CHAPTER I

PRELIMINARIES

Let S be a semigroup. An element a of S is said to be *regular* if $a = aba$ for some $b \in S$. If every element of S is regular, then S is said to be a *regular semigroup.*

For any set A, we let 1_A denote the identity map on A. If $\alpha : A \to B$, then for $C \subseteq A$, let $\alpha|_C$ denote the restriction of α to C.

Let X be a set. A *partial transformation* of X is a map from a subset of X into X. The *empty transformation* of X is the partial transformation with empty domain and it is denoted by 0. For a partial transformation α of X, let $\Delta\alpha$ and $\nabla\alpha$ denote the domain and the range of α, respectively. Let $PT(X)$ be the set of all partial transformations of X. For $\alpha, \beta \in PT(X)$, define the product $\alpha\beta$ as follows: If $\nabla\alpha \cap \Delta\beta = \phi$, let $\alpha\beta = 0$. If $\nabla\alpha \cap \Delta\beta \neq \phi$, let $\alpha\beta$ be the composition of the maps $\alpha|_{(\nabla\alpha \cap \Delta\beta)\alpha^{-1}}$ and $\beta|_{\nabla\alpha \cap \Delta\beta}$. Then $PT(X)$ is a semigroup having 0 and 1_X as its zero and identity, respectively and for $\alpha, \beta \in PT(X)$, $\Delta\alpha\beta = (\nabla\alpha \cap \Delta\beta)\alpha^{-1} \subseteq \Delta\alpha$, $\nabla\alpha\beta = (\nabla\alpha \cap \Delta\beta)\beta \subseteq \nabla\beta$. The semigroup $PT(X)$ is called the *partial transformation semigroup* on X.

By a *transformation semigroup* on X, we mean a subsemigroup of $PT(X)$.

By a *transformation* of X, we mean a map of X into itself. Let $T(X)$ be a set of all transformations of X, that is,

$$T(X) = \{\alpha \in PT(X) \mid \Delta\alpha = X.\}.$$

Then $T(X)$ is a subsemigroup of $PT(X)$ containing 1_X and it is called the *full transformation semigroup* on X.

Let $I(X)$ be the set of all 1-1 partial transformations of X, that is,

$$I(X) = \{\alpha \in PT(X) \mid \alpha \text{ is 1-1.}\}.$$

Then $I(X)$ is a subsemigroup of $PT(X)$ containing 0 and 1_X, and it is called the *1-1 partial transformation semigroup* or the *symmetric inverse semigroup* on X. We have that for $\alpha \in I(X)$, $\alpha = \alpha\alpha^{-1}\alpha$, $\alpha^{-1} = \alpha^{-1}\alpha\alpha^{-1}$, $\Delta\alpha^{-1} = \nabla\alpha$, $\nabla\alpha^{-1} = \Delta\alpha$,

$\alpha\alpha^{-1} = 1_{\Delta\alpha}$ and $\alpha^{-1}\alpha = 1_{\nabla\alpha}$.

It is well-known that $PT(X)$, $T(X)$ and $I(X)$ are all regular.

The *shift* of $\alpha \in PT(X)$ is defined to be the set $\{x \in \Delta\alpha \mid x\alpha \neq x\}$ and it is denoted by $s(\alpha)$. For $\alpha \in PT(X)$, α is said to be *almost identical* if $s(\alpha)$ is finite.

From the definition of shift, the following two propositions are obtained and they will be used later. The first one clearly holds.

Proposition 1.1. *Let X be a set. Then the following statements hold.*

(1) For $\alpha \in PT(X)$, $\Delta\alpha - \nabla\alpha \subseteq s(\alpha)$.

(2) For $\alpha \in PT(X)$, if $s(\alpha)$ is finite, then for each $x \in \nabla\alpha$, $x\alpha^{-1}$ is finite and $\{x \in \nabla\alpha \mid x\alpha^{-1} \neq \{x\}\}$ is finite.

Proposition 1.2. *Let X be a set. Then the following statements hold.*

(1) If $\alpha \in PT(X)$, then for every $a \in \nabla\alpha - s(\alpha)\alpha$, $a\alpha^{-1} = \{a\}$.

(2) If $\alpha \in I(X)$, then $s(\alpha^{-1}) = s(\alpha)\alpha$.

Proof. (1) Let $\alpha \in PT(X)$ and $a \in \nabla\alpha - s(\alpha)\alpha$. Let $x \in a\alpha^{-1}$. Then $x\alpha = a$. Since $a \notin s(\alpha)\alpha$, $x \notin s(\alpha)$. Therefore $x\alpha = x$ which implies that $x = a$. This proves that $a\alpha^{-1} \subseteq \{a\}$. Since $a \in \nabla\alpha$, $a\alpha^{-1} \neq \phi$. It follows that $a\alpha^{-1} = \{a\}$.

(2) Let $\alpha \in I(X)$. From (1) we have that $a\alpha^{-1} = a$ for all $a \in \nabla\alpha - s(\alpha)\alpha$. This implies that $s(\alpha^{-1}) \subseteq s(\alpha)\alpha$ since $\Delta\alpha^{-1} = \nabla\alpha$. This proves that $s(\beta^{-1}) \subseteq s(\beta)\beta$ for all $\beta \in I(X)$. Then $s(\alpha) = s((\alpha^{-1})^{-1}) \subseteq s(\alpha^{-1})\alpha^{-1}$, so $s(\alpha)\alpha \subseteq s(\alpha^{-1})\alpha^{-1}\alpha = s(\alpha^{-1})$ since $\alpha^{-1}\alpha = 1_{\nabla\alpha}$. □

Let X be a set and let

$U(X)$ = the set of all almost identical partial transformations of X,

$V(X)$ = the set of all almost identical transformations of X

and

$W(X)$ = the set of all almost identical 1-1 partial transformations of X.

Then

$$U(X) = \{\alpha \in PT(X) \mid s(\alpha) \text{ is finite.}\},$$
$$V(X) = \{\alpha \in T(X) \mid s(\alpha) \text{ is finite.}\}$$

and

$$W(X) = \{\alpha \in I(X) \mid s(\alpha) \text{ is finite.}\}.$$

We have that $0,\ 1_X \in U(X)$, $1_X \in V(X)$ and $0,\ 1_X \in W(X)$. In fact, $U(X)$, $V(X)$ and $W(X)$ are regular semigroups.

The following notations will be used.

$\mathbf{R}$ = the set of real numbers,

$\mathbf{R}^+$ = the set of positive real numbers,

$\mathbf{Z}$ = the set of integers

$\mathbf{N}$ = the set of positive integers and

$\mathbf{Z}^-$ = the set of negative integers.

In this research, the partial order on any subset of $\mathbf{R}$ always mean the natural partial order on $\mathbf{R}$ if we do not define a particular partial order for it.

Let X and Y be partially ordered sets. A map $\varphi : X \to Y$ is said to be *order-preserving* if for all $a,\ b \in X$, $a \le b$ in X implies $a\varphi \le b\varphi$ in Y. We call a map φ an *order-isomorphism* from X onto Y if φ is a bijection from X onto Y and φ and φ^{-1} are order-preserving. X and Y are said to be *order-isomorphic* if there exists an order-isomorphism from X onto Y. Then the following statements hold.

(1) X is a finite chain if and only if X is order-isomorphic to $\{1, 2, 3, \ldots, n\}$ for some positive integers n.

(2) X is order-isomorphic to a subset of $\mathbf{Z}$ if and only if X is one of the four following forms:

$\{x_1, x_2, x_3, \ldots, x_n\}$ where $n \in \mathbf{N}$ and $x_1 < x_2 < x_3 < \ldots < x_n$,

$\{x_i \mid i \in \mathbf{N}\}$ where $x_i < x_j$ if $i < j$,

$\{x_i \mid i \in \mathbf{Z}^-\}$ where $x_i < x_j$ if $i < j$

and

$\{x_i \mid i \in \mathbf{Z}\}$ where $x_i < x_j$ if $i < j$.

Let X be a partially ordered set. An element a of X is said to be an *isolated point* if for every $x \in X$, $x \leq a$ or $x \geq a$ implies $x = a$. X is said to be *isolated* if every point of X is isolated. For $A \subseteq X$, let $inf(A)$, $sup(A)$, $min(A)$ and $max(A)$ denote the infimum, the supremum, the minimum element and maximum element of A, respectively if they exist. By a *subposet* of X, we mean a partially ordered set Y such that $Y \subseteq X$ and for a, $b \in Y$, $a \leq b$ in Y if and only if $a \leq b$ in X. By a *chain* of X we mean a subposet of X which is a chain. For $a, b \in X$, let $a \leq^* b$ denote $a \leq b$ or $b \leq a$. X is said to be *connected* if for all a, $b \in X$, there exist $x_1, x_2, x_3, \ldots, x_n \in X$ such that $a \leq^* x_1 \leq^* x_2 \leq^* x_3 \leq^* \ldots \leq^* x_n \leq^* b$. By a *component* of X, we mean a maximal connected subposet of X.

A transformation semigroup on X is said to be *order - preserving* if all of its elements are order - preserving. For a transformation semigroup $S(X)$ on X, let

$$S_{OP}(X) = \{\alpha \in S(X) \mid \alpha \text{ is order - preserving.}\}$$

which is a subsemigroup of $S(X)$ if it is nonempty. Then $PT_{OP}(X)$, $T_{OP}(X)$, $I_{OP}(X)$, $U_{OP}(X)$, $V_{OP}(X)$ and $W_{OP}(X)$ are subsemigroups of $PT(X)$, $T(X)$, $I(X)$, $U(X)$, $V(X)$ and $W(X)$, respectively and

$$U_{OP}(X) = \{\alpha \in PT_{OP}(X) \mid s(\alpha) \text{ is finite.}\},$$

$$V_{OP}(X) = \{\alpha \in T_{OP}(X) \mid s(\alpha) \text{ is finite.}\}$$

and

$$W_{OP}(X) = \{\alpha \in I_{OP}(X) \mid s(\alpha) \text{ is finite.}\}.$$

In general, $PT_{OP}(X)$, $T_{OP}(X)$, $I_{OP}(X)$, $U_{OP}(X)$, $V_{OP}(X)$ and $W_{OP}(X)$ need not be regular. It will be shown in Chapter II and Chapter III.

The following statements are known.

Proposition 1.3 ([3]). *If X is a finite chain, then $T_{OP}(X)$ is regular.*

Proposition 1.4 ([1]). *If X is a finite chain, then $I_{OP}(X)$ is regular.*

Let $(X, \leq)$ be a partially ordered set. Define the partial order $\leq_{opp}$ on X as follows: For $x, y \in X$,

$$x \leq_{opp} y \text{ if and only if } y \leq x.$$

We call $\leq_{opp}$ the *opposite partial order* of $\leq$. Then we have that for $\alpha \in PT(X)$, α is order – preserving with respect to $\leq$ if and only if α is order – preserving with respect to $\leq_{opp}$. Hence following propositions hold.

Proposition 1.5. *Let $(X, \leq)$ be a partially ordered set and $\leq_{opp}$ the opposite partial order of $\leq$*

1. $PT_{OP}((X, \leq)) = PT_{OP}((X, \leq_{opp}))$ and $PT_{OP}((X, \leq))$ is regular if and only if $PT_{OP}((X, \leq_{opp}))$ is regular.

2. $T_{OP}((X, \leq)) = T_{OP}((X, \leq_{opp}))$ and $T_{OP}((X, \leq))$ is regular if and only if $T_{OP}((X, \leq_{opp}))$ is regular.

3. $I_{OP}((X, \leq)) = I_{OP}((X, \leq_{opp}))$ and $I_{OP}((X, \leq))$ is regular if and only if $I_{OP}((X, \leq_{opp}))$ is regular.

4. $U_{OP}((X, \leq)) = U_{OP}((X, \leq_{opp}))$ and $U_{OP}((X, \leq))$ is regular if and only if $U_{OP}((X, \leq_{opp}))$ is regular.

5. $V_{OP}((X, \leq)) = V_{OP}((X, \leq_{opp}))$ and $V_{OP}((X, \leq))$ is regular if and only if $V_{OP}((X, \leq_{opp}))$ is regular.

6. $W_{OP}((X, \leq)) = W_{OP}((X, \leq_{opp}))$ and $W_{OP}((X, \leq))$ is regular if and only if $W_{OP}((X, \leq_{opp}))$ is regular.

For any set A, let $|A|$ denote the cardinality of A.