K.

Proof Let $x, y \in A$. Then $|x| \wedge |a| = |y| \wedge |a| = 1$. Then $1 \leq |xy^{-1}| \wedge |a| \leq |x|\,|y^{-1}| \wedge |a| = |x|\,|y| \wedge |a|$. By Remark 3.15. (1), $|x|\,|y| \wedge |a| = 1$, so $|xy^{-1}| \wedge |a| = 1$. Hence $xy^{-1} \in A$, so A is a multiplicative subgroup. Let $z \in K$. Then $1 \leq |(x + z)(y + z)^{-1}| \wedge |a| \leq |xy^{-1}| \wedge |a| = 1$, so $(x + z)(y + z)^{-1} \in A$. By Proposition 3.14., A is an a-convex subgroup of K. Let $z \in K$ and $x \in A$ be such that $|z| \leq |x|$. Then $1 \leq |z| \wedge |a| \leq |x| \wedge |a| = 1$, so $|z| \wedge |a| = 1$. Hence $z \in A$. By proposition 3.18., A is an L-ideal of K. #

Definition 3.22. Let K be a positive lattice ordered semifield and $a \in K^*$. The principal L-ideal generated by a, denoted by $< a >_L$, is the smallest

L-ideal of K containing a.

Remark 3.23. Let K be a positive lattice ordered semifield and $a \in K^*$. Then $< a >_L = \{x \in K \mid |x| \leq |a|^m$ for some $m \in Z^+\}$.

Proof Let $B = \{x \in K \mid |x| \leq |a|^m$ for some $m \in Z^+\}$. To show that B is L-ideal, let $x, y \in B$. Then there exist $m, n \in Z^+$ such that $|x| \leq |a|^m$ and $|y| \leq |a|^n$. Then $|xy^{-1}| \leq |x||y^{-1}| = |x||y| \leq |a|^{m+n}$, $xy^{-1} \in B$. Let $z \in K$. Since $|(x + z)(y + z)^{-1}| \leq |xy^{-1}| \leq |a|^{m+n}$, $(x + z)(y + z)^{-1} \in B$. Hence B is a-convex subgroup.

Let $w \in K$ be such that $|w| \leq |x|$ for some $x \in B$. Then $|x| \leq |a|^m$ for some $m \in Z^+$, hence $|w| \leq |a|^m$. Thus $w \in B$. By Proposition 3.18., B is an L-ideal of K. Clear that B contains a.

Let J be an L-ideal of K containing a. Then $|a|^m \in J$ for all $m \in Z^+$. Let $x \in B$. Then $|x| \leq |a|^n$ for some $n \in Z^+$.
Since $|x| \leq |a|^n = ||a|^n|$, by Proposition 3.27., $x \in J$. Hence $B \subseteq J$. Therefore $< a >_L = \{x \in K \mid |x| \leq |a|^m$ for some $m \in Z^+\}$. #

Proposition 3.24. Let K be a positive lattice ordered semifield and $a, b \in K^*$. Then $|a| \perp |b|$ if and only if $< a >_L \cap < b >_L = \{1\}$.

Proof Assume that $|a| \perp |b|$. Then $|a| \wedge |b| = 1$.
Let $< a >_L \cap < b >_L$. Then there exists $m \in Z^+$ such that $|x| \leq |a|^m$ and $|x| \leq |b|^m$. Since $|a| \perp |b|$ and $|b| \geq 1$, by Remark 3.15. (2), $|a| \wedge |b|^m = |a| \wedge |b| = 1$. Thus $|a| \perp |b|^m$. By Remark 3.15. (2), $|a| \perp |b|^m$ and $|a| \geq 1$, we have that $|a|^m \wedge |b|^m = |a| \wedge |b|^m = 1$. So $|a|^m \perp |b|^m$ Hence $|x| \leq |a|^m \wedge |b|^m = 1$, so $x = 1$.

Conversely, assume that $< a >_L \cap < b >_L = \{1\}$. Since $|a| \wedge |b| \leq |a|$ and $|a| \wedge |b| \leq |b|$, $|a| \wedge |b| \in < a >_L \cap < b >_L$. By assumption, $|a| \wedge |b| = 1$. Therefore $|a| \perp |b|$. #

Definition 3.25. Let K and M be positive lattice ordered semifield. A map $f : K \rightarrow M$ is said to be an L-homomorphism iff f is a homomorphism and for every $x, y \in K$, $f(x \vee y) = f(x) \vee f(y)$.

The definitions of L-epimorphisms, L-monomorphisms and L-isomorphisms are defined as one would expect. If there is an L-isomorphism from K onto M, we denoted by $K \cong_L M$.

Remark 3.26. Let $f : K \rightarrow M$ be an L-homomorphism between positive lattice ordered semifields. Then the following statements hold :

(1) f is isotone.

(2) ker f is an L-ideal of K.

(3) $f(x \wedge y) = f(x) \wedge f(y)$ for all $x, y \in K$.

(4) If I' is an L-ideal of M then $f^{-1}(I')$ is an L-ideal of K containing ker f.

Proof (1) Let $x, y \in K$ be such that $x \leq y$. Then $x \vee y = y$. Hence $f(y) = f(x \vee y) = f(x) \vee f(y)$, so $f(x) \leq f(y)$.

(2) By Remark 2.15. (2), ker f is a convex subgroup of K. Let $x \in \ker f$. Then $f(x \vee 1) = f(x) \vee f(1) = 1 \vee 1 = 1$, $x \vee 1 \in \ker f$. Hence ker f is an L-ideal of K.

(3) Let $x, y \in K$. If $x = 0$ or $y = 0$ then done. Suppose that x, y are nonzero. $f(x \wedge y) = f(xy(x \vee y)^{-1}) = f(x)f(y)[f(x) \vee f(y)]^{-1} = f(x) \wedge f(y)$.

(4) Let I' be an L-ideal of M. By Remark 2.15. (3), we have $f^{-1}(I')$ is a convex subgroup of K containing ker f. Let $x \in f^{-1}(I')$. Then

$f(x) \in I'$. Since I' is an L-ideal, $f(x \vee 1) = f(x) \vee f(1) = f(x) \vee 1 \in I'$. Thus $x \vee 1 \in f^{-1}(I')$, hence $f^{-1}(I')$ is an L-ideal of K. #

Let K be a positive lattice ordered semifield and I an L-ideal of K. Then $K/_I$ is a positive ordered semifield.

To prove that $K/_I$ is a lattice, let $x \in K$. If $x = 0$ then $xI \vee I = I$ exists. Suppose that $x \neq 0$. Claim that $xI \vee I = (x \vee 1)I$. Choose $a \in xI$ and $b \in I$. Then $a = xi$ for some $i \in I$. Since $ix(bx)^{-1} = ib^{-1} \in I$, by Corollary 3.20, $(a \vee b)[b(x \vee 1)]^{-1} = (ix \vee b)(bx \vee b)^{-1} \in I$. Since $b \in I$, $(a \vee b)(x \vee 1)^{-1} \in I$. Hence $(a \vee b)I = (x \vee 1)I$. Therefore $\vee$ is well-defined. Clearly, $(x \vee 1)I$ is an upper bound of xI and I. Let $\alpha \in K/_I$ be such that $xI, I \leq \alpha$. Then there exist $i, j \in I$ and $a, b \in \alpha$ such that $xi \leq a$ and $j \leq b$. Then $(i \wedge j)(x \vee 1) = (i \wedge j)x \vee (i \wedge j) \leq ix \vee j \leq a \vee b = a(1 \vee ba^{-1})$. Since $ba^{-1} \in I$, $ba^{-1} \vee 1 \in I$. This prove that $(x \vee 1)I \leq aI = \alpha$. Hence $xI \vee I = (x \vee 1)I$. By Proposition 3.5. (2), $K/_I$ is a positive lattice ordered semifield. #

Note that the projection map Π defined by $\Pi(x) = xI$ for all $x \in K$ is an L-epimorphisdm of K onto $K/_I$.

<u>Theorem 3.27.</u> (First Isomorphism Theorem)

Let $f : K \to M$ be an L-epimorphism of positive lattice ordered semifields. Then $K/_{\ker f} \cong_L M$.

<u>Proof</u> By Theorem 2.20., we have that φ is an epimorphism. To show that φ is a lattice epimorphism, let $x, y \in K$. Then $\varphi(x\ker f \vee y\ker f) = \varphi((x \vee y)\ker f) = f(x \vee y) = f(x) \vee f(y) = \varphi(x\ker f) \vee \varphi(y\ker f)$. #

Lemma 3.28. Let H a subsemifield of a positive lattice ordered semifield K, and I an L-ideal of K. Then $H \cap I$ is an L-ideal of H and HI is subsemifield of K.

Proof This proof is similar to the proof of Lemma 2.21.. #

Theorem 3.29. (Second Isomorphism Theorem) Let H be a subsemifield of a positive lattice ordered semifield K. Let I be an L-ideal of K such that $P_{HI} \subseteq P_H$. Then $H/_{H \cap I} \cong_L HI/_I$.

Proof This proof is similar to the proof of Lemma 2.22.. #

Lemma 3.30. Let D and I be L-ideals of a positive lattice ordered semifield K such that $I \subseteq D$. Then $D/_I$ is an L-ideal of $K/_I$.

Proof This proof is similar to the proof of Lemma 2.23.. #

Theorem 3.31. (Third Isomorphism Theorem)

Let K be a positive lattice ordered semifield, D and I are L-ideals of K such that $I \subseteq D$. Then $(K/_I)/_{(D/_I)} \cong_L K/_D$.

Proof This proof is similar to the proof of Theorem 2.24.. #

Proposition 3.32. Let $f : K \to M$ be an L-epimorphism of a positive lattice ordered semifields. If I' is an L-ideal of M then $K/_{f^{-1}(I')} \cong_L M/_{I'}$.

Proof This proof is similar to the proof of Theorem 2.25.. #

Proposition 3.33. ([3]) Let $\{K_i \mid i \in I\}$ be a family of positive ordered semifields. Then $\prod_{i \in I} K_i$ is a lattice iff K_i is a lattice for all $i \in I$.

Proof See [3], pp. 46.. #

Definition 3.34. Let K be a positive lattice ordered semifield. A congruence ρ on K is said to be an L-congruence for every $x, y, z \in K$, $x \rho y$ implies that $(x \vee z) \rho (y \vee z)$.

Remark 3.25 Let K be a positive lattice ordered semifield.

1) for every $x, y \in K^*$, $x \rho y$ implies $x^{-1} \rho y^{-1}$.

2) for every $x, y, z \in K$, $(x \wedge z) \rho (y \wedge z)$.

Examples 3.35. (1) Every positive lattice ordered semifield has the trivial L-congruence , that is for every $x, y \in K$, $x \rho y$ iff $x = y$.

(2) Let I be an L-ideal of positive lattice ordered semifield K. Define a relation ρ_I on K by $x \rho_I y$ iff $xy^{-1} \in I$ or $x = y = 0$ for all $x, y \in K$. Then we have that ρ_I is a congruence on K. We must show that $(x \vee z) \rho_I (y \vee z)$ and $(x \wedge z) \rho_I (y \wedge z)$ for all $x, y, z \in K$.
Let $x, y, z \in K$ be such that $x \rho_I y$. If $x = 0$ then done. Suppose that $x \neq 0$. Then $xy^{-1} \in I$. By Corollary 3.20., $(x \vee z)(y \vee z)^{-1}$, $(x \wedge z)(y \wedge z)^{-1} \in I$. Hence $(x \vee z) \rho_I (y \vee z)$ and $(x \wedge z) \rho_I (y \wedge z)$. Therefore ρ_I is an L-congruence of K induced by I.

Note that I is an equivalence class of K/ρ_I and ρ_I is a unique L-congruence on K such that $I \in K/\rho_I$.

To prove uniqueness, let ρ^* be an L-congruence on K such that $I \in K/\rho^*$. Let $x, y \in K$ be such that $x \rho^* y$. If $x = 0$ then done. Suppose that $x \neq 0$. Then $yx^{-1} \in I$. Since $1 \in I$, $yx^{-1} \in I$. Thus $x \, \rho_I \, y$. Thus $\rho^* \subseteq \rho_I$. Obviously, $\rho_I \subseteq \rho^*$. Therefore $\rho_I = \rho^*$. #

Let ρ be an L-congruence on a positive lattice ordered semifield K. Let $I_\rho = \{ x \in K \mid x \rho 1\}$.Then we have that I_ρ is a convex subgroup of K. Let $x \in I_\rho$. Then $x \rho 1$, so $(x \vee 1) \rho (1 \vee 1) = 1$. Thus $x \vee 1 \in I_\rho$, so I_ρ is an L-ideal of K.

Proposition 3.36. Let K be a positive lattice ordered semifield, $\mathscr{I}$ the set of all L-ideals of K and $\mathscr{C}$ the set of all L-congruences on K. Then there exists an order isomorphism from $\mathscr{I}$ onto $\mathscr{C}$.

Proof Similar to the proof of Proposition 1.41.. #

Let K be a positive lattice ordered semifield and I an L-ideal of K. Then $P \cap I$ is a convex subsemigroup of P.

Let S be a convex subsemigroup of P containing S. Let $< S >$ be a multiplicative subgroup of K generated by S. Then $< S >$ is an L-ideal of K.

To prove this, let $x, y \in < S >$. Then $x = st^{-1}$ and $y = uv^{-1}$ for some $s, t, u, v \in S$. Let $z \in K$ be such that $x \leq z \leq y$. Then $st^{-1} \leq z \leq uv^{-1}$, $sv \leq ztv \leq ut$. By the o-convexity of S, $ztv \in S$. So $z = ztv(tv)^{-1} \in < S >$, hence $< S >$ is o-convex.

Let $a, b \in K$ be such that $a + b = 1$. By the a-convexity of S, $asv + but \in S$. Then $ax + by = a(st^{-1}) + b(uv^{-1}) = (asv + but)(vt)^{-1} \in < S >$. Hence $< S >$ is a-convex.

Next, to show that $< S >$ is lattice, let $x \in < S >$, Then there exist $s, t \in S$ such that $x = st^{-1}$ and $s \wedge t = 1$. Thus $x \vee 1 = (st^{-1}) \vee 1 = (s \vee t)t^{-1} = [st(s \wedge y)]t^{-1} = s(s \wedge t)^{-1} = s \in S$. Therefore $< S >$ is an L-ideal of K.

Proposition 3.37. Let K be a positive lattice order semifield. Let $\mathcal{A}$ be the set of all L-ideals of K and $\mathcal{B}$ the set of all a-convex subsemigroups of P containing 1. Then there exists a bijection from $\mathcal{A}$ onto $\mathcal{B}$.

Proof Define $\varphi : \mathcal{A} \to \mathcal{B}$ by $\varphi(I) = I \cap P$ for all $I \in \mathcal{A}$ and define $\Psi : \mathcal{B} \to \mathcal{A}$ by $\Psi(S) = < S >$ for all $S \in \mathcal{B}$. To show that $\varphi \circ \Psi = Id_{\mathcal{B}}$ and $\Psi \circ \varphi = Id_{\mathcal{A}}$, let $S \in \mathcal{B}$. Then $\varphi \circ \Psi(S) = \varphi(\Psi(S)) = \varphi(< S >) = < S > \cap P$. Since $S \subseteq P$, $S \subseteq < S > \cap P$. Let $x \in < S > \cap P$. Then $x \geq 1$ and $x = ab^{-1}$ for some $a, b \in S$ and $a \wedge b = 1$. So $x = x \vee 1 = (ab^{-1}) \vee 1 = (a \vee b)b^{-1} = a \in S$ since $a \vee b = ab$. Hence $< S > \cap P \subseteq S$, so $< S > \cap P = S$. Next, let $I \in \mathcal{A}$. $\Psi \circ \varphi(I) = \Psi(\varphi(I)) = \Psi(I \cap P) = < I \cap P >$. Clear that $< I \cap P > \subseteq I$. Let $x \in I$. Then $(x \wedge 1)^{-1}$, $x \vee 1 \in I \cap P$. Thus $x = (x \vee 1)(x \wedge 1) \in < I \cap P >$. Therefore so $I \subseteq < I \cap P >$. Hence $I = < I \cap P >$. Therefore φ is a bijection. #