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จุฬาลงกรณ์มหาวิทยาลัย

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A COMPARATIVE ANALYSIS OF CDO PRICING MODELS

Mr. Kridsda Nimmanunta

สถาบันวิทยบริการ
จุฬาลงกรณ์มหาวิทยาลัย

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วิทยานิพนธ์ ฉบับ นี้ ทำ การ ศึกษา การ ประเมิน มูลค่า และ การ วัด ความ เสี่ยง ของ ตราสารอนุพันธ์ความน่าเชื่อถือซีดีโอโดยประยุกต์ใช้การกระจายและการดำเนินแบบเลวี รายงานฉบับนี้นำเสนอขั้นตอนการประเมินมูลค่าซึ่งอาศัยการกระจายแบบไมคซ์เนอร์ ซึ่งเป็นกระจายเลวีชนิดหนึ่ง เพื่อสร้างโครงสร้างความเกี่ยวเนื่องแบบอสมมาตรด้วยเหตุผลที่ว่า การกระจายแบบไมคซ์เนอร์มีคุณสมบัติเหมาะสม กล่าวคือ มีความหนาของหางและมีการเบ้ของกระจาย และการดำเนินเป็นไปอย่างกระโดด ซึ่งสอดคล้องกับที่พบจริงในการกระจายของอัตราผลตอบแทนและการเปลี่ยนแปลงของมูลค่ากิจการ นอกจากนี้ยังง่ายต่อการนำไปใช้งานจริงเมื่อเทียบกับการกระจายแบบเลวีอื่นๆ ไมคซ์เนอร์สามารถนำไปประยุกต์ใช้ได้กับวิธีการแบบคอปูลาและวิธีการแบบโครงสร้างในการประเมินมูลค่าของซีดีโอ เมื่อนำข้อมูลการซื้อขายของซีดีโอมาจากตลาดอนุพันธ์ของสหรัฐฯ ซึ่งมีชื่อว่า CDX NA IG มาทดสอบ ประสิทธิภาพของแบบจำลองที่นำเสนอโดยใช้ซีเทสต์แบบคู่เปรียบเทียบกับแบบจำลองมาตรฐาน อาทิ เช่น แบบจำลองเกาส์เซียนคอปูลา แบบจำลองดดับเบิลทีคอปูลา และ แบบจำลองโครงสร้างบราวเนียนโมชันแบบสัมพันธ์ ผลการศึกษาพบว่าแบบจำลองที่ใช้การกระจายแบบไมคซ์เนอร์มีประสิทธิภาพในการประเมินมูลค่าซีดีโอมากกว่าแบบจำลองอื่นๆเมื่อพิจารณาความผิดพลาดสัมบูรณ์เฉลี่ย นอกจากนี้มาตรวัดความเสี่ยงต่างๆยังถูกวิเคราะห์และแสดงอีกด้วย

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ภาควิชา	การธนาคารและการเงิน	ลายมือชื่อนิสิต ....
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KRIDSDA NIMMANUNTA : A COMPARATIVE ANALYSIS OF CDO PRICING MODELS. THESIS ADVISOR : ANANT CHIARAWONGSE, PH.D., AND ASSOC. PROF. SUNTI TIRAPAT, PH.D., 74 pp.

This study concerns pricing and risk measuring of Collateralized Debt Obligations (CDOs) by using a Lévy distribution/process. The paper provides a framework to price CDOs using an asymmetric depencence structure based on the Meixner distribution since it posesses desirable properties such as fat-tail, skewness, and jump component. Moreover, it is relatively simple to implement comparing to other Levy processes. It is shown that the Meixner distribution can be applied to both copula and structural form approaches. Using the prices of CDOs on the CDX NA IG, the performances of the proposed models are examined and compared to those of standard models such as Gaussian copula model, double-t copula model, and correlated Brownian motion structural model. It is found that the Meixner-based models have the edge over the standard models in all cases in terms of the mean absolute pricing errors (MAPEs). Using the paired Z-test also confirms that the proposed models seem to outperform the standard ones. Additionally, the risk measures of the models are examined.

สถาบันวิทยบริการ
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CHAPTER I

INTRODUCTION

1.1 Background and Problem Review

Collateralized Debt Obligation (CDO) is a type of credit derivatives written on a portfolio of defaultable instruments such as bonds, loans, credit default swaps (CDSs). It is divided into a number of tranches defined by a set of lower and upper bounds called attachment points and detachment points. When some firms in the portfolio defaults, the investors holding CDO tranches are responsible for only the loss incurs in between the attachment and detachment points of their holding tranches. For example, an investor holding an equity CDO tranche must be responsible for the portfolio default loss that incurs in between 0% and 3% of the whole portfolio value. Because of the non-existence of free lunch, the investors get compensated with periodic fees, called premium, until the expiry or the notional principal of that tranche reaches zero.

CDOs have been growing rapidly in the industry for recent years. "The market size for CDOs was estimated to be $2 trillion by the end of 2006 which covers around 40% of the total credit derivative market." (Hull et al.; 2005) Banks use this kind of financial instruments to mitigate their exposure toward credit risk by transferring some parts of the risk to others. Because of its popularity, pricing and risk management for CDOs are essential for market participants. However, the industry's standard model that is widely used to price CDOs is still questionable. It cannot capture the premium structure of CDO tranches well enough. Hence, many researchers attempt to extend this standard model in various ways.

In general, there are two major approaches to model credit risk. The first one is called reduced-form model; for example, Duffie and Singleton (1999); Jarrow and Turnball (1995). The other is called structural model such as Merton (1974); Black and Cox (1976). There are a lot of controversial toward which model is better. Some shows that reduced-form and structural models are almost the same except information perceived. (Arora et al.; 2005; Jarrow and Protter; 2004)

CDOs require the model of credit portfolios, which can be modeled directly or indirectly. Top-down approaches model the portfolio losses directly and find the marginal entities that are consistent if necessary. On the other hand, bottom-up approaches model each individual entity and combined together to construct the joint default risk. In this paper, we focus on the later approaches.

In bottom-up approaches, the marginals can be joined via a dependence function called copula or via a factor model. In most copula models, the marginal default probabilities are estimated via a reduced-form model and then are combined using a copula to derive the portfolio loss distributions, although the marginals from structural model can be used. By contrast, most correlated structural models are based on Monte Carlo by which the paths of each firm value are simulated by assuming that they have some correlations in their values. When the firm value falls below some thresholds, the firm defaults. In some cases, the analytical solution for the structural models is available. These two approaches are used to construct the portfolio loss distributions and hence the fair premiums of CDOs.

Both copula models and correlated structural models are based on many assumptions that researchers attempt to relax to improve the performance of the models. On the one hand, one of the most imperative assumptions for copula models is the choice of copulas. In addition to Gaussian copula which is the standard one, there are Student-t, double-t, Clayton, Marshall-Olkin, and Lévy copulas. On the other hand, one of the most important assumptions for structural models is the choice of firm value processes. Not to mention the geometric Brownian motion, or generally Wiener processes, which is the standard one, there are a lot of Lévy processes that can be applied to structural models.

Lévy distributions/processes provide more flexibility to the models. Most of the Lévy distribution's properties are that they can have skewness and excess kurtosis, known as fat-tails. Therefore, when applying to copula, the Lévy copula can capture not only the symmetric co-dependence structure but also the asymmetric e.g. left heavy tail. When considering its associated process, Lévy process has three components i.e. deterministic, Brownian, Lévy measure which can be interpreted as a mixture of drift, diffusion, and jump parts. Meanwhile, the Wiener process has only drift and diffusion parts. See Schoutens (2003) for Lévy distributions/processes and its applications in finance.

As well as pricing, risk management is imperative. Investors need to be aware of both the expected credit loss that can incur and the potential losses in extreme cases. In general, Value-at-Risk (VaR), the maximum potential loss in a period at a confident level, is the most popular tool for this affair; nevertheless, VaR, or credit VaR in this setting, fails to apply in CDOs. The expected shortfall, known as conditional VaR which is the expected loss given an occurrence of extreme case is used in this paper. It is interesting to note that these risk measures depend on the loss distribution assumptions. Calculating risk measures with the loss distribution implied from market quotes is believed to provide more sensible risk measures. In this case, Lévy distributions/processes can handle this issue.

There are many choices of Lévy distributions[1] and Lévy processes to be applied such as Compound Poisson, Normal Inverse Gaussian, Variance Gamma, CGMY, Generalized Hyperbolic, Meixner, Generalized Z and etc. In jump-diffusion (Merton; 1976), the diffusion which is of infinite variation represents the very active small moves while the jump of finite activity represents rare large moves. However, both small and large moves can be modeled simultaneously by using infinite activity jump processes. Carr et al. (2002) using an extended CGMY model found the Brownian component in individual stock returns are insignificant and that in stock indices are absent because the diffusions are diversifiable. Moreover, they found the empirical risk-neutral processes derived from option prices lack diffusions too. Hence, they argued that asset returns essentially follow pure jump processes with infinite activity.[2] Although market risk derived from frequent small moves seems to be diversifiable, it may or may not be in credit risk. The reason is that the part of credit risk originated from active small moves may not be diversifiable as in market risk, though it may has little attention from the market when compared with large jumps. In this study, we model asset returns with a pure jump process of infinite activity and infinite variation, called Meixner, to capture frequent small moves, rare large moves, and also very high degree of activity near zero. By using Meixner process in first passage models, it is equivalent to assume that the firm value is driven completely by jumps. When new information arrives, the firm value jumps. Firms can default unexpectedly anytime because the firm values can jump over the barrier suddenly. Another advantage is that Meixner distribution is one of the not many Lévy distributions that its density function is analytically expressible. Usually, only analytical characteristic functions are available and need Fourier Transformation. Hence, it is considered less complicated and more convenient to implement than the

[1]Or in general, infinitely divisible distributions.

[2]Furthermore, the results show that the jump processes are mainly finite variation.

other distributions with the similar features. Another reason we choose Meixner is that it is has not yet been studied in this field before while the alternatives such as Generalized Hyperbolic and their special cases have been investigated by Kalemanova et al. (2005); Moosbrucker (2006b); Brunlid (2006); Baxter (2006).

This study is an exploratory investigation of the pricing models of CDOs. The objective is to provide a framework for incorporating the Meixner distributions into the copula and structural approaches in modeling credit risk. The paper also examines the performance of the Meixner-based models comparing to the traditional ones (Gaussian copula, double-t copula, correlated-Brownian motion structural) in pricing the CDOs. In addition, it compares the performances between copula models and the structural models too. Furthermore, it also investigates the characteristics of the risk measures based on expected discounted loss (EDL), Value-at-Risk (VaR), and expected shortfall (ES) for each model.

1.2 Statement of Problem / Research Questions

1. Are Meixner-based models superior to the traditional ones in pricing CDOs?
2. Which model is the best in pricing CDOs, copula models or correlated structural models?
3. How about the risk measures? Are there any significant differences in the risk measures derived by each model?

1.3 Objective of the Study

- To investigate the incremental performance of the models used Meixner distribution and its associated Lévy process in the extension of the standard models.
- To investigate a number of CDO pricing models and to figure out which model is the best in pricing CDOs.
- To investigate risk measures computed from each of the five models as to whether they are significantly different.

1.4 Scope of the Study

In this study, we investigate prices and risk measures from five CDO pricing models i.e. Gaussian copula (GC), double-t copula (DC), Meixner copula (MC), correlated-Brownian motion structural (BS), and correlated-Meixner structural models (MS). GC, DC, and BS have been proposed by Li (2000), Hull and White (2004), and Hull, Predescu and White (2005), respectively, but the other two (MC and MS), to the best of our knowledge, have not yet been investigated. Furthermore, the risk measures investigated are expected discounted loss (EDL), Value-at-Risk (VaR), and expected shortfall (ES).

Our study is based on the U.S. credit derivative market. Only the synthetic CDOs having the North American investment-graded CDS index (CDX NA IG) as the underlying portfolio are investigated. Although there are many other underlying portfolios such as CDX NA HY (High Yield), iTraxx Asia and iTraxx CJ (Japan), the CDX NA IG tranches seem the most actively traded, while the others are being improved over the last few years.

1.5 Contributions

- Investigate a new alternative distribution/process called Meixner.
  - Extend the Gaussian copula approach to capture tail-dependence using Meixner copula.
  - Extend the standard structural model to capture the fat-tail of credit events using Meixner process.
- Comprehensively inform decision-makers at credit desks and academicians about the empirical performance of a number of CDO pricing models comparing with the industry's standard model.
- Inform and discuss the risk measures derived from the Meixner-based models and other models to risk managers and academicians.

CHAPTER II

LITERATURE REVIEW

2.1 Single-Name Credit Risk Modeling

To model credit risk, there are generally two different types of models: reduced-form models and structural models. For the reduced-form model, we implies the default probability directly from the market quotes of such defaultable securities as bonds or CDSs by modeling a hazard rate function. On the other hand, the structural model estimates the probability of firm default from the likelihood of an event that the firm's asset value falls below a threshold called a default barrier.

2.1.1 Reduced-Form Models

Jarrow and Turnball (1995) assume that the stopping time (default time) follows a Cox process. When the information sets are given, it becomes a Poisson process in which the intensity is equal to hazard rate function. By using this process, the stopping time is unpredictable.

Duffie and Singleton (1999) extend the reduced-form approach by modeling the losses as a fractional reduction in the market value at the time of default. By using this approach, credit securities are valued by discounting the pay-off with the new adjusted short rate that also takes into account the losses in market value at default in the risk neutral settings.

Hull and White (2000) uses the reduced-form model to price credit default swaps. They argue that the ways the two above papers assuming on recovery rate are not consistent with bankruptcy laws in most country. When the default occurs, the face value of the bond plus its accrued interest rate should be claimed.

2.1.2 Structural Models

Merton (1974) views the firm's equity as an European call option written on the firm value with the strike price of the debt. Therefore, the probability of default is the likelihood of the events that the firm value is below the debt level at the maturity. This model is celebrated but is unrealistic. The major flaw is that the firm can default anytime in between a period not just at the end of the considering period.

Black and Cox (1976) extend the Merton's model with the first passage time. By using this model, a firm is considered default immediately when the firm value fall below the threshold. Note also that the default barrier of this model is invent to grow over time.

Geske (1977) extends the Merton's model to risky coupon bonds by viewing it as compound options, whereas Longstaff and Schwartz (1995) assume stochastic interest rate when value risky coupon bonds. Leland and Toft (1996) assume that the firm continuously issues a fixed amount of debt so that the debt structure is stationary; by contrast, Collin-Dufresne and Goldstein (2001) let the debt structure follows the mean-reverting process. For empirical comparison of these models, see Eom et al. (2003).

2.1.3 Reconciliation of These Two Models

Jarrow and Protter (2004) show that the reduced-form and structural model are the same. The only differences are the assumptions on information. The reduced-form model assumes that investors cannot observe some information such as the current level of firm value and the default boundary, while these information are known to firm managers which are used in structural model.

2.2 Multiple-Name Credit Risk Modeling

2.2.1 Earlier Works

Gupton et al. (1997) invented the JPMorgan's CreditMetrics asset correlation approach. In this methodology, the default probabilities are derived from the transition matrix. However, since they also assume that the asset value follows the normal distribution, the probability of credit quality migration is interpreted as the likelihoods that the asset value falls below a threshold. Hence, the probability that the firms will jointly default can be computed by simulating from the multi-variate normal distribution by using the equity correlations as an estimate of the asset correlations.

2.2.2 Copula Models

Li (2000) proposed the default (survival) time copula model which has become the industry's standard model. The default probabilities are implied from the default-able instruments such as bonds, and linked those marginal default probabilities together with the standard normal Gaussian copula. Similar to the CreditMetrics, the equity correlation matrix is used as the Gaussian copula's parameter. This method is equivalent to the simulation method in the CreditMetrics in marginal-probability joining perspective. In spite of the Gaussian copula, other copulas such as t-student and Archimedean have been studied in this framework as in, for example, Galiani (2003). Until now, the copula has been used in the form of a single factor copula as in Laurent and Gregory (2003); Schönbucher (2000); Finger (1999).

Elizalde (2005) explained the model, called Vasicek asymptotic single factor model, proposed by Vasicek (1987, 1991) and Vasicek (2002). It said that under the Vasicek (1987)'s model if the underlying portfolio was assumed to be a homogeneous infinitely large portfolio (LHP), the distribution function of the portfolio loss would have a closed-form formula.

Hull and White (2004) proposed the one-factor double-t copula model to price CDOs and n^{th}-to-default CDSs. They found it could fit the prices reasonably well since it can capture the fat-tail of a pool of credit risk. Instead of assuming some forms of copula and implying the copula parameters, Hull and White (2006b) implied the copula itself from the tranche's quotes. They called it the perfect copula because it can perfectly fit the market quotes. However, this model is suitable for pricing non-standard CDO tranches.

Burtschell et al. (2005) compared Gaussian, stochastic correlation, Student-t, double-t, Clayton and Marshall-Olkin copula models. They found the Student-t and Clayton copula models' results are similar to the Gaussian copula, whereas Marshall-Olkin copula model results in overly fattening of the tail of the loss distributions. The double-t model lies in between and gives a better fit to market quotes, while the stochastic correlation copula model can also achieve a reasonable skew as in the market.

2.2.3 Correlated-Structural Models

For the structural model, Hull, Predescu and White (2005) proposed the structural correlated-Brownian motion model based on the first passage model suggested by Black and Cox (1976). They assume that the firm's asset value followed the geometric Brownian motion, while the Wiener components of each firm are correlated. Moreover, they also extended the model to incorporate the empirical evidences that default correlations are positively dependent on the default rates, and the recovery rates are negatively dependent on the default rates. This study found that the stochastic correlation version of the structural model make the model much improve. The stochastic recovery rate version also has some improvements but less than the stochastic correlation.

2.2.4 Reconciliation of These Two Models

Not only the reduced-form model that can be based on the copula approach, but also the structural models because the survival time can be modeled by both reduced-form and structural models. (see Giesecke; 2004)

Nevertheless, joining the probability of survival time with copula is far from economic support as to which copula should be used. In Appendix D, we show that, under some restrictions, a correlated structural model which is usually based on Monte Carlo can be simplified to a Bernoulli mixture framework like in a one-factor copula model. As a result, a correlated structural model can be used without simulation and provides more economic underpinnings than copula approaches.

2.3 The Emerging of Lévy Models

2.3.1 Copula Models

Kalemanova et al. (2005) are ones who used a Lévy distribution called the Normal Inverse Gaussian (NIG). They followed the large homogeneous portfolio assumptions (LHP) which are much popular in practice. In the study, they extended the Gaussian copula model and the t-student copula model to the Normal Inverse Gaussian copula. The model fitted the market quotes reasonably well.

Albrecher et al. (2006) pointed out that one can use any Lévy processes and their associated distributions such as Shifted Gamma, Shifted Inverse Gaussian, Variance Gamma, Normal Inverse Gaussian, Meixner, and many more for the one-factor copula model. Like Albrecher et al. (2006), Moosbrucker (2006a) showed a number of infinitely divisible processes/distributions can be used in the one-factor copula framework.

Brunlid (2006) compares a number of Hyperbolic Lévy copula models i.e. Normal Inverse Gaussian, Variance Gamma, Skewed Student t copula models. The results suggest that the distribution of credit risk should have negative skewness and high kurtosis.

2.3.2 Correlated Structural Models

Luciano and Schoutens (2005) modeled the correlated default by using a modified structural model. In a single-factor setting, they assumed that the factor component followed the Gamma process, whereas the idiosyncratic part followed the Wiener process. Hence, they can estimate the conditional default probabilities by using the Merton (1974) and Black and Cox (1976) models. Then, they computed the joint unconditional default probability using the Gamma distribution. A Brownian motion with drift time-changed by a Gamma process leads to the Variance Gamma process which is in the Lévy process's family.

Moosbrucker (2006b) investigated a number of CDOs pricing models by extending the structural model of Luciano and Schoutens (2005) by using some variations of the correlated Variance Gamma process. Luciano and Schoutens (2005) used identical Gamma process with independent Brownian motions, while his study examined correlated Gamma processes with independent Brownian motions, identical Gamma processes with dependent Brownian motions, and sum of two independent Variance Gamma processes. In addition, he also examined the pricing of CDOs via the Variance Gamma copula model. By using the structural Variance Gamma model, they found the model is better in fitting the market quotes than the Gaussian and double-t copula. Moreover, their Variance Gamma copula model is the best in his study for fitting the market quotes over the period.

Baxter (2006) studies a number of Lévy structural models such as Catastrophe Gamma, Variance Gamma, Gamma, Brownian Gamma structural models, and many combinations, etc. toward single-name and multi-name credits.

2.4 Risk Measurement

There are not many literatures that study the risk measurement for basket credit portfolio especially CDOs.

Mashal et al. (2003) investigate Value-at-Risk and Expected Shortfall of CDO tranches using Gaussian copula and t copula. The results from different models are substantially different.

Antonov et al. (2005) calculate the portfolio risk measures, VaR and ES, under the saddlepoint approach. They also assess the marginal contributions of individual assets to these two risk measures. In addition, Moosbrucker (2006a) investigates the credit VaR of a credit portfolio under different Lévy copula models.

2.5 Potential Extensions

Schoutens (2002) uses Meixner distribution to fit the daily stock returns. It can fit accurately well. Furthermore, he extended the Black and Scholes (1973) model by using the Meixner process instead of the Brownian motion to price a plain vanilla call option. Then, he put the price derived from the Meixner process to the Black and Scholes (1973) model and back the implied volatility out for different strike prices. The model results in volatility smiles familiar to backing the implied volatility out from the market quotes. This concludes that the Meixner distribution is more realistic than Gaussian distribution.

In this study, we extend the standard models in credit risk field with Meixner distribution/process and calculate their risk measures. The next chapter discusses the models.

CHAPTER III

MODELS

In this section, the models used in this study are discussed. The first one is Gaussian copula model proposed by Li (2000), which has become the industry's standard model. The second model is double-t copula proposed by Hull and White (2004), which has the capability to capture the premium structure via the fat-tails. The third model is called Meixner copula model which we make use of Meixner distribution to relax the assumption of normality in Gaussian copula model. These three models are considered as static models which are not suitable in pricing some instruments such as an option on a CDO tranche.

For the fourth model, we are based on the structural model proposed by Hull, Predescu and White (2005) which has the economic rationale supported and is considered as a dynamic model. By using this model, one can simulate possible paths of each firm value in the future and hence the evolution of the likelihoods of default; ultimately, we can value the instruments that the static models cannot. For the last model, we apply Meixner process to relax the assumption of path continuity to the standard structural model. However, we use an approximation approach for long-maturity series to avoid using Monte Carlo simulation which is cost expensive. This model is called correlated-Meixner structural model, or generally a correlated-Lévy structural model. For more details on Meixner distribution and process, see Appendix B.

In short, we extend Gaussian copula model to Meixner copula model, or generally a Lévy copula model, and extend Hull, Predescu and White (2005)'s structural model by using Meixner process instead of the Brownian motion.

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3.1 CDO Pricing Framework

Credit derivatives have a similar structure to interest rate swaps: both have a floating leg and a fixed leg. The floating leg for credit derivatives, also known as the default leg, represents the amount of losses the investors have to pay when defaults occur; while the fixed leg, also known as the premium leg, represents the amount investors receive periodically[1]. At initiation, a parameter called a spread is chosen so

[1]CDOs have fixed payment dates which are on 20th of March, June, September and December of

that these legs have equal value.

Let t_i denote the time at the end of an arbitrary period i, and $P(t_i)$ the corresponding notional principal of one of the tranches. The value of the default leg at time t_i is the sum of discounted change of the expected notional principal of the tranche from the period before.

$$default\ leg = \sum_{i=1}^{n} \{E[P(t_{i-1})] - E[P(t_i)]\} e^{-rt_i}$$

For the premium leg, the premium paid at time t_i is an annualized fixed proportion s, called spread, to the sum of the expected outstanding notional principal at that particular default time and the accrued premium. The accrued premium arises because the investors need compensation for the protection they provide from the end of the previous period up until the default time. For the purpose of illustration, suppose as in Hull and White (2006b) that the defaults, if any, occur at the middle of each period . The value of the premium leg is the present value of all the cash flows:

$$\begin{aligned} premium\ leg &= s \sum_{i=1}^{n} (t_i - t_{i-1}) E[P(t_i)] e^{-rt_i} \\ &+ s \sum_{i=1}^{n} (\frac{t_i - t_{i-1}}{2}) \{E[P(t_{i-1})] - E[P(t_i)]\} e^{-r(\frac{t_i + t_{i-1}}{2})} \end{aligned}$$

Then, s is chosen to equate the value of the two legs:

$$s = \frac{\sum_{i=1}^{n} \{E[P(t_{i-1})] - E[P(t_i)]\} e^{-rt_i}}{\sum_{i=1}^{n} (t_i - t_{i-1}) E[P(t_i)] e^{-rt_i} + (\frac{t_i - t_{i-1}}{2})(E[P(t_{i-1})] - E[P(t_i)]) e^{-r(\frac{t_i + t_{i-1}}{2})}} \tag{3.1}$$

Now we explain how $E[P(t_i)]$ can be computed. Let $P_j(t_i)$ denote the outstanding notional principal of tranche j at time t_i, while K_{L_j} and K_{U_j} denote the lower and upper loss limits of tranche j, respectively. Let Z_{t_i} stand for the *portfolio loss rate* at time t_i, it represents the percentage of the cumulative loss in the *portfolio* value at time t_i. Given Z_{t_i}, the value of the outstanding notional principal of tranche j is:

$$P_j(t_i; Z_{t_i}) = P_j(0) \times min\left(1, max\left(0, \frac{(K_{U_j} - Z_{t_i})}{(K_{U_j} - K_{L_j})}\right)\right) \tag{3.2}$$

---

each year.

where $P_j(0)$ is the initial notional principal of tranche j.

The expected outstanding notional principal of the tranche j at the future time t_i is then:

$$E[P_j(t_i)] = \int_{z=0}^{z=1} P_j(t_i; z) dF_{t_i}(z) \approx \sum_z P_j(t_i; z) \cdot Pr(Z = z) \qquad (3.3)$$

where F_{t_i} is the cumulative distribution function of the portfolio losses z at time t_i. See Elizalde (2005); Hull and White (2006b); Laurent and Gregory (2003) for more details.

It is apparent that the component that drives the value of CDOs is the distribution of portfolio loss rate. All models follow the above framework, but they differ on how the portfolio loss distributions are modeled. In this study, we will rely on the following assumptions which are common in practice

1. The underlying portfolio is homogeneous, so their recovery rates, marginal default probabilities, and their default correlations are the same among all firms.
2. The default correlations ρ are assumed to be constant over time.
3. The recovery rates R of each firm are all constant at 40%, conforming the research of Varma and Cantor (2004).
4. The weights of firms in the portfolio are equal and are constant over time.

As a consequence, the portfolio loss rate Z_{t_i} at time t_i can be expressed as follows

$$Z_{t_i} = \frac{1}{N} \times (1 - R) \times \sum_{k=1}^{N} 1_{k,t_i,\{default\}} \qquad (3.4)$$

where $1_{k,t_i,\{default\}}$ is the indicator function of the default event of firm k by time t_i and N is the number of firms in the portfolio. The portfolio loss distribution function F_{t_i} can be derived from the distribution function of the number of defaults or the default rate[2].

In the next two sections, we will discuss how the default rate are derived, and

[2]Note that the loss rate and default rate are not the same. The loss rate is the percentage of loss over the notional principal value whereas the default rate is the ratio of the number of defaulted firm to the number of all firms in the portfolio.

hence the portfolio loss distribution, by a single-factor copula and a correlated structural model.

3.2 Copula Models

3.2.1 A One-Factor Copula Model

The first three models are based on a one-factor copula model. see Appendix C for briefly discussion of copula. Now, we explain the general framework for copula approaches. We introduce the model in an independent world and then extend to the more realistic one. Gaussian copula, double-t copula and Meixner copula model are formulated afterward.

Let X be the default rate i.e. $X = \sum_{k=1}^{N} 1_{k,t_i,\{default\}}$. If we assume that the individual default probabilities at time t_i are all p_{t_i} and the defaults occur *independently* among each other, the probability that m out of N firms default will follow a binomial distribution.

$$Pr(X = m) = \begin{pmatrix} N \\ m \end{pmatrix} p_{t_i}^{m}(1 - p_{t_i})^{N-m} \tag{3.5}$$

$$Pr(X \leq m) = \sum_{k=1}^{m} \begin{pmatrix} N \\ k \end{pmatrix} p_{t_i}^{k}(1 - p_{t_i})^{N-k} \tag{3.6}$$

As a result, the portfolio loss distribution becomes

$$F_{t_i}(z) = Pr(X \leq \left\lfloor \frac{zN}{(1-R)} \right\rfloor) = \sum_{k=1}^{\lfloor zN/(1-R) \rfloor} \begin{pmatrix} N \\ k \end{pmatrix} p_{t_i}^{k}(1 - p_{t_i})^{N-k} \tag{3.7}$$

Though the assumption of default independence is extremely unrealistic, we will explain later on how it can work in pricing CDOs.

In the real world, the default is dependent. We model the default dependence via a single-factor model. Let ρ be the degree of dependency. The proxy of the default of firm j, X_j, which can represent the survival time, the first passage time, the firm value, or the asset return, etc., can be decomposed into the common factor M and the unsystematic part Z_j:

$$X_j = \sqrt{\rho}M + \sqrt{1-\rho}Z_j$$

If M is given, each pair of X_j is conditionally independent. If the distribution function of the random variables X_j, M and Z_j are F_{M+Z}, F_M, and F_Z respectively, the probability distribution of the random variable X_j conditioned on M is as follows

$$\begin{aligned} Pr\left(X_j \leq x \middle| M\right) &= Pr\left(\sqrt{\rho}M + \sqrt{1-\rho}Z_j \leq x \middle| M\right) \\ &= Pr\left(Z_j \leq \frac{x-\sqrt{\rho}M}{\sqrt{1-\rho}} \middle| M\right) \\ Pr(X_j \leq x|M) &= F_Z\left(\frac{F_{M+Z}^{-1}\left(Pr(X_j \leq x)\right) - \sqrt{\rho}M}{\sqrt{1-\rho}}\right) \end{aligned}$$

Since the conditional defaults are independent, the margins can be multiplied together forming a copula shown below

$$C(u_1, u_2, \ldots, u_N) = \int_{m=-\infty}^{m=+\infty} \prod_{j=1}^{N} F_Z\left(\frac{F_{M+Z}^{-1}\left(u_j\right) - \sqrt{\rho}m}{\sqrt{1-\rho}}\right) dF_M(m)$$

When F_M and F_Z are the Gaussian distribution, F_{M+Z} also become the Gaussian distribution. The copula becomes the multivariate Gaussian copula as follows

$$C_{\mathbf{a}}^{Ga}(u_1, u_2, \ldots, u_N) = \int_{m=-\infty}^{m=+\infty} \prod_{j=1}^{N} \Phi\left(\frac{\Phi^{-1}\left(u_j\right) - \sqrt{\rho}m}{\sqrt{1-\rho}}\right) d\Phi(m)$$

It is imperative to note that although F_M and F_Z are the same probability distribution, the probability distribution function of the sum of those two random variables F_{M+Z} is not necessary to be that same distribution. For example, if F_M and F_Z are Student-t distributions. F_{M+Z} is not the Student-t distribution since this distribution is not stable under convolution[3]. Hence, the empirical distribution of X_j must be used instead.

Due to the conditional independence in this framework, the equation (3.7) can be used by its nature. Thus, the portfolio loss distribution based on our assumptions

[3]The distribution function of the sum of two random variables is a convolution of their distribution functions.

becomes[4]

$$F_{t_i}(z) = \int_{m=-\infty}^{m=+\infty} \sum_{k=1}^{\lfloor zN/(1-R) \rfloor} \begin{pmatrix} N \\ k \end{pmatrix} p_{t_i|m}^{k} (1 - p_{t_i|m})^{N-k} dF_M(m) \quad (3.8)$$

$$p_{t_i|m} = F_Z \left(\frac{F_{M+Z}^{-1}(p_{t_i}) - \sqrt{\rho} m}{\sqrt{1-\rho}} \right) \quad (3.9)$$

Furthermore, if we assume that the portfolio is infinitely large, a closed-form formula to approximate the price of CDOs exists. See Elizalde (2005) for more details.

3.2.2 Gaussian Copula Model

For modeling the default probability marginally, we follow Li (2000) who proposed the model called the survival time Gaussian copula. A random variable called "time-until-default" is defined as the survival time of each firm before it defaults and the default correlation is defined as the correlation of those survival times. Default events are assumed to follow a Poisson process with a parameter $\lambda(t)$ called a hazard rate function which interpreted as an instantaneous default probability at that time t. The probability that a firm survives beyond time t_i is $q(t_i) = e^{-\int_0^{t_i} \lambda(t) dt}$. If we assume that the hazard rate function is constant, we will get the survival probability $q(t_i) = e^{-\lambda t_i}$ and thus the default probability of a firm in time t_i is

$$p(t_i) = 1 - q(t_i) = 1 - e^{-\lambda t_i} \quad (3.10)$$

Like many studies such as Finger (1999); Schönbucher (2000); Laurent and Gregory (2003), we used one-factor Gaussian copula to price CDOs, thereby specialize the arbitrary distributions to the normal distributions. When the unconditional default probability in the equation (3.9) is substituted with (3.10), the conditional default probability is

$$Pr(\tau \leq t_i | M) = p(t_i | M) = \Phi \left(\frac{\Phi^{-1}\left(1 - e^{-\lambda t_i}\right) - \sqrt{\rho} M}{\sqrt{1-\rho}} \right) \quad (3.11)$$

where $\Phi(.)$ is the standardized normal distribution function. As a result, the portfolio

[4]Note also that we use the quadratures called Gaussian-Hermite and Gaussian-Laguerre for numerical integration.

loss distribution function becomes

$$F_{t_i}(z) = \int_{m=-\infty}^{m=+\infty} \sum_{k=1}^{\lfloor zN/(1-R) \rfloor} \begin{pmatrix} N \\ k \end{pmatrix} p_{t_i|m}^{k} (1 - p_{t_i|m})^{N-k} d\Phi(m) \tag{3.12}$$

Although this model is the industry standard, it cannot capture the capital structure of CDOs well enough. This model needs some extensions. However, this model is the simplest in implementation and fastest in speed of computation.

3.2.3 Double-t Copula Model

Hull and White (2004) proposed this model. In this context, we specialize the arbitrary distributions of M and Z_j to the Student-t distributions, while the others remain the same. In this study, we will find the degree of freedom ν that make the model best fit to all of the market quotes observed. Thus, the model becomes

$$p(t_i|M) = T_{\upsilon} \left(\left(\frac{\upsilon}{\upsilon - 2} \right)^{\frac{1}{2}} \frac{F_{T+T}^{-1}\left(1 - e^{-\lambda t_i}\right) - \sqrt{\rho}\left(\frac{\upsilon-2}{\upsilon}\right)^{\frac{1}{2}} M}{\sqrt{1-\rho}} \right) \tag{3.13}$$

$$F_{t_i}(z) = \int_{m=-\infty}^{m=+\infty} \sum_{k=1}^{\lfloor zN/(1-R) \rfloor} \begin{pmatrix} N \\ k \end{pmatrix} p_{t_i|m}^{k} (1 - p_{t_i|m})^{N-k} dT_{\upsilon}(m) \tag{3.14}$$

where F_{T+T} is the empirical distribution of the sum of two independent Student-t random variables since the sum of them does not distribute as Student-t, whileT_{υ} is a standard Student-t distribution. Note also that we find the empirical distribution by convolution.

The advantage of this model is that it is more consistent with the capital structure of CDO quotes; nonetheless, it is not perfect fit. This model requires a lot of computation time since each time we change the value of correlation in the model, we must perform convolution and inversion.

3.2.4 Meixner Copula Model

For the Meixner copula, we specialize the distributions of M and Z_j to the Meixner distributions, while the others remain the same. Actually, the distributions of M and Z_j are not necessary to be the same. However, if the distribution of M and

Z_j are the same, the distribution of X_j i.e. F_{M+Z} is a Meixner distribution since it is stable under convolution. In this study, we will find the Meixner's parameters α and β that minimize the pricing error among all of the market quotes. Consequently, the portfolio loss distribution function (3.8) and (3.9) becomes

$$p(t_i|M) = \Upsilon_{\alpha,\beta}\left(\frac{\Upsilon^{-1}_{\alpha,\beta}\left(1-e^{-\lambda t_i}\right)-\sqrt{\rho}M}{\sqrt{1-\rho}}\right) \tag{3.15}$$

$$F_{t_i}(z) = \int_{m=-\infty}^{m=+\infty}\sum_{k=1}^{\lfloor zN/(1-R)\rfloor}\left(\begin{array}{c} N \\ k \end{array}\right) p^k_{t_i|m}(1-p_{t_i|m})^{N-k}d\Upsilon_{\alpha,\beta}(m) \tag{3.16}$$

where $\Upsilon_{\alpha,\beta}(.)$ is the zero-mean and unit-variance Meixner distribution function as described in Appendix B.

This model is the second to Gaussian copula model in speed of computation given that we know the optimal parameters. Before the computation starts, two look-up tables are constructed for a cumulative probability function and a inverse function from the analytical probability density function. Although the speed is faster than double-t copula model in run-time, when we consider calibration time, it consumes a lot because the two parameters are real numbers with bounds.

3.3 Correlated Structural Models

In this section, we discuss a generalized correlated structural model. Then, the correlated-Brownian motion structural and the correlated-Meixner structural models are presented. The approximation approach for the correlated structural model is in Appendix D.

3.3.1 A Correlated Structural Model

To begin with, a generalized correlated structural model adapted from the frameworks in Hull, Predescu and White (2005); Schoutens (2002, 2003); Luciano and Schoutens (2005); Black and Cox (1976); Hull and White (2001) is presented. Let consider the structural model in an *independent* world first. Then, we extends to the correlated structural case.

Assuming that the firm's asset value follows an exponential of a Lévy process, under the risk-neutral setting, the asset value follows

$$V_j(t) = V_j(0)exp\Big((\mu_j - \theta_j)t + \mathbb{Z}_j^{(L)}(t)\Big) \tag{3.17}$$

where $V_j(t)$ is the value of firm j at time t
μ_j is the adjusted risk-neutral rate of return
θ_j is the mean-correcting parameter that makes the firm value process becomes a martingale
$\mathbb{Z}_j^{(L)}$ is a Lévy process

By factorizing the scaling factor of Lévy process, the redundant parameters of the model can be reduced. Hence,

$$V_j(t) = V_j(0)exp\Big((\mu_j - \theta_j)t + \alpha X_j^{(L)}(t)\Big)$$

where α is the scaling factor of the process and $X_j^{(L)}(t)$ is the scaled Lévy process of $\mathbb{Z}_j^{(L)}$. When viewing the scaled Lévy component as a proxy of the asset return's process, the equation becomes

$$X_j^{(L)}(t) = \frac{lnV_j(t) - lnV_j(0)}{\alpha} + \frac{-(\mu_j - \theta_j)}{\alpha}t$$

We change the focus to the proxy because it will be easy for simulation.

Based on our assumptions that the firms are homogeneous, the parameters are the same. The equation rewrites as

$$X_j^{(L)}(t) = \frac{lnV_j(t) - lnV(0)}{\alpha} + \frac{-(\mu - \theta)}{\alpha}t \tag{3.18}$$

Also, the default barriers for each firm are the same. According to Black and Cox (1976), the default barrier K can grow over time

$$K(t) = Kexp(\gamma t)$$

where γ is the growth parameter. Hence, the new default barrier that corresponds to the proxy of asset return becomes

$$
\begin{aligned}
K^* &= \frac{lnK(t) - lnV(0)}{\alpha} + \frac{-(\mu-\theta)}{\alpha}t \\
&= \frac{lnK - lnV(0)}{\alpha} + \frac{\gamma-(\mu-\theta)}{\alpha}t \\
&= a + bt
\end{aligned}
$$

where $a = \frac{lnK - lnV(0)}{\alpha}$ and $b = \frac{\gamma-(\mu-\theta)}{\alpha}$. Whenever $X_j^{(L)}(t)$ falls below $a + bt$, the firm j is considered default. Thus, the default time of firm j is

$$\tau_j = \inf_t \left\{ X_j^{(L)}(t) \leq a + bt \right\}$$

When the firm values are dependent, the model need modifications. The proxy of asset return is assumed to follow a correlated Lévy process with the degree of dependency ρ. Based on our assumptions, the formulas become

$$
\begin{aligned}
X_j^{(L)}(t+dt) &= X_j^{(L)}(t) + dX_j^{(L)}(t) \\
dX_j^{(L)}(t) &= \sqrt{\rho}dM(t) + \sqrt{1-\rho}dZ_j(t)
\end{aligned} \tag{3.19}
$$

where M and Z_j are independent Lévy processes which $M(0) = Z_j(0) = 0$ and they have independent and stationary increments. To construct the portfolio loss distribution, the distribution of default rate is converted via (3.4). To construct the default rate distribution, we sample the paths of M and Z_j and calculate $X_j^{(L)}$; whenever the $X_j^{(L)}$ falls below K^*, the firm j is considered default after that time. The default rate is counted to create a frequency table representing the default rate distribution.

Economically, if we model the market factor with heavier-downsided jumps, while the firm specific is heavier-upsided jumps. It is interpreted as the firm values tend to decrease dependently, while they rise independently. However, if we model the market factor with Brownian motion but the firm specific as heavier-downsided jumps, that means the large plunge/soar represents market crash/boom is unlikely to occur, yet a firm can suffer from large loss individually. To sum up, the market factor determine how firm will act interdependently, while the idiosyncratic indicates how firm will act independently.

3.3.2 The Correlated-Brownian Motion Structural Model

For the standard multi-variate structural model, we follow Hull et al. (2005). Thus, in this context, we specialize the correlated Lévy process $X_j^{(L)}$ to a correlated Brownian motion X_j , replace the drift parameter with r_f[5], and specify the mean-correcting parameter θ to $\frac{\sigma^2}{2}$:

$$V_j(t) \;=\; V(0)exp\left((r_f - \frac{\sigma^2}{2})t + \sigma X_j(t)\right) \tag{3.20}$$

where r_f is the risk-free rate, σ is the asset volatility of the homogeneous firm. After some arithmetic transformations,

$$X_j(t) = \frac{lnV_j(t) - lnV(0) - (r_f - \frac{\sigma^2}{2})t}{\sigma}$$

and then the default barrier becomes

$$\begin{aligned} K_i^* &= \frac{lnK_i - lnV(0) - (r_f - \frac{\sigma^2}{2})t}{\sigma} \\ &= \left(\frac{lnK_i - lnV(0)}{\sigma}\right) + \left(-\frac{(r_f - \frac{\sigma^2}{2})}{\sigma}\right)t \\ &= a + bt \end{aligned} \tag{3.21}$$

where $a = \frac{lnK - lnV(0)}{\sigma}$ and $b = -\frac{(r_f - \frac{\sigma^2}{2})}{\sigma}$.

We can use this continuous default barrier; nevertheless, as suggested in Hull et al. (2005); Hull and White (2001), a discrete version of the default barrier is constructed thereafter to reduce the computation time.

3.3.3 The Structural Correlated Meixner Model

Unlike the structural model proposed by Hull, Predescu and White (2005), we assume that the firm's asset value follows a Lévy process called the Meixner process with parameters of α, β, δ. Hence, we specialize the correlated Lévy process $X_j^{(L)}$ to a correlated Meixner process $X_j^{(Meixner)}$, replace the drift parameter with r_f[6], and

[5]Note that if the firm pays dividends, $r_f - q_j$ must be used where q_j is dividend yield of firm j.
[6]Note that if the firm pays dividends, $r_f - q_j$ must be used where q_j is dividend yield of firm j.

specify the mean-correcting parameter θ:

$$V_j(t) = V(0)exp\left((r_f - \theta)t + \alpha X_j^{(Meixner)}\right) \tag{3.22}$$

where $\theta = 2\delta \ln\left(\frac{\cos(\beta/2)}{\cos((\alpha+\beta)/2)}\right)$. The term θ corrects the firm's asset value so that the discounted asset value process become a martingale. However, since it is incomplete model, the market participants may not view as this formula. Indeed, we need no special care of how market participants view because we imply the parameters from CDS quotes. As a consequence, we have

$$X_j^{(Meixner)}(t) = \frac{lnV_j(t) - lnV(0)}{\alpha} + \frac{-(r_f - \theta)}{\alpha}t \tag{3.23}$$

and the default barrier becomes

$$\begin{aligned} K^* &= \frac{lnK(t) - lnV(0)}{\alpha} + \frac{-(r_f - \theta)}{\alpha}t \\ &= \frac{lnK - lnV(0)}{\alpha} + \frac{\gamma - (r_f - \theta)}{\alpha}t \\ &= a + bt \end{aligned}$$

where $a = \frac{lnK - lnV(0)}{\alpha}$ and $b = \frac{\gamma-(r_f-\theta)}{\alpha}$. When all parameters are known, we can price CDOs by sampling the paths of each firm j, $X_j^{(Meixner)}$ by sampling the paths of market factor F and the firm specific Z_j:

$$\begin{aligned} X_i(t + dt) &= X_i(t) + dX_i(t) \\ dX_i(t) &= \sqrt{\rho}dF(t) + \sqrt{1-\rho}dZ_i(t) \end{aligned}$$

where F and Z_j are independent Meixner processes which $F(0) = U_i(0) = 0$ and their increments distribute as Meixner(α_i,β_i,δdt,$-\alpha\delta dt \tan(\beta/2)$).

3.4 Calibration

3.4.1 Copula Models

In copula models, there are generally three components to calibrate the model. The first is a hazard rate function of each firm which can be implied from CDS market. The second are the correlation parameter which will be implied from the equity tranche, and the last are copula's parameters which will be chosen so that the overall mean absolute pricing error are minimum.

3.4.1.1 Implying the hazard rate function

In this study, we assume that the underlying portfolio is homogeneous and finite. Because of homogeneity and equally-weighted in the underlying portfolio, the firms' spreads are all the same and equal to the CDS index spread. Thus, we can view the CDS index as a CDS of a representative company. Assuming the recovery rate to be 40% for all companies and a constant hazard rate, by using a reduced-form model, we can imply the hazard rate and, in turn, finding the default probability on any maturities from the CDS index. To price a CDS, we can use the same concept as in section 3.1 with some modifications. The equation (3.1) will become

$$s = \frac{(1-R)\sum_{i=1}^{n}\{E[P(t_{i-1})] - E[P(t_i)]\}e^{-rt_i}}{\sum_{i=1}^{n}(t_i - t_{i-1})E[P(t_i)]e^{-rt_i} + \sum_{i=1}^{n}(\frac{t_i - t_{i-1}}{2})\{E[P(t_{i-1})] - E[P(t_i)]\}e^{-r(\frac{t_i + t_{i-1}}{2})}} \tag{3.24}$$

where $E[P(t_i)] = e^{-\lambda t_i}$. For illustration, considering a day on 5YS6 series, one can imply the hazard rate λ by setting s with a 5YS6 CDX NA IG quote as the input.

3.4.1.2 Calibrating the correlation parameter

We will imply the correlation parameter from the equity tranche since, as Hull, Predescu and White (2005) described, this tranche is the most sensitive to the correlation parameter. For each day and with the given copula parameters (if any), we will equal the model price and the market price of equity tranche and back the correlation parameter out. After that , we will use this parameter to price the other tranches on that day.

3.4.1.3 Calibrating the copula's parameters

We will choose the copula parameters i.e. v of student t, and α and β of the Meixner distribution so that the mean absolute pricing error across all data observed is minimum.

3.4.2 Correlated Structural Models

To calibrate the correlated structural models, instead of implying the hazard rate, we imply the default barrier in which the firm defaults when the firm value falls below on day-by-day basis. Despite the differences, the correlation parameter and the process's parameters will be estimated similar to the cases of the copula models.

3.4.2.1 Implying the default barriers

Like in the copula models, we view the CDS index as a CDS of a representative company underlying the portfolio. For correlated-BS model, the parameter $a = \frac{lnK - lnV(0)}{\sigma}$ and $b = \frac{\gamma - \left(r_f - \frac{\sigma^2}{2}\right)}{\sigma}$ of the default barrier K^* will be estimated by using 5- and 10- year CDS spread and the equation (3.24) where $E[P(t_i)] = 1 - p(0, t_i)$ in which $p(0, t_i)$ is the probability of default from the first passage time's equation (Harrison; 1990) i.e. $p(0, t_i) = \Phi\left(\frac{a+bt_i}{\sqrt{t_i}}\right) + exp\left(-2ab\right)\Phi\left(\frac{a-bt_i}{\sqrt{t_i}}\right)$. If we need to know the asset volatility σ and the relative difference between the default barrier and the asset value $K/V(0)$, we can calculate them when the USD zero curves and the decay parameter γ are known. However, it is not necessary to further find these values, only the values of a and b are sufficient to price CDOs.

For MS model, the parameters $a = \frac{lnK - lnV(0)}{\alpha}$ and $b = \frac{\gamma - (r_f - \theta)}{\alpha}$ of the default barrier K^*will be estimated similar to correlated-BS model. Unfortunately, we have to estimate two more parameters which are the β and δ of the Meixner process. Moreover, the first passage time distribution of the Meixner process has no closed-form as in the case before; this makes calibration harder. To estimate a and b, we have to assume the value of β and δ first and then simulate until we figure out the barrier parameters that make the CDS prices match the market quotes. Note that in all cases we will use time step $\Delta t = 0.25$.

For illustration, considering a day on series 6 with maturity of 5 and 10 years, we sample the paths that follow Meixner process$(1, \beta, \delta)$ over time horizon of 10 years and then we find the probability of first hitting the barrier $a + bt_i$ by time t_i i.e. $p(0, t_i)$. After that the 5Y and 10Y CDS premium are computed using the equation (3.24); we adjust the barrier parameters (a, b) until these CDS premiums are matched with the market quotes. However for 7YS6, we use the data from 5YS6 and 7YS6 to find the suitable barrier.

3.4.2.2 Calibrating the correlation parameter and the process's parameters

We do the same way as in copula models. However, instead of calibrating copula's parameters, we find the process' parameters β and δ that minimize the overall mean absolute pricing error.

3.5 Risk Measurement

3.5.1 Expected Discount Loss

Expected Discount Loss is the first risk measure we compute. It is not a good risk measure since it provides only the average of the present value of losses that can be incurred. However, investors may want to know how much the expected loss is.

The expected discount loss is equal to the default leg in CDO pricing framework i.e.

$$EDL = default\ leg = \sum_{i=1}^{n} \{E[P(t_{i-1})] - E[P(t_i)]\} e^{-rt_i}$$

where $E[P(t_i)]$ is the expected outstanding principal of a CDO tranche at time t_i. Recall that the default leg is equal to premium leg at initiation.

3.5.2 Value-at-Risk

From Moosbrucker (2006a), the VaR of the portfolio was computed under the assumption of large homogeneous portfolio (LHP). In this study, we compute the VaR of portfolios and each tranche based on the assumption of homogeneous finite portfolio. Suppose L_p is a random variable of the portfolio loss and $P(0)$is the total value of the portfolio. The maximum potential loss of the portfolio at a confidence level $(1-\alpha)\%$, VaR_p can be calculated as follows

$$\begin{aligned} Pr(L_p > VaR_p) &= \alpha \\ F_p(\frac{VaR_p}{P(0)}) &= 1-\alpha \\ VaR_p &= P(0)F_p^{-1}(1-\alpha) \end{aligned}$$

where F_p is the distribution of the portfolio loss rate. Let $P_j(0)$ is the total value of the single tranche j. The credit VaR of the tranche j is

$$\begin{aligned} VaR_j &= P_j(0) \times min\left(1, max\left(0, \frac{K_{U_j} - \frac{VaR_p}{P(0)}}{K_{U_j} - K_{L_j}}\right)\right) \\ VaR_j\,(1-\alpha) &= P_j(0) \times min\left(1, max\left(0, \frac{K_{U_j} - F_p^{-1}\,(1-\alpha)}{K_{U_j} - K_{L_j}}\right)\right) \end{aligned} \quad (3.25)$$

3.5.3 Expected Shortfall

Mashal et al. (2003); Antonov et al. (2005) suggests that the expected shortfall of CDOs can be calculated as:

$$ES_j = \frac{\int_{1-\alpha}^{1} VaR_j(x)dx}{\alpha}$$

where ES_j is the expected shortfall or known as conditional VaR of single tranche j and α is the significant level.

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CHAPTER IV

DATA AND METHODOLOGY

4.1 Data

Synthetic CDOs is a combination of the securitization techniques and credit derivatives. There are many types of CDOs. The Synthetic CDOs is the CDOs written on Credit Default Swaps (CDSs). In this study, the data is based on CDOs written on the CDX NA IG, which is a credit default swap index consisting of the equally-weighted of the most 125 actively-traded CDSs of investment-graded firms in North America. Whenever a firm in the index default, the investor will get compensated and the firm will be removed from the index, thereby reducing the outstanding value. CDX NA IG is divided into five standard tranches: 0-3%, 3-7%, 7-10%, 10-15% and 15-30% which one can buy or sell each tranche separately, called the single tranche trade. The market is made by a global group of broker-dealers. In every six months—on September 20th and March 20th, it will roll over into a new series which represents the firms that currently have highest volumes in their CDS trades.

The historical data used in this study are the series 5 and 6 of CDX NA IG which are issued on September 20, 2005 and March 23, 2006, respectively. For CDX NA IG series 5, we use the daily mid bid-ask spreads of all standard tranches and the index quotes between September 20, 2005 and October 13, 2006, whereas for series 6, the data observed are between March 23, 2006 and October 16, 2006. Not only CDOs with maturity of 5-year but also the CDOs with maturity of 7- and 10-year are investigated. These series are denoted as 5YS5, 7YS5, 10YS5, 5YS6, 7YS6, and 10YS6. Note further that we include only the data from the days in which there were both bid and ask of all tranches quoted. Therefore, the number of observations for 5YS5 is 128, while 113 for 5YS6, 118 for 7YS5, 110 for 7YS6, 115 for 10YS5, and 101 for 10YS6. The CDO data are from Reuters (GFI), while CDS index and USD Zero Curves are downloaded from DataStream. We use the linear interpolation for the interest rates in between the available maturities.

4.2 Research Hypotheses

1. **Meixner copula model is preferable to double-t and Gaussian copula mod-**

els.

This is because the Meixner distribution can have both non-zero skewness and excess kurtosis conforming the real world. Although the double-t model can capture the fat-tail, its distribution has zero skewness and, worse comes to worst, the Gaussian distribution has none of these features.

2. **The correlated-Meixner structural model is preferable to the correlated-GBM structural model.**

   In reality, the asset value does not just drifts and diffuses; it can jump. Relaxing the assumption of normality by using a Lévy process, which the firm value can jump or be driven by jumps, is more sensible.

3. **The copula models and the structural models are approximately equivalent in performance.**

   Some may argue that the reduced-form models, or in this case the copula models, are preferable to the structural models in pricing a defaultable security owing to the information based perspective. (Jarrow and Protter; 2004) Others may, by contrast, argue that some sophisticated structural models are surprisingly better than the reduced-form models, although the basic structural model i.e. Merton (1974) is worse. (Arora, Bohn and Zhu; 2005) In our opinion, the reduced-form model is better in general since they can be calibrated to the market quotes which reflect the credit risk that market participants expect, whereas the structural model based on unobservable firm's asset value. However, in pricing CDOs, we can calibrate the structural model as well by following Hull and White (2001); Hull, Predescu and White (2005). Thus, we anticipate the performance of these two approaches should be approximately the same.

4. **Risk measures derived from each model should be slightly different.**

   However, the EDL, VaR, and ES from the models which are able to capture the fat-tail should be more pessimistic than the one from the Gaussian-based models, especially in case of the high level of confidence.

4.3 Methodology

In this section, we describe the methodology to test our hypotheses. Recall that GC represents the Gaussian copula model, DC is the double-t copula model, MC means the Meixner copula model, BS represents the correlated-Brownian motion structural model, and finally, MS is the correlated-Meixner structural model.

4.3.1 Mean Absolute Pricing Error (MAPE)

MAPE indicates how well a model can capture the tranche's premium structures. The lower the MAPE, the better the model. The MAPE of each model will be calculated from the formula as follows

$$MAPE = \frac{\sum_{i=1}^{n} APE_{t_i}}{n}$$

where $APE_{t_i} = \sum_{tranche2}^{tranche5} |ModelSpread_{t_i} - MarketSpread_{t_i}|$,
n is the number of period until maturity

To make sure that one model significantly outperforms the other, test statistics should be used. According to Houweling and Vorst (2005), a paired Z-test is used to compare the performance of Model A and Model B as follows

$$Z_{A,B} = \sqrt{n}\frac{\bar{d}_{A,B}}{s_{A,B}}$$

where $d_{A,B,t_i} = APE_{A,t_i} - APE_{B,t_i}$,
$\bar{d}_{A,B}$ is the sample mean of d_{A,B,t_i},
$s_{A,B}$ is the sample standard deviation of d_{A,B,t_i},
n is the sample size.

$Z_{A,B}$ has asymptotically the standard normal distribution.

In addition, we use MAPE to measure the performance of each model for each tranche by

$$MAPE_j = \frac{\sum_{i=1}^{n} APE_{j,t_i}}{n}$$

where $APE_{j,t_i} = |ModelSpread_{j,t_i} - MarketSpread_{j,t_i}|$,
n is the number of period until maturity
j is the tranche j

4.3.2 Hypothesis I

Meixner copula model is preferable to double-t and Gaussian copula models. We will investigate two pairs of models i.e. MC vs. GC and MC vs. DC. Thus, the hypotheses are

$$\begin{aligned} H_0 : \quad & MAPE_{GC} \leq MAPE_{MC} \\ H_1 : \quad & MAPE_{GC} > MAPE_{MC} \end{aligned}$$

and

$$\begin{aligned} H_0 : \quad & MAPE_{DC} \leq MAPE_{MC} \\ H_1 : \quad & MAPE_{DC} > MAPE_{MC} \end{aligned}$$

At the first place, we presume that GC is better so that its MAPE is less than the MC's. After that if MC model is better the null hypothesis will be rejected. The same will be applied for the pair of DC and MC.

4.3.3 Hypothesis II

The correlated-Meixner structural model is preferable to the correlated-Brownian motion structural model.

$$\begin{aligned} H_0 : \quad & MAPE_{BS} \leq MAPE_{MS} \\ H_1 : \quad & MAPE_{BS} > MAPE_{MS} \end{aligned}$$

At the first place, we presume that BS is better so that its MAPE is less than MS's. After that if BS model is better the null hypothesis will be rejected.

4.3.4 Hypothesis III

The copula models and the structural models are equivalent in performance. We will investigate the MAPEs of the structural models and the copula models whether which one is better.

4.3.5 Hypothesis IV

Risk measures of each CDO from each model should be slightly difference or roughly the same. We will investigate the results from each model whether they are much different or not.

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CHAPTER V

RESULTS

In this chapter, we discuss the results and the interpretations of the results. First, we investigate the fitting performance of each model via MAPEs. Among the copula models, which one is the best?, and among the two correlated structural models, which one is the better?. After that, we discuss the results as to which approach between the copula model and the correlated structural model is better. Next, we discuss the three risk measures computed from each model.

According to our implementation, the degree of freedom of double-t copula model that minimizes MAPE is 3 for all time series except 10YS5 and 10YS6 which are 6. For Meixner copula model, the optimal parameters (α, β) are vary among the time series; (2.6753,-0.70036) for 5YS6, (1.3724,-1.8162) for 7YS6, (0.013162,-3.0949) for 10YS6, and others shown in Table 5.1. Also, the optimal parameters for the correlated Meixner structural model, (β, δ) are shown in Table 5.1; for example, (-0.6435, 0.0400) for 5YS6, (-0.8261, 0.0280) for 7YS6, and (-1.3694, 0.0400) for 10YS6. Notice that the βs of MC and MS are all negative. Note that some of the correlated Meixner structural's parameters i.e. 10YS5 and 10YS6 are best effort and not guaranteed the global optimum since MS model is based on Monte Carlo which is hard to calibrate. It is cost expensive that it consumes a lot of time in calibration compared with the other models that are semi-analytic. Although MS model is not calibrated to its best in some cases, with these parameters, MS model's performance is sufficient to be judged.

Table 5.2 shows the MAPEs of each model for all time series. For 5YS6, the MAPE of GC is 99.35, while DC has very low MAPE of 24.36. MC is even better with MAPE of 13.74. BS's is 78.84, whereas MS's is 10.55. As a consequence, the model's rank is MS, MC, DC, BS, GC. The rank of each model for 7YS6 are the same; they have MAPEs of 152.61, 39.98, 25.94, 146.60, and 25.81 from the top of the table to the bottom. For 10YS6, the results are a bit more different; BS has MAPE of 235.55 which is more than that of GC i.e. 185.40. Thus, the rank is MS, MC, DC, GC, BS. The ranks for series 5 are similar except that the rank of MS and MC in 5YS5 and 7YS5 are swapped as compared with the series 6. Generally, the models based on Meixner distribution/process i.e. MC and MS have the least MAPEs. Double-t copula model, which takes into account the symmetric tail dependence, performs second to

Table 5.1: The optimal parameters of each model calibrated from the market quotes of CDX NA IG index and its CDOs.

		CDX NA IG					
		5YS5	5YS6	7YS5	7YS6	10YS5	10YS6
GC		NA	NA	NA	NA	NA	NA
DC	df	3	3	3	3	6	6
MC	α	3.2225	2.6753	1.6657	1.3724	0.0180	0.0132
	β	-0.6911	-0.7004	-1.9726	-1.8162	-3.0644	-3.0949
BS		NA	NA	NA	NA	NA	NA
MS	β	-0.6435	-1.0472	-0.7499	-0.8261	-1.3694	-1.3694
	δ	0.0400	0.0500	0.0268	0.0280	0.0400	0.0400

the two former models that incorporate asymmetric features. The standard models, GC and BS, are the worst. The following statistical tests will prove these obvious results.

5.1 Results I

All of the copula models are compared using the paired z-test; the mean differences, their standard deviations, and their t-stats are shown in Table 5.3. Recall that a significant positive t-stat means the fist model has MAPE more than the second model; therefore, the first model underperforms the second significantly, and vice versa for a significant negative t-stat.

According to the t-stats in the table, Meixner copula model significantly outperforms both Gaussian copula and double-t copula models in all cases. When double-t copula model and Gaussian copula model are considered, the former is significantly better in all cases too. For 5YS6, the t-stat of GC vs MC is 37.73 which means MC is extremely significantly better than GC, while the t-stat of DC vs MC is 8.02, thereby suggesting that MC is again better significantly. Between GC and DC, the t-stat of 65.01 also presents that DC is superior to GC. For 7YS6, the results is similar; the t-stats of GC vs MC, DC vs MC, and GC vs DC are 51.26, 12.45, 74.60. This suggests that MC is significantly superior to the other two, while DC is the second. Likewise, the t-stats of 10YS6 shows that the rank is MC, DC, and GC, respectively. For series 5, the results are very similar as shown in Table 5.3.

Table 5.2: The minimum MAPEs and their standard deviations of each model for all time series using the optimal parameter shown in Table 5.1.

	CDX NA IG					
	5YS5	5YS6	7YS5	7YS6	10YS5	10YS6
GC	102.65 (33.70)	99.35 (19.18)	168.31 (54.17)	152.61 (26.00)	200.15 (48.29)	185.40 (24.01)
DC	25.53 (17.32)	24.36 (9.91)	59.98 (31.19)	39.98 (16.13)	158.47 (33.22)	158.15 (43.64)
MC	21.93 (13.76)	13.74 (8.02)	43.76 (24.66)	25.94 (13.00)	154.66 (31.69)	151.11 (42.81)
BS	84.68 (40.73)	78.84 (37.36)	156.72 (47.89)	146.60 (73.78)	255.14 (73.07)	235.55 (44.50)
MS	22.12 (10.64)	10.55 (6.34)	53.86* (34.05)	25.81* (17.13)	98.53* (58.65)	72.97* (32.97)

Note: All numbers are in basis point except the t-stats that is in unit of one.
* This results from the approximation approach.

Table 5.3: The results of the paired z-test among the copula models.

		CDX NA IG					
		5YS5	5YS6	7YS5	7YS6	10YS5	10YS6
GC vs MC	Mean Diff.	80.72	85.61	124.55	126.68	45.49	34.29
	S.D. of Diff.	(34.65)	(24.12)	(59.81)	(25.92)	(37.24)	(42.76)
	t-stats	26.36	37.73	22.62	51.26	13.10	8.06
DC vs MC	Mean Diff.	3.60	10.63	16.22	14.04	3.81	7.05
	S.D. of Diff.	(16.22)	(14.09)	(26.65)	(11.83)	(9.22)	(3.17)
	t-stats	2.51	8.02	6.61	12.45	4.43	22.35
GC vs DC	Mean Diff.	77.12	74.98	108.33	112.63	41.68	27.25
	S.D. of Diff.	(20.93)	(12.26)	(36.59)	(15.84)	(38.29)	(43.00)
	t-stats	41.68	65.01	32.16	74.60	11.67	6.37

Note: The number of observations of 5YS5 is 128, while 5YS6's is 113 observations; 7YS5's, 7YS6's, 10YS5's, and 10YS6's are 118, 110, 115, and 101, respectively. All numbers are in basis point except the t-stats that is in unit of one.

This suggests that the model taking account of the tail dependence of each asset in the baskets will improve the capability to capture the capital structure of CDOs. Moreover, the more flexible model that can handle the asymmetric tail dependence will further increase the performance. This confirms that the assets in the portfolio are more correlated in lower tail than in upper tail because the parameters of MC that makes the MAPE minimum has negative β for all data samples. We suggest that, for CDO valuation, considering only the symmetric tail dependence is not sufficient since firms tends to default together in downturn but be more independent in upturn.

To further discuss the results, we focus on CDX NA IG 5YS6 for all copula models. The three copula models are based on the reduced-form model marginally. They have the same hazard rate over time; Figure 5.1 shows the level of hazard rate implied from the market quotes on day-by-day basis. However, the three models use the different copulas which result in the differences in the dependence structures. Recall that Gaussian copula has no tail dependence, while the double-t copula has symmetric tail dependence, and Meixner copula has more flexibility that it can supports an asymmetric dependence structure. Figure 5.2 represents the probability density distributions used to construct the dependence structures (copulas), whereas Figure 5.3 reveals the dependence structures of Gaussian copula, double-t copula, and Meixner copula, respectively. For illustration, we show the plots of two 3000-sample variates with different degree of dependency—ρ = 0.2, 0.5, and 0.9. The more the degree of dependency, the more noticeable the shapes of the dependence structures. For Gaussian copula, the shape of bi-variate dependence structure is ellipse; on the other hand, in case of double-t copula, it is more scattered in the middle but more clustered at the both tails. This is known as the fat-tail. Moreover, Meixner copula has the shape that even more scattered in the middle and even more clustered at the both tails like star-shape. Due to the negative β, the PDF of Meixner has fatter left tail than the right, and the lower tail of Meixner copula is, in turn, more clustered than the upper. Focusing on the tail of the plot, double-t and Meixner copula have tail dependence because their tails are quite sharp so that the trends at the tails are obvious.

Nevertheless, the figures of dependence structures above using the degree of dependence (ρ) for illustration purposes, the actual degree of dependence over time implied from the market quotes are shown in Figure 5.4. Note that ρ is not indeed the default correlation, the asset correlation of firms, nor even the correlation of the default times, though it may be estimated from those ways. It actually represents the

Figure 5.1: The movement of hazard rates implied from CDX NA IG 5YS6

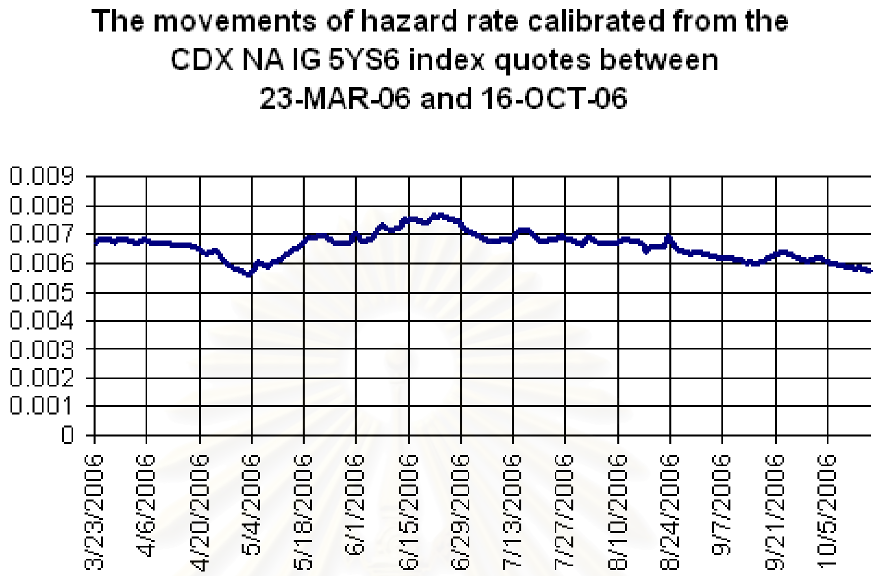


Figure 5.2: The probability density function of the normal, unit-variance student-t, and Meixner distribution behind the copula models that optimally fit the whole samples of the market quotes of CDX NA IG 5YS6

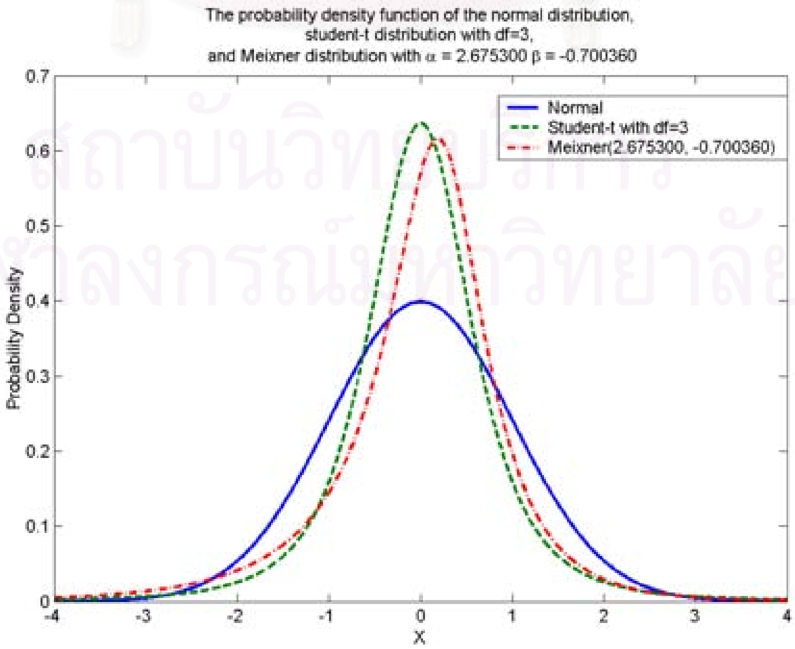
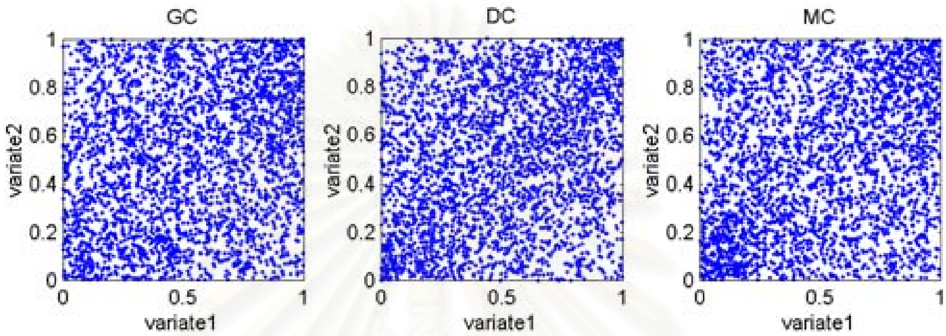
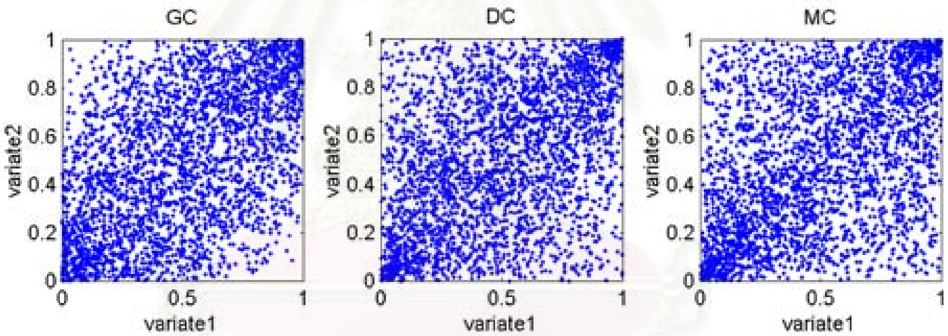


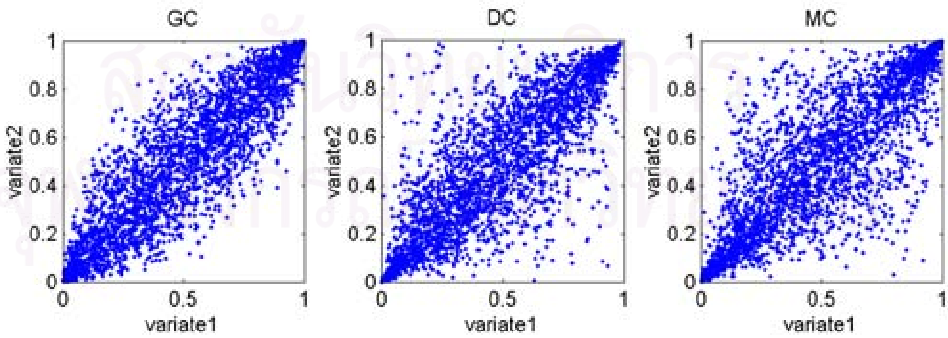
Figure 5.3: The dependence structure of Gaussian copula, double-t copula with df=3, and Meixner copula with parameters $\alpha = 2.6753$, $\beta = -0.70036$.



(a) ρ = 0.2

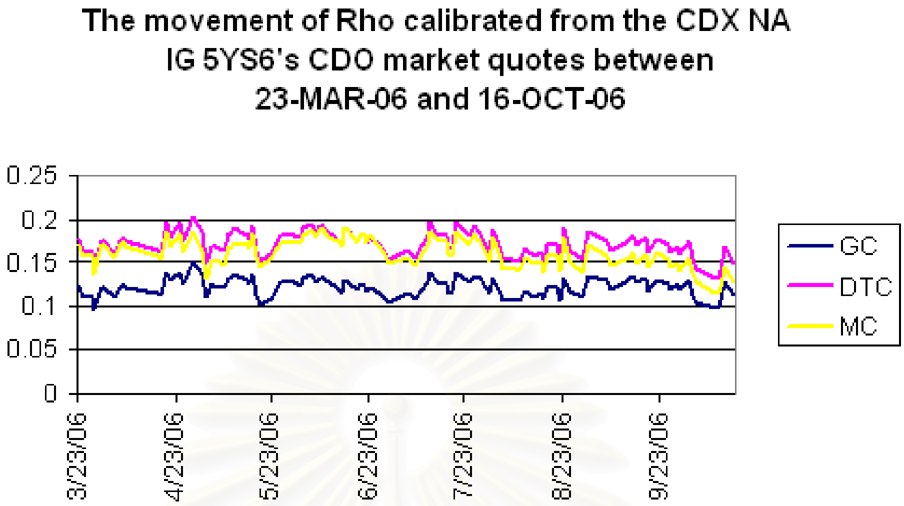


(b) ρ = 0.5



(c) ρ = 0.9

Figure 5.4: The movement of ρ



level of dependency in the copulas. As shown in the figure, the degrees of dependency implied from each model are very highly correlated; the only difference is the overall level of the paths.

5.2 Results II

Among correlated structural models, the results are similar to those of the copula models as shown in Table 5.4. The correlated Meixner structural model significantly outperforms the standard structural model in all cases. For 5YS6, the t-stat is 36.44, while 7YS6's is 13.30, and 10YS6's is 14.89. Likewise, the t-stats in series 5 for 5Y, 7Y, and 10Y are 16.86, 18.91, and 14.98, respectively.

This suggests that, for CDO valuation, the assumption that firm value follows the Brownian motion is not justified. The firm value can have rare large moves governed by jumps in addition to the frequent small moves provided by diffusion, or in this study, we assume that the firm value movement is driven wholly by jumps of infinite activity and infinite variation which can capture both the rare events, small moves, and the extremely frequent small moves. The result shows that modeling the asset return as Meixner process is more realistic. Furthermore, the optimal parameters having negative β's for all cases, imply that a firm is more likely to suffer from large,down-sided jumps than the large,up-sided jumps. The large,down-sided jumps can be from market factor or firm specific. When it is from market factor, all firms tremendously increase the

Table 5.4: The results of the pair z-test between BS and MS models

		CDX NA IG					
		5YS5	5YS6	7YS5	7YS6	10YS5	10YS6
BS vs MS	Mean Diff.	62.57	65.10	94.82	103.80	140.70	128.29
	S.D. of Diff.	(41.99)	(18.99)	(54.48)	(81.85)	(100.72)	(86.61)
	t-stats	16.86	36.44	18.91	13.30	14.98	14.89

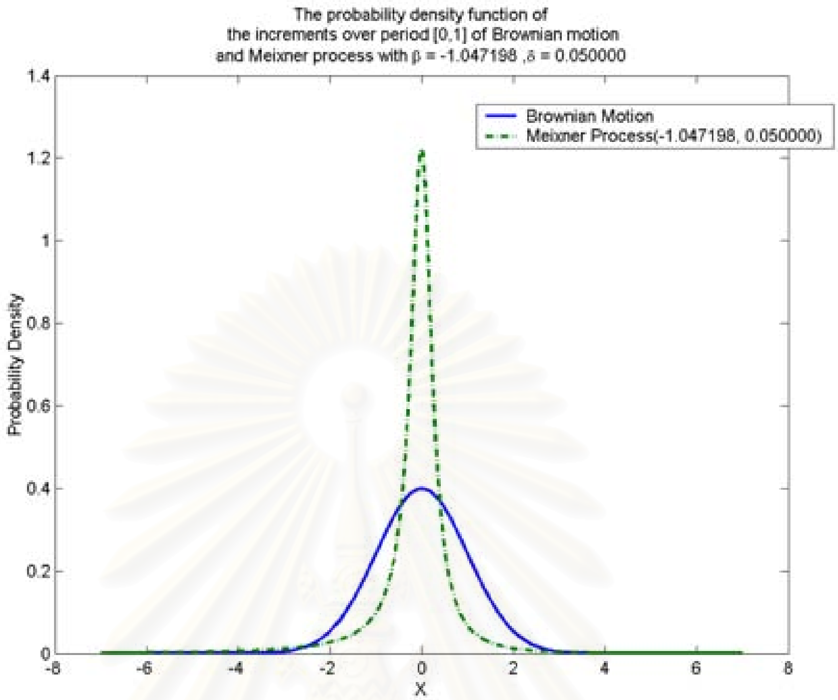
Note: All numbers are in basis point except the t-stats that is in unit of one.

chance of default.

We further investigate CDX NA IG 5YS6 for both of the correlated structural models. Both BS and MS are based on the first passage model marginally. However, MS has more flexibility than BS in that it can adjust its process to have heavier/lighter upper/lower tail, while BS has no fat-tail at both sides. Unfortunately, there is an analytical solution for the probability of first hitting time in Brownian case derived via Reflection principle but not in Meixner case. Because Meixner is a pure jump process, this principle cannot be applied.

Figure 5.5 represents the probability distribution of the increments of Brownian motion and Meixner process, whereas the movements of default barrier parameters calibrated using one-by-one basis are shown in Figure 5.6 and 5.7. From the sketch of PDF over time, there is more probability for Meixner process that the cumulative asset return will be around the earlier level or have a large move than for Brownian motion. This is known as leptokurtic of the return distribution which is found in reality. Moreover, Meixner process moves asymmetrically which represents the skewness of the actual return distribution. For the movements of default barrier's parameter of both models, they are highly correlated; the dramatic difference is the level of the values. The sources of this differences are contributed to the different volatility, skewness, and kurtosis of the asset return distribution. If we assume that $K/V_0 = 0.5$, the volatility σ can implied directly from (3.21) i.e. $\sigma = 15.82\%$ p.a. on 23-Mar-2006. In the case of Meixner, the same procedure can be applied but not directly because the α is not the volatility or variance; it is just a scaling factor of the process. To determine the volatility of asset return, we must take β and δ into account. According to Table B.1, volatility is equal to $\alpha\sqrt{\frac{\delta}{1+cos(\beta)}} = 11.03\%$ p.a. on 23-Mar-2006 which is sensible.

Figure 5.5: The probability distribution of the increments over period [0,1] of Brownian motion and Meixner process (-1.0472, 0.05) implied from 5YS6



In multi-variate settings, correlated asset returns follows Brownian motion in BS and Meixner process in MS. For illustration, Figure 5.8 shows the sample paths of the asset return simulated from the correlated Brownian motion and the correlated Meixner process (-1.0472, 0.05) with the degree of dependency of 0.2, 0.5, and 0.9. In BS, both market factor and the idiosyncratic diffuse without jumps, whereas the market factor and the idiosyncratic in MS jumps through the time.

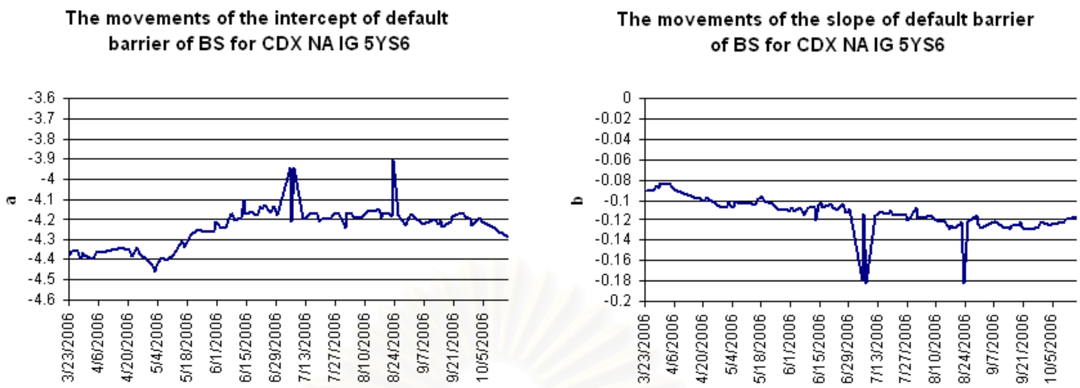
Figure 5.9 shows the degree of dependency of all models over time; it is highly correlated regardless the types of models. Note that it is not indeed the asset correlation; it is just its proxy.

5.3 Results III

This section compares the copula models with the correlated structural models. First, we compare the copula models with BS. Then, MS and the copula models are compared.

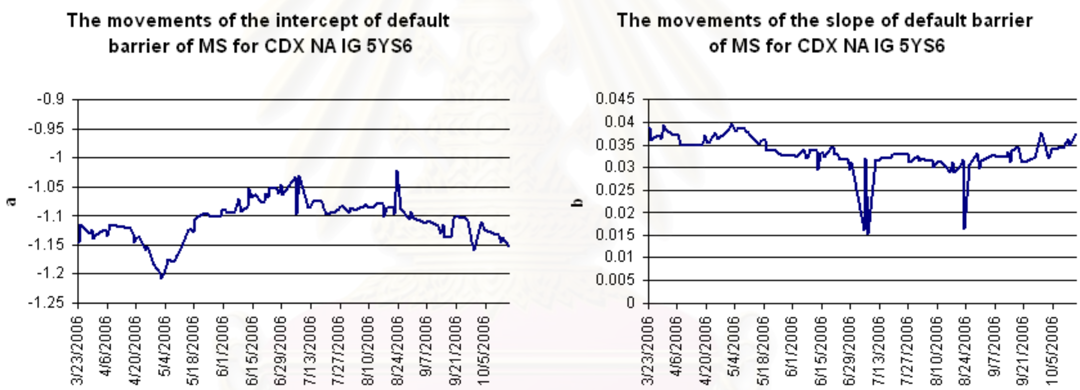
The results of copula models and BS comparison reveals in Table 5.5. The performance of BS is poorer in almost all of the cases. BS is only better than GC

Figure 5.6: The movement of barrier parameters of BS for CDX NA IG 5YS6



(a) The intercepts of GBMS (b) The slopes of GBMS

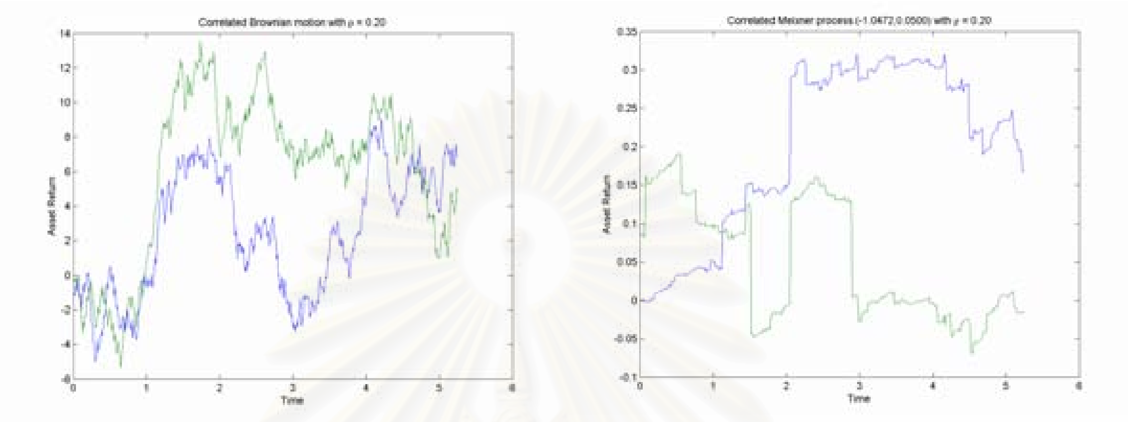
Figure 5.7: The movement of barrier parameters of MS for CDX NA IG 5YS6



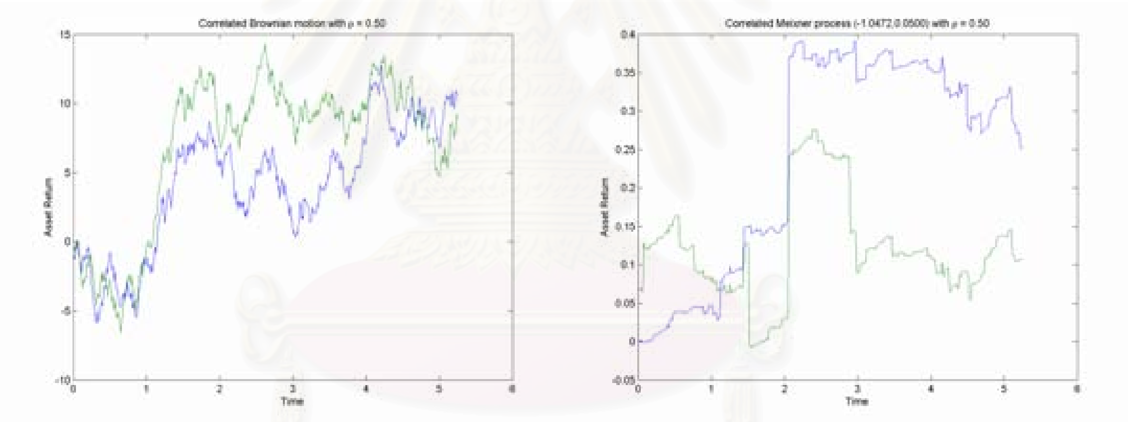
(a) The intercepts of GBMS (b) The slopes of GBMS

in 5YS5, 5YS6, 7YS5, and 7YS6 cases. For GC vs BS, the t-stats of 5YS6 is 36.72, while 7YS6's is 0.89; however, 10YS6's is -18.17. This implies that BS performs better than GC in short and medium maturity. For DC vs BS, the t-stats of 5YS6, 7YS6, and 10YS6, which are -40.30, -15.65, and -9.40, shows that DC significantly outperforms BS in all cases. Likewise in MC vs BS, the t-stats of -28.88, -17.15, and -16.50 for series 6 reveals the outperformance of MC. The results of series 5 is very consistent with those of series 6. As a consequence, the models taking tail dependence into account perform better regardless of the approaches. The results is almost what we anticipated except the 10Y series. In our opinion, the BS should be better than GC because it joins the dependency through the periods, while copula approach joins at the end of periods.

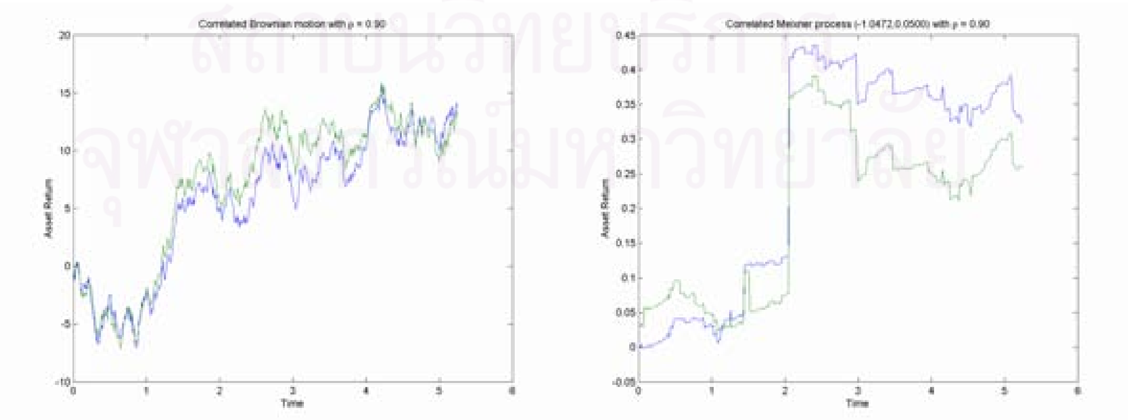
Figure 5.8: The paths of the cumulative asset returns of two sample firms that follow correlated Brownian motion and correlated Meixner process(-1.0472, 0.05) with different degree of dependency.



(a) ρ = 0.2



(b) ρ = 0.5



(c) ρ = 0.9

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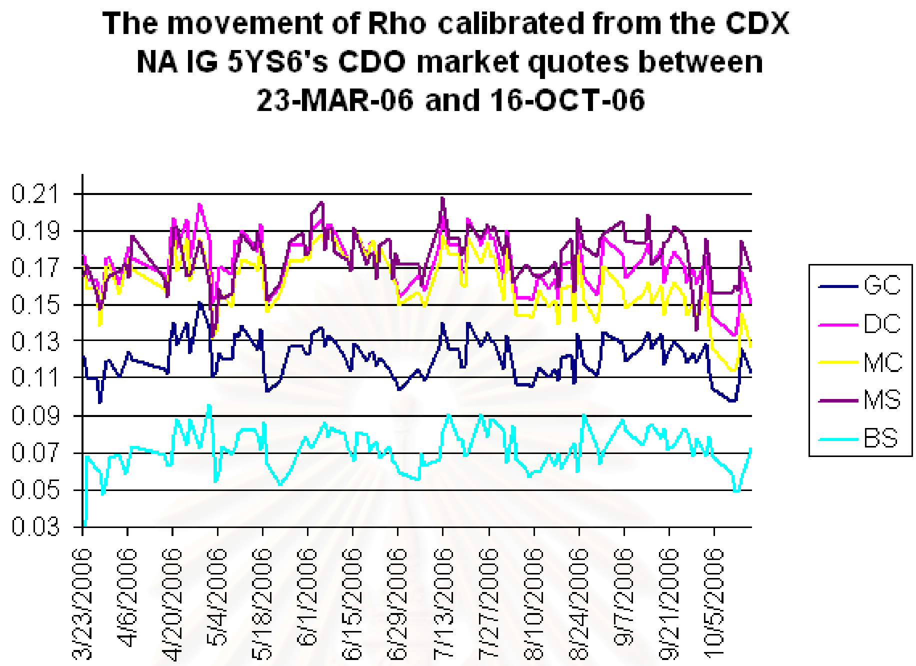
Figure 5.9: The movement of ρ's



The results of the comparison of the copula models and MS are shown in Table 5.6. MS can beat all the models in 5Y and 10Y series, while in 7Y series, it is considerably worse only than MC.

The conclusion as to which approach between a copula model and a correlated structural model is better is ambiguous; however, in most cases MC is the best, while MS is the second; double-t is the third and the second to last and the last are BS and GC. It seems that, from this study, the performance does not depend directly on how the marginals are modeled but rather on the dependence structure the models provide. As a consequence, modeling the marginals with either reduced-form approach or structural approach does not quite different in CDOs.

In general, a copula model is faster in computation than a correlated structural model, while the correlated structural model has economic underpinnings and usually based on Monte Carlo. Economically, a correlated structural model is better than a copula model in that it joins the firm value with market factor through out the time via correlated first passage time, while the copula model joins the firm value via the level of the market factor at the end of the time. Moreover, correlated structural models are dynamic models in that the conditional default probability can evolve over time;

Table 5.5: Paired Z-test for the copula models and BS

		5YS5	5YS6	7YS5	7YS6	10YS5	10YS6
GC vs BS	Mean Diff.	17.97	23.70	11.59	6.01	-54.99	-50.15
	S.D. of Diff.	(11.72)	(6.86)	(13.28)	(70.50)	(35.63)	(27.74)
	t-stats	17.35	36.72	9.48	0.89	-16.55	-18.17
DC vs BS	Mean Diff.	-59.15	-51.28	-96.74	-106.62	-68.88	-48.67
	S.D. of Diff.	(28.34)	(13.52)	(30.25)	(71.45)	(66.89)	(52.06)
	t-stats	-23.61	-40.30	-34.74	-15.65	-11.04	-9.40
MC vs BS	Mean Diff.	-58.92	-61.91	-112.96	-120.66	-100.48	-84.45
	S.D. of Diff.	(47.11)	(22.79)	(53.20)	(73.81)	(57.09)	(51.44)
	t-stats	-14.15	-28.88	-23.06	-17.15	-18.87	-16.50

Note: All numbers are in basis point except the t-stats that is in unit of one.

Table 5.6: Paired Z-test for the copula models and MS

		5YS5	5YS6	7YS5	7YS6	10YS5	10YS6
GC vs MS	Mean Diff.	80.54	88.80	114.45	126.80	101.62	112.44
	S.D. of Diff.	(35.78)	(20.05)	(57.68)	(32.15)	(79.79)	(45.47)
	t-stats	25.46	47.08	21.55	41.36	13.66	24.85
DC vs MS	Mean Diff.	3.42	13.82	6.12	14.17	59.94	74.82
	S.D. of Diff.	(18.47)	(10.04)	(25.98)	(18.89)	(58.44)	(40.93)
	t-stats	2.09	14.63	2.56	7.86	11.00	18.37
MC vs MS	Mean Diff.	3.65	3.19	-10.10	0.12	56.13	78.14
	S.D. of Diff.	(10.45)	(9.10)	(22.25)	(10.93)	(53.30)	(30.07)
	t-stats	3.95	3.73	-4.93	0.12	11.29	26.12

Note: All numbers are in basis point except the t-stats that is in unit of one.

they can price some exotic credit derivatives that a simple copula model cannot such as Forward Starting CDO and Options on CDOs. However, copula models can also increase the dynamics by modeling the evolution of hazard rates. (Hull and White; 2006a)

5.4 Results IV

This section discusses the risk measures derived from each model: EDL, VaR, ESs of CDX NA IG by assuming that investors hold the CDOs until the expiry. First, the portfolio loss distribution of 5YS6, 7YS6, and 10YS6 implied from each model are investigated. Then, we discuss the risk measures of each CDO tranche at confidence level of 95%. It is important to bear in mind that we imply the risk measures under risk-neutral world; the results may be different in the real world.

Figure 5.10 reveals the portfolio loss distribution of CDX NA IG 5YS6, 7YS6, and 10YS6 implied from the market quotes on the issued date (23 Mar 2006) for each model. For 5 years, GC is the most optimistic

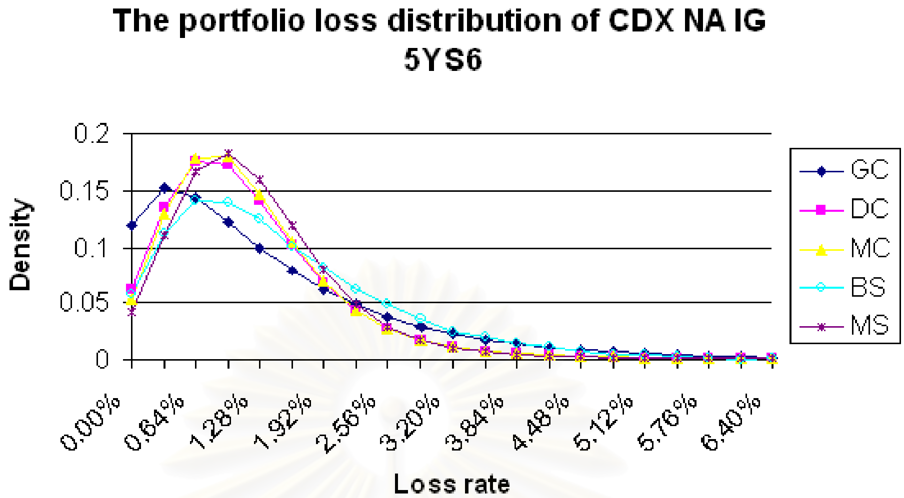
Table 5.7 represents the risk measures for all data samples. Recall that these risk measures have different interpretations. Expected Discount Loss (EDL) is the expected loss derived from each model while VaR is the potential loss in extreme case at a confidence level of 95%. In contrast, ES is the average of the potential losses given that a extreme event occurs at the same confidence level. In addition, the undiscount total premium (UTP) is an approximate indicating actually how much investors get compensated totally. Therefore, VaR/EDL indicates how much the loss in extreme case is relative to the expected loss. For ES/EDL, it indicates how much the average potential loss given an extreme case is, compared to the expected loss. Similarly, VaR/UTP and ES/UTP indicate how much the loss in extreme case and the average loss given extreme case are relative to what market views.

The VaR and ES is relatively high in some tranches when comparing with EDL and the market premiums (Undiscounted Total Premium). This is interesting because EDL is what we expect in average and it indicates directly how much investors should fairly get in compensation, but in extreme case, the investors can end up with a hugh loss than what is expected. We will discuss the risk measures from 5YS6; however, the other series have only slightly different interpretations.

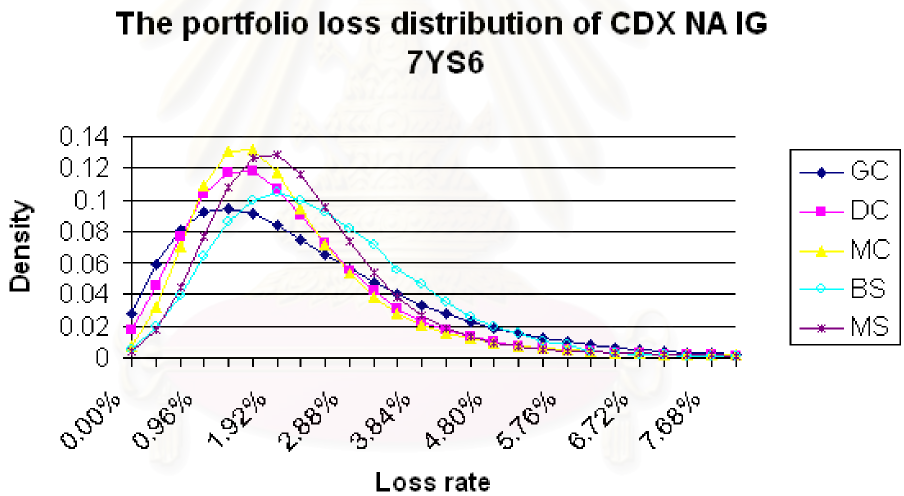
For the equity tranches (0-3%), it is quite certain that in extreme cases the investors holding this tranche will loss all of their investments. The risk measures computed from different models provide approximately the same. In extreme cases, the investors holding 5YS6 equity tranches can loss about twice of the expected loss while it is around 1.8 times of what market views. Given an extreme event occur, it is quite sure that all the value of the notional principal is wiped out.

For junior mezzanine tranches (3-7%), investors holding this tranche until the expiry are exposure to large losses in extreme cases comparing to what they expect to get in average. For 5YS6 as an example, investors can loss around 7-8 times of the

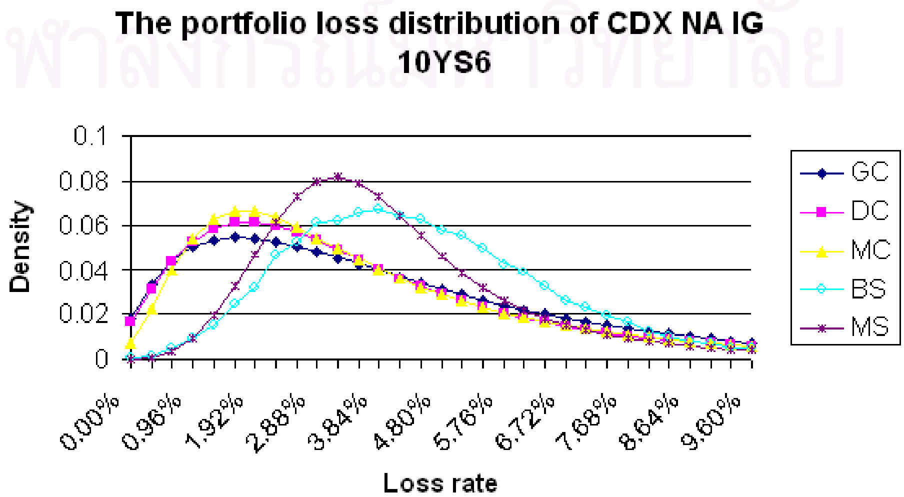
Figure 5.10: The portfolio loss distribution of CDX NA IG Series 6



(a)



(b)



(c)

expected loss while about 5-14 times of what market views. Given that the extreme cases occur, the loss is approximately 12-18 times of the expected loss; whereas it is about 14-20 times of what market views. For this tranche, each of the models provides slightly different risk measures except GC and BS; they are the most pessimistic. It is interesting that although tranche 0-3% can worthless in extreme case, the premium is high. Even though this tranche losses less than tranche 0-3% in extreme cases, the premium is even less.

For mezzanine tranche 7-10%, GC and DC provide similar higher risk measures; whereas, MC's and MS's are close to each other; BS has significant less risk measures. Though, the losses in extreme case are zeros at confidence level of 95%, the average potential loss given the extreme events is about 24-26 times of the expected loss. Moreover, it is about 11 times of what market views for BS but 20-28 times for the other models. Notice that BS's risk measures start to fall sharply because the model neglects extreme cases.

For tranche 10-15% of 5YS6, DC provides the highest risk measures; whereas, MC's and MS's is very similar. GC has less risk measures and BS has much less ones. In extreme cases, investors holding this tranche is quite safe; the potential loss are zeros for all models. However, given extreme events occur, the average potential loss can be 24-26 times of the expected loss. For BS and GC, it is around 1.7 and 9 times of what market views; whereas, it is 18-35 times for MS, MC, and DC. Notice that GC and BS fall sharply; the probable reason is they ignore the extreme cases.

Like in tranche 10-15%, DC provides the highest risk measures for tranche 15-30%. MC's and MS's are very similar while GC has less risk measures. BS has much less ones. Moreover, the potential losses in extreme cases are zeros. However, given extreme events, the average potential loss is 24-26 times of the expected loss. When comparing with what market views, BS and GC provide only 0.0330-0.2317 times; whereas, MC and MS suggest about 12-14 times. It is 33 folds for DC. In this tranche, the interpretation is similar to the previous tranche i.e. BS and GC neglect the extreme cases; hence, the risk measures for later tranche is too low.

Risk measures derived from 5YS5 is very similar to the case of 5YS6. Moreover, the risk measures for 7YS5, 7YS6, 10YS5, and 10YS6 reveal consistency with the interpretation of 5YS6. Firstly, BS and GC neglect the extreme cases so that their risk measures are higher in earlier tranches and then plummet later. Secondly, DC provide relatively high risk measures in most cases. Finally, both MC's and MS's risk measures tend to be less than DC's in most cases. However, the noticeable differences among different series are: in case of 5Y, only tranche 0-3% is wiped out in extreme cases; for 7Y, the first two tranches are wiped out much likely in rare cases; lastly, the first three tranches of 10Y are worthless in extreme events quite surely, and rather surely for the fourth tranche.

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Table 5.7: The risk measures for various Tranches

Panel A: Tranche 0-3%

		Risk Measures (unit)						Risk Measures / EDL						Risk Measures / UTP					
		5YS5	5YS6	7YS5	7YS6	10YS5	10YS6	5YS5	5YS6	7YS5	7YS6	10YS5	10YS6	5YS5	5YS6	7YS5	7YS6	10YS5	10YS6
Mean Mkt. Quotes		0.3252	0.2970	0.4910	0.4635	0.5750	0.5501	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA
UTP		0.5752	0.5470	0.7410	0.7135	0.8250	0.8001	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA
	GC	0.4718	0.4518	0.6491	0.6310	0.7417	0.7228	NA	NA	NA	NA	NA	NA	0.8204	0.8259	0.8761	0.8844	0.8989	0.9034
	DC	0.4731	0.4532	0.6508	0.6330	0.7412	0.7224	NA	NA	NA	NA	NA	NA	0.8225	0.8285	0.8783	0.8871	0.8984	0.9030
EDL	MC	0.4753	0.4554	0.6564	0.6377	0.7448	0.7267	NA	NA	NA	NA	NA	NA	0.8264	0.8326	0.8859	0.8937	0.9028	0.9083
	BS	0.4959	0.4744	0.6775	0.6618	0.7700	0.7531	NA	NA	NA	NA	NA	NA	0.8622	0.8672	0.9143	0.9274	0.9332	0.9413
	MS	0.4834	0.4633	0.6714	0.6569	0.7697	0.7520	NA	NA	NA	NA	NA	NA	0.8405	0.8470	0.9061	0.9206	0.9329	0.9399
	GC	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	2.1193	2.2135	1.5405	1.5847	1.3483	1.3836	1.7387	1.8282	1.3496	1.4015	1.2121	1.2498
	DC	0.9972	1.0000	1.0000	1.0000	1.0000	1.0000	2.1081	2.2065	1.5366	1.5798	1.3491	1.3842	1.7338	1.8282	1.3496	1.4015	1.2121	1.2498
VaR	MC	0.9929	1.0000	1.0000	1.0000	1.0000	1.0000	2.0889	2.1959	1.5234	1.5682	1.3426	1.3760	1.7263	1.8282	1.3496	1.4015	1.2121	1.2498
	BS	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	2.0165	2.1081	1.4761	1.5111	1.2988	1.3278	1.7387	1.8282	1.3496	1.4015	1.2121	1.2498
	MS	0.9978	1.0000	1.0000	1.0000	1.0000	1.0000	2.0641	2.1583	1.4894	1.5224	1.2992	1.3298	1.7348	1.8282	1.3496	1.4015	1.2121	1.2498
	GC	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	2.1193	2.2135	1.5405	1.5847	1.3483	1.3836	1.7387	1.8282	1.3496	1.4015	1.2121	1.2498
	DC	0.9996	1.0000	1.0000	1.0000	1.0000	1.0000	2.1130	2.2065	1.5366	1.5798	1.3491	1.3842	1.7379	1.8282	1.3496	1.4015	1.2121	1.2498
ES	MC	0.9988	1.0000	1.0000	1.0000	1.0000	1.0000	2.1012	2.1959	1.5234	1.5682	1.3426	1.3760	1.7365	1.8282	1.3496	1.4015	1.2121	1.2498
	BS	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	2.0165	2.1081	1.4761	1.5111	1.2988	1.3278	1.7387	1.8282	1.3496	1.4015	1.2121	1.2498
	MS	0.9997	1.0000	1.0000	1.0000	1.0000	1.0000	2.0679	2.1583	1.4894	1.5224	1.2992	1.3298	1.7381	1.8282	1.3496	1.4015	1.2121	1.2498

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Table 5.7: The risk measures for various Tranches (Continued)

Panel B: Tranche 3-7%

		Risk Measures (unit)						Risk Measures / EDL						Risk Measures / UTP					
		5YS5	5YS6	7YS5	7YS6	10YS5	10YS6	5YS5	5YS6	7YS5	7YS6	10YS5	10YS6	5YS5	5YS6	7YS5	7YS6	10YS5	10YS6
Mean Mkt. Quotes		0.0088	0.0090	0.0227	0.0230	0.0572	0.0554	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA
UTP		0.0442	0.0452	0.1590	0.1613	0.5716	0.5542	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA
EDL	GC	0.0707	0.0727	0.1815	0.1828	0.3697	0.3591	NA	NA	NA	NA	NA	NA	1.6006	1.6106	1.1421	1.1330	0.6467	0.6480
	DC	0.0415	0.0402	0.1353	0.1317	0.3415	0.3290	NA	NA	NA	NA	NA	NA	0.9396	0.8902	0.8510	0.8165	0.5974	0.5937
	MC	0.0361	0.0369	0.1169	0.1249	0.3460	0.3300	NA	NA	NA	NA	NA	NA	0.8187	0.8179	0.7352	0.7743	0.6054	0.5954
	BS	0.0598	0.0618	0.1860	0.1850	0.4508	0.4355	NA	NA	NA	NA	NA	NA	1.3545	1.3686	1.1701	1.1469	0.7887	0.7859
	MS	0.0409	0.0355	0.1315	0.1318	0.3772	0.3640	NA	NA	NA	NA	NA	NA	0.9261	0.7860	0.8273	0.8170	0.6599	0.6568
VaR	GC	0.5513	0.6182	0.9786	1.0000	1.0000	1.0000	7.8008	8.5004	5.3902	5.4705	2.7050	2.7847	12.486	13.690	6.1563	6.1980	1.7494	1.8045
	DC	0.2800	0.2905	0.9246	0.9844	1.0000	1.0000	6.7480	7.2272	6.8348	7.4725	2.9283	3.0393	6.3406	6.4334	5.8167	6.1013	1.7494	1.8045
	MC	0.2321	0.2604	0.8683	0.9752	1.0000	1.0000	6.4200	7.0514	7.4308	7.8058	2.8899	3.0306	5.2558	5.7675	5.4628	6.0440	1.7494	1.8045
	BS	0.4326	0.4915	0.9317	0.9840	1.0000	1.0000	7.2344	7.9527	5.0098	5.3177	2.2181	2.2960	9.7988	10.8844	5.8617	6.0989	1.7494	1.8045
	MS	0.2841	0.2497	0.8412	0.9283	1.0000	1.0000	6.9475	7.0348	6.3968	7.0430	2.6509	2.7473	6.4338	5.5297	5.2920	5.7538	1.7494	1.8045
ES	GC	0.8385	0.8985	0.9960	1.0000	1.0000	1.0000	11.865	12.355	5.4862	5.4705	2.7050	2.7847	18.992	19.899	6.2660	6.1980	1.7494	1.8045
	DC	0.6711	0.7200	0.9887	0.9990	1.0000	1.0000	16.176	17.913	7.3085	7.5835	2.9283	3.0393	15.199	15.946	6.2199	6.1919	1.7494	1.8045
	MC	0.6064	0.6733	0.9765	0.9986	1.0000	1.0000	16.777	18.231	8.3564	7.9936	2.8899	3.0306	13.735	14.912	6.1433	6.1893	1.7494	1.8045
	BS	0.7134	0.7837	0.9844	0.9982	1.0000	1.0000	11.930	12.681	5.2929	5.3943	2.2181	2.2960	16.159	17.356	6.1930	6.1867	1.7494	1.8045
	MS	0.6608	0.6477	0.9651	0.9935	1.0000	1.0000	16.162	18.247	7.3389	7.5373	2.6509	2.7473	14.967	14.343	6.0714	6.1576	1.7494	1.8045

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Table 5.7: The risk measures for various Tranches (Continued)

Panel C: Tranche 7-10%

		Risk Measures (unit)						Risk Measures / EDL						Risk Measures / UTP					
		5YS5	5YS6	7YS5	7YS6	10YS5	10YS6	5YS5	5YS6	7YS5	7YS6	10YS5	10YS6	5YS5	5YS6	7YS5	7YS6	10YS5	10YS6
Mean Mkt. Quotes		0.0021	0.0020	0.0043	0.0046	0.0110	0.0114	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA
UTP		0.0104	0.0100	0.0301	0.0321	0.1099	0.1144	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA
EDL	GC	0.0085	0.0109	0.0356	0.0404	0.1604	0.1608	NA	NA	NA	NA	NA	NA	0.8182	1.0879	1.1805	1.2551	1.4587	1.4064
	DC	0.0102	0.0118	0.0308	0.0330	0.1420	0.1411	NA	NA	NA	NA	NA	NA	0.9819	1.1788	1.0226	1.0274	1.2920	1.2336
	MC	0.0081	0.0086	0.0282	0.0316	0.1451	0.1417	NA	NA	NA	NA	NA	NA	0.7856	0.8596	0.9356	0.9833	1.3198	1.2386
	BS	0.0028	0.0042	0.0191	0.0218	0.1418	0.1396	NA	NA	NA	NA	NA	NA	0.2688	0.4182	0.6327	0.6781	1.2899	1.2209
	MS	0.0097	0.0077	0.0221	0.0251	0.1077	0.1118	NA	NA	NA	NA	NA	NA	0.9348	0.7653	0.7337	0.7807	0.9798	0.9773
VaR	GC	0.0000	0.0000	0.3148	0.4204	0.9948	1.0000	0.0000	0.0000	8.8443	10.419	6.2035	6.2170	0.0000	0.0000	10.440	13.077	9.0488	8.7435
	DC	0.0000	0.0000	0.1714	0.1861	0.9967	1.0000	0.0000	0.0000	5.5585	5.6357	7.0165	7.0878	0.0000	0.0000	5.6840	5.7899	9.0657	8.7435
	MC	0.0000	0.0000	0.1357	0.1692	0.9949	1.0000	0.0000	0.0000	4.8112	5.3532	6.8570	7.0590	0.0000	0.0000	4.5013	5.2635	9.0495	8.7435
	BS	0.0000	0.0000	0.0862	0.0982	0.9558	0.9946	0.0000	0.0000	4.5220	4.5039	6.7402	7.1227	0.0000	0.0000	2.8609	3.0542	8.6941	8.6959
	MS	0.0000	0.0000	0.0796	0.0826	0.9572	1.0000	0.0000	0.0000	3.6000	3.2926	8.8854	8.9463	0.0000	0.0000	2.6412	2.5705	8.7063	8.7435
ES	GC	0.2066	0.2692	0.7385	0.8502	0.9996	1.0000	24.375	24.706	20.752	21.071	6.2331	6.2170	19.943	26.877	24.497	26.447	9.0919	8.7435
	DC	0.2413	0.2819	0.7264	0.8070	0.9998	1.0000	23.722	23.869	23.563	24.433	7.0382	7.0878	23.294	28.136	24.095	25.102	9.0937	8.7435
	MC	0.1952	0.2099	0.6834	0.7823	0.9996	1.0000	23.982	24.370	24.228	24.748	6.8897	7.0590	18.839	20.949	22.668	24.334	9.0927	8.7435
	BS	0.0703	0.1081	0.4608	0.5596	0.9861	0.9997	25.242	25.791	24.160	25.670	6.9535	7.1596	6.7858	10.7870	15.2846	17.407	8.9693	8.7409
	MS	0.2358	0.1908	0.5791	0.6841	0.9886	1.0000	24.345	24.895	26.181	27.254	9.1776	8.9463	22.758	19.051	19.208	21.278	8.9926	8.7435

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Table 5.7: The risk measures for various Tranches (Continued)

Panel D: Tranche 10-15%

		Risk Measures (unit)						Risk Measures / EDL						Risk Measures / UTP					
		5YS5	5YS6	7YS5	7YS6	10YS5	10YS6	5YS5	5YS6	7YS5	7YS6	10YS5	10YS6	5YS5	5YS6	7YS5	7YS6	10YS5	10YS6
Mean Mkt. Quotes		0.0010	0.0009	0.0022	0.0020	0.0054	0.0054	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA
UTP		0.0049	0.0047	0.0153	0.0140	0.0541	0.0544	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA
	GC	0.0011	0.0018	0.0062	0.0080	0.0610	0.0640	NA	NA	NA	NA	NA	NA	0.2196	0.3751	0.4079	0.5695	1.1283	1.1773
	DC	0.0055	0.0070	0.0144	0.0162	0.0621	0.0642	NA	NA	NA	NA	NA	NA	1.1361	1.4933	0.9458	1.1586	1.1480	1.1806
EDL	MC	0.0041	0.0041	0.0141	0.0140	0.0605	0.0624	NA	NA	NA	NA	NA	NA	0.8399	0.8675	0.9225	1.0008	1.1194	1.1475
	BS	0.0002	0.0003	0.0014	0.0021	0.0231	0.0243	NA	NA	NA	NA	NA	NA	0.0342	0.0697	0.0950	0.1464	0.4264	0.4478
	MS	0.0049	0.0035	0.0091	0.0105	0.0365	0.0432	NA	NA	NA	NA	NA	NA	1.0001	0.7383	0.5936	0.7517	0.6754	0.7939
	GC	0.0000	0.0000	0.0000	0.0000	0.7798	0.8963	0.0000	0.0000	0.0000	0.0000	12.779	14.003	0.0000	0.0000	0.0000	0.0000	14.418	16.486
	DC	0.0000	0.0000	0.0000	0.0000	0.8713	0.9635	0.0000	0.0000	0.0000	0.0000	14.032	15.011	0.0000	0.0000	0.0000	0.0000	16.110	17.723
VaR	MC	0.0000	0.0000	0.0000	0.0000	0.8256	0.9366	0.0000	0.0000	0.0000	0.0000	13.636	15.013	0.0000	0.0000	0.0000	0.0000	15.264	17.227
	BS	0.0000	0.0000	0.0000	0.0000	0.2319	0.2695	0.0000	0.0000	0.0000	0.0000	10.0543	11.0669	0.0000	0.0000	0.0000	0.0000	4.2867	4.9563
	MS	0.0000	0.0000	0.0000	0.0000	0.5278	0.7634	0.0000	0.0000	0.0000	0.0000	14.447	17.686	0.0000	0.0000	0.0000	0.0000	9.7583	14.041
	GC	0.0264	0.0439	0.1660	0.2170	0.9367	0.9847	24.582	24.945	26.662	27.183	15.349	15.385	5.3985	9.3570	10.876	15.480	17.318	18.113
	DC	0.1311	0.1659	0.3649	0.4167	0.9663	0.9958	23.635	23.685	25.280	25.654	15.562	15.513	26.853	35.369	23.910	29.724	17.865	18.316
ES	MC	0.0985	0.0993	0.3604	0.3673	0.9515	0.9920	24.015	24.396	25.596	26.177	15.716	15.902	20.170	21.165	23.612	26.197	17.593	18.247
	BS	0.0042	0.0084	0.0400	0.0582	0.5451	0.6017	25.194	25.799	27.571	28.379	23.637	24.711	0.8611	1.7982	2.6185	4.1546	10.0778	11.0666
	MS	0.1176	0.0855	0.2504	0.2997	0.8587	0.9687	24.084	24.692	27.637	28.438	23.504	22.442	24.088	18.229	16.406	21.378	15.876	17.817

Table 5.7: The risk measures for various Tranches (Continued)

Panel E: Tranche 15-30%

		Risk Measures (unit)						Risk Measures / EDL						Risk Measures / UTP					
		5YS5	5YS6	7YS5	7YS6	10YS5	10YS6	5YS5	5YS6	7YS5	7YS6	10YS5	10YS6	5YS5	5YS6	7YS5	7YS6	10YS5	10YS6
Mean Mkt. Quotes		0.0005	0.0005	0.0008	0.0007	0.0016	0.0015	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA
UTP		0.0025	0.0025	0.0053	0.0050	0.0160	0.0150	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA
	GC	0.0000	0.0001	0.0002	0.0003	0.0070	0.0080	NA	NA	NA	NA	NA	NA	0.0094	0.0234	0.0390	0.0634	0.4372	0.5333
	DC	0.0025	0.0036	0.0057	0.0068	0.0151	0.0166	NA	NA	NA	NA	NA	NA	0.9672	1.4175	1.0776	1.3392	0.9425	1.1077
EDL	MC	0.0017	0.0015	0.0061	0.0046	0.0109	0.0129	NA	NA	NA	NA	NA	NA	0.6657	0.5846	1.1412	0.9128	0.6781	0.8573
	BS	0.0000	0.0000	0.0000	0.0000	0.0005	0.0004	NA	NA	NA	NA	NA	NA	0.0013	0.0038	0.0039	0.0029	0.0318	0.0298
	MS	0.0024	0.0013	0.0024	0.0028	0.0083	0.0116	NA	NA	NA	NA	NA	NA	0.9273	0.5220	0.4570	0.5490	0.5207	0.7705
	GC	0.0000	0.0000	0.0000	0.0000	0.0148	0.0106	0.0000	0.0000	0.0000	0.0000	2.1199	1.3291	0.0000	0.0000	0.0000	0.0000	0.9269	0.7087
	DC	0.0000	0.0000	0.0000	0.0000	0.0444	0.0545	0.0000	0.0000	0.0000	0.0000	2.9434	3.2802	0.0000	0.0000	0.0000	0.0000	2.7741	3.6333
VaR	MC	0.0000	0.0000	0.0000	0.0000	0.0292	0.0330	0.0000	0.0000	0.0000	0.0000	2.6864	2.5659	0.0000	0.0000	0.0000	0.0000	1.8217	2.1996
	BS	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	MS	0.0000	0.0000	0.0000	0.0000	0.0011	0.0024	0.0000	0.0000	0.0000	0.0000	0.1334	0.2089	0.0000	0.0000	0.0000	0.0000	0.0695	0.1610
	GC	0.0006	0.0015	0.0056	0.0088	0.2061	0.2437	24.768	25.178	27.019	27.579	29.427	30.449	0.2317	0.5883	1.0536	1.7477	12.867	16.237
	DC	0.0583	0.0843	0.1434	0.1714	0.4160	0.4693	23.665	23.499	25.047	25.374	27.557	28.225	22.889	33.310	26.991	33.982	25.972	31.263
ES	MC	0.0409	0.0363	0.1558	0.1209	0.3111	0.3783	24.132	24.513	25.697	26.261	28.637	29.401	16.064	14.330	29.326	23.970	19.420	25.204
	BS	0.0001	0.0002	0.0006	0.0004	0.0162	0.0148	25.445	25.988	27.747	28.785	31.706	33.072	0.0330	0.0975	0.1081	0.0841	1.0089	0.9842
	MS	0.0563	0.0321	0.0679	0.0799	0.2670	0.3825	23.835	24.257	27.976	28.835	32.011	33.073	22.103	12.663	12.785	15.831	16.667	25.483

CHAPTER VI

CONCLUSION AND RECOMMENDATION

6.1 Conclusion

This research concerns pricing and risk measurement of CDOs. We illustrate how to apply a Lévy distribution/process named Meixner to the standard models for pricing CDOs. Moreover, a comparative analysis among the copula models i.e. Gaussian copula model, double-t copula model, Meixner copula model, and the correlated structural models i.e. correlated Brownian motion structural model and correlated Meixner structural model is performed. Among the copula models, we hypothesize that MC is the best while DC is the second, and the last is GC. The reason is that Meixner distribution has fat-tail and skewness; hence, its associate copula provides asymmetric dependence structure which is more flexible than the others. By contrast, double-t copula provides only symmetric one and Gaussian copula has none of these features. Furthermore, we hypothesize that MS is better than BS because of the similar reason. MS assumes that firm values are driven by jumps which can be decomposed to jumps from the market factor and from the firm specific; whereas, BS assumes that firm values and also its common factor and firm specific parts continuously diffuse over time. Modeling the market factor with jumps can be interpreted as unanticipated market crashes/booms while modeling it with Brownian motion results in predictable market movement which is not realistic. Moreover, modeling the firm specific with jumps is interpreted as unanticipated independent movement of each firm caused by, for instance, fraud, announcement, operation results, etc. in which the Brownian motion is not suitable. In addition, we hypothesize that both copula and correlated structural models are approximately equivalent in performance since in this setting both types of models can be calibrated from the market quotes. For risk measurement, we think the risk measures computed from each of the different models are approximately the same.

According to the results, Meixner-based models have the edge over the other models. 12 out of 12 cases revels that Meixner copula model is the best while 6 out of 6 cases reveals that correlated Meixner structural model is the better. Meixner copula model has negative β in all 6 cases which means that the dependence structure is more intense in lower tail while less intense in upper tail. This is sensible because the default seems more correlated in downturn and less correlated in the normal situations.

Correlated Meixner structural model has also negative β; this suggests that a firm has more likelihood of large down-sided jumps than large up-sided jumps, and it can be attribute to the market factor and the idiosyncratic component. Thus, neglecting the tail dependence and/or fat-tail in pricing CDOs is unacceptable. On top of that, considering also asymmetric dependence structures, particularly left skewness, can enrich the performance. As we have shown in Copula, double-t copula model which include only symmetric dependence structure are better than Gaussian copula model while Meixner copula model is the best. Similarly, structural model used the correlated Meixner structural model is the better than the standard one.

Next, the answer to the question which type of models are better between copula models and correlated structural models is vary among the cases. Copula models are better than correlated structural models 15 out of 36 cases while the reverse is true for 18 out of 36 cases. The other 3 cases are insignificant. It seems that, in pricing CDOs, the performance of the models does not directly depend on how we model the marginals i.e. reduced-form approach or first passage time approach, nor which approach to be used between copula models and correlated structural models. It depends apparently on whether the dependence structure takes into account the extreme events; the models taking account of fat-tail/tail dependence are superior to the others.

For risk measurement, different models provide different risk measures. It depends majorly on how the extreme events are considered. DC considerably provides higher risk measures in most cases; whereas, GC and BS claim higher likelihood of losses in the earlier tranche and fall sharply afterward. MC's and MS's provides risk measures that tend to be less than DC's but still intuitive in most cases. Thus, including tail dependence and/or fat-tail result(s) in more sensible risk measures.

In summary, in addition to fat-tails, skewness is also an important property which should also be considered when valuing the credit dependent securities. Imposing a negative skewness in the distribution of market factor results in more dependence in lower tail of its associated copula and also more likelihood of common large-downsided jumps in its associated process. These extensions using Meixner distribution/process are more realistic and improve the performance of CDO pricing models. Moreover, the risk measures have more validity because they are based on the portfolio distribution that fits the CDO capital structure more than the standard approaches.

6.2 Recommendation

There are many issues relevant to modeling CDOs that is interesting and important left for further studies. First of all, the performance of Meixner-based models in pricing CDOs should be tested out of samples. Secondly, other Lévy models should also be investigated and compared including in-samples and out-of-samples with Meixner models. Thirdly, the copula models are known as the static models; there are some extensions that makes them dynamic, called dynamic Lévy copula models. Fourthly, it is interesting to investigate Meixner models with stochastic correlation. Fifthly, Meixner distribution and process should be further investigated in other relevant areas such as bonds, CDSs, and non-standard credit derivative such as CDO-Squard, options on CDOs and forward starting CDOs etc. Finally, if the Meixner's parameters can be estimated from other sources, it will increase the fascination of this model.

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APPENDICES

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APPENDIX A

COLLATERALIZED DEBT OBLIGATIONS

A.1 What are CDOs?

For illustration, CDX NA IG is one of the most popular reference portfolios for single tranche trading. It consists of 125 equally weighted North American investment grade companies. It is divided into five standard tranches: 0-3%, 3-7%, 7-10%, 10-15% and 15-30%. Suppose an initial notional principal of the underlying portfolio is \$100 million. The initial notional principle of each tranche will become \$3, \$4, \$3, \$5 and \$15 million respectively. When the cumulative losses incur at 3%, only the investors holding the 0-3% tranche, called equity tranche, are responsible for those losses. Suppose now the cumulative losses increase to 8%. The investors holding the second tranche, called mezzanine tranche, are responsible for the additional cumulative losses of 4%, while the ones holding the third tranche are responsible for the remaining 1% of those losses. Up to this point, the outstanding notional principals of the equity and mezzanine tranches are both zero whilst the third tranche has \$2 out of \$3 million left; however, the others are not affected.

On the other hand, the investors will receive the premiums at the end of each period. The premiums will be calculated from the outstanding principle of each tranche at that time[1]. For example, suppose the third tranche spread is 35.5 basis points per year and the payments are made semi-annually. The outstanding notional principle of the third tranche is \$2 million left. At the end of this period, the investors holding this tranche will receive the premium of $.00355 \times .5 \times \$2$ million = \$3,550.

It should be noted that the cumulative losses that we are considering here is not the default rate in the portfolio. It is, in fact, the loss rate. By assuming the recovery rate of 40%, when 10 out of 100 firms have defaulted, the cumulative losses should be 6% not 10%.

For more details about the mechanics of CDOs, see *Synthetic CDO Primer* (2005); Hull (2006); Elizalde (2005).

[1]In addition to the regular premiums, there is also the accrual premiums which compensate for the protection period just before the firms default.

A.2 CDO Pricing Models

To price CDOs, there are, generally, three major steps as follows: estimate the default probability of each firm; combine these default probabilities together to derive a portfolio loss distribution; and finally, compute the CDO spreads. Let us explain the first step first. The individual default probability estimation can be separated into two major classes—an intensity-based model, also called a reduced-form model, and a structural model. For the reduced-form model, we imply the default probability directly from the market quotes of such defaultable securities as bonds or CDSs by modeling a hazard rate function. On the other hand, the structural model estimates the probability of firm default from the likelihood of an event that the firm's asset value falls below a threshold called a default barrier. Secondly, to derive the portfolio loss distribution, there are many models ranging from a closed-form formula, semi-analytical models to Monte Carlo simulation approaches depending on the level of assumptions. Under some assumptions, Copula, a statistical tool, can be used to join the default probabilities together instead of a Monte Carlo simulation, and so do some other techniques. Ultimately, the fair CDO spreads can be computed in the same way as an interest-rate derivative.

There are many models ranging from an analytical model, semi-analytical models to Monte Carlo simulation approaches depending on the level of assumptions. Put it another way, they can also be separated into two major classes of models—the structural models and the intensity-based models, also called the reduced-form models. The structural models use firms' asset value and their default barrier to estimate the default probabilities, whereas the reduced-form models use the market quotes of some defaultable securities such as bonds and/or CDSs. The copula approach is ones of the reduced-form models. Despite many models, the objectives are the same which is to construct the distribution function of the portfolio losses in order to compute the fair CDO spread.

APPENDIX B

MEIXNER DISTRIBUTION/PROCESS

B.1 Meixner Distribution

Alternative to the standard normal distribution, Meixner distribution can have skewness and excess kurtosis. Meixner distribution is an infinitely divisible distribution. That means for each positive integer n, the Meixner characteristic function $\phi(u)$ is the *n*-th power of a Meixner characteristic function i.e. $\phi(u;\alpha,\beta,\delta,\mu) = [\phi(u;\alpha,\beta,\delta/n,\mu/n)]^n$. Hence, it is stable under convolution i.e. the distribution function of the sum of independent Meixner random variables is still the Meixner distribution; see Schoutens (2003).

Meixner distribution is one of the not many Lévy distribution that has an analytical probability density function (PDF). The PDF of Meixner distribution is

$$f_{Meixner}(x;\alpha,\beta,\delta,\mu) = \frac{(2cos(\beta/2))^{2\delta}}{2\alpha\pi\Gamma(2\delta)}\exp\left(\frac{\beta(x-\mu)}{\alpha}\right)\left|\Gamma\left(\delta+\frac{i(x-\mu)}{\alpha}\right)\right|^2,\ x\in\mathbb{R}$$

where $\alpha > 0, -\pi < \beta < \pi, \delta > 0$ and $\mu \in \mathbb{R}$. In practice, we found that calculating the probability density via an exponential of logarithmized PDF gives more arithmetically accurate since it prevents from register overflows.

According to Schoutens (2003) and Albrecher et al. (2006), the moments of this distribution is shown in table B.1 . In the case we want the Meixner distribution to have zero mean and unit variance, we can set the parameters $\delta = \frac{1+\cos(\beta)}{\alpha^2}$ and $\mu = -\frac{\sin(\beta)}{\alpha}$. Now, we have two parameters left which are α that is mainly used to control kurtosis or fat-tail, and β mainly used to control skewness. However, changing one parameter affects both kurtosis and skewness as in table B.2

B.2 Meixner Process

Since Meixner distribution is infinitely divisible, we can find its associated Lévy process, called Meixner process. Strictly speaking, a Meixner process $X^{(Meixner)} = \{X_t^{(Meixner)}, t \geq 0\}$ is a stochastic process in which $X_0^{(Meixner)} = 0$, and it has an independent and stationary increments over $[t, t+\Delta t]$ i.e. $X_{t+\triangle t}^{(Meixner)} - X_t^{(Meixner)}$ dis-

Table B.1: The moments of the Meixner distribution

	Meixner(α,β,δ,μ)
mean	$\mu + \alpha\delta\tan(\beta/2)$
variance	$\alpha^2\delta/(1+\cos(\beta))$
skewness	$\sin(\beta/2)\sqrt{2/\delta}$
kurtosis	$3 + (2-\cos(\beta))/\delta$

Table B.2: The moments of the zero-mean and unit-variance Meixner distribution

	Meixner(α,β,$\frac{2\cos^2(\beta/2)}{\alpha^2}$,$-\frac{\sin(\beta)}{\alpha}$)
mean	0
variance	1
skewness	$\alpha\tan(\beta/2)$
kurtosis	$3 + (\frac{3-2\cos^2(\beta/2)}{2\cos^2(\beta/2)})\alpha^2$

tributes as Meixner(α, β, $\delta\triangle t$,$\mu\triangle t$). Because of its infinitely divisible, the distribution of $X_t^{(Meixner)}$ follows Meixner(α, β, δt,μt).

Each of Lévy process has a triplet characteristics $[\gamma, \sigma^2, \nu\,(dx)]$. The first one is deterministic part; the second is Brownian components, and the last one is Lévy measure that indicates how the jumps occur. In case of Meixner process, there is no Brownian part, thereby being a pure jump process. Moreover, the first parameter in the Lévy triplet is

$$\gamma = \alpha\delta tan\,(\beta/2) - 2\delta\int_1^{\infty}\frac{sinh\,(\beta x/\alpha)}{sinh\,(\pi x/\alpha)}dx$$

and the Lévy measure is

$$\nu\left(dx\right)=\delta\frac{exp\left(\beta x/\alpha\right)}{xsinh\left(\pi x/\alpha\right)}dx$$

Meixner process is of infinite variation since $\int_{-1}^{+1}\left|x\right|\nu\left(dx\right)=\infty$. More detail on the Meixner distribution and process see Schoutens (2003); Albrecher et al. (2006); Schoutens (2002).

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APPENDIX C

COPULA

Copula is a *dependence structure* linking a number of standard uniform random variables together. The result is a multivariate distribution function. For n uniform random variables, $U_1, U_2, \ldots, U_n$, the joint distribution function C, defined as

$$C(u_1, u_2, \ldots, u_n; \Sigma) = Pr([U_1 \leq u_1, U_2 \leq u_2, \ldots, U_n \leq u_n]$$

where Σ is an arbitrary set of dependence parameters.

Because any distribution functions of a random variable distribute as standard uniform $F_i(X) \sim U_i$. Thus, the copula function can be used to join any marginal probability distribution functions together in order to generate a multivariate distribution function.

$$\begin{aligned} C(F_1(x_1), F_2(x_2), \ldots, F_n(x_n); \Sigma) &= Pr([U_1 \leq F_1(x_1), U_2 \leq F_2(x_2), \ldots, U_n \leq F_n(x_n)] \\ &= Pr([F_1^{-1}(U_1) \leq x_1, F_2^{-1}(U_2) \leq x_2, \ldots, F_1^{-1}(U_n) \leq x_n] \\ &= Pr([X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n] \\ &= F(x_1, x_2, \ldots, x_n) \end{aligned}$$

Product copula $C^{\perp}$, for example, is a dependence structure reflecting independence among margins.

$$C^{\perp}(u_1, u_2, \ldots, u_n) = u_1 u_2 \cdots u_n$$

Or equivalently,

$$F(x_1, x_2, \ldots, x_n) = C^{\perp}(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)) = F_1(x_1)F_2(x_2) \cdots F_n(x_n)$$

From Sklar (1959)'s theorem as explained in Cherubini et al. (2004), for a given n-dimensional multivariate distribution function F with margins $F_1, F_2, \ldots, F_n$, one can find a copula C by

$$C(u_1, u_2, \ldots, u_n) = F(F_1^{-1}(u_1), F_2^{-1}(u_2), \ldots, F_n^{-1}(u_n))$$

Therefore, for a multivariate Gaussian copula, we get

$$\begin{aligned} C^{Ga}(u_1, u_2, \ldots, u_n; \Sigma) &= \Phi_n\left(\Phi^{-1}(u_1), \Phi^{-1}(u_2), \ldots, \Phi^{-1}(u_n); \Sigma\right) \\ &= \int_{-\infty}^{\Phi^{-1}(u_1)} \cdots \int_{-\infty}^{\Phi^{-1}(u_n)} \frac{exp\left(-\frac{1}{2}\mathbf{x}^T\Sigma^{-1}\mathbf{x}\right)}{(2\pi)^{\frac{n}{2}}|\Sigma|^{\frac{1}{2}}} dx_1 \ldots dx_n \end{aligned}$$

For more details on copula theories in a statistical perspective see Nelsen (1999), see Cherubini et al. (2004) for how copulas be used in finance.

Copula, which can be used for deriving the joint default probability, can be applied to the CDO pricing models. For example in equation (3.5), if the probabilities that the survival time τ of each firm are less than time t_i are all equal to p_{t_i}, and providing that the default correlations are all the same denoted by a correlation matrix Σ_{t_i}. The probability that k out of N firms default will be

$$Pr(X = k) = \begin{pmatrix} N \\ k \end{pmatrix} Pr(\tau_1 \leq t_i, \ldots, \tau_k \leq t_i, \tau_{k+1} > t_i, \ldots, \tau_N > t_i; \Sigma_{t_i}) \quad \text{(C.1)}$$

We can derive a modified Gaussian copula to be used for this purpose.

$$C_k^{Ga^*}(u_1, u_2, \ldots, u_N; \Sigma) = \int_{-\infty}^{\Phi^{-1}(u_1)} \cdots \int_{-\infty}^{\Phi^{-1}(u_k)} \int_{\Phi^{-1}(u_{k+1})}^{+\infty} \cdots \int_{\Phi^{-1}(u_N)}^{+\infty} \frac{e^{\left(-\frac{1}{2}\mathbf{x}^T\Sigma^{-1}\mathbf{x}\right)}}{(2\pi)^{\frac{N}{2}}|\Sigma|^{\frac{1}{2}}} dx_1 \ldots dx_N$$

Therefore, the equation (C.1) becomes

$$Pr(X = k) = \begin{pmatrix} N \\ k \end{pmatrix} C_k^{Ga^*}(p_{t_t}, \ldots, p_{t_t}; \Sigma_{t_i})$$

Consequently, the portfolio loss distribution become

$$F_{t_i}(z) = Pr(X \leq \left\lfloor \frac{zN}{(1-R)} \right\rfloor) = \sum_{k=1}^{\lfloor zN/(1-R) \rfloor} \begin{pmatrix} N \\ k \end{pmatrix} C_k^{Ga^*}(p_{t_t}, \ldots, p_{t_t}; \Sigma_{t_i}) \quad \text{(C.2)}$$

APPENDIX D

A SIMPLIFIED CORRELATED STRUCTURAL MODEL

We explain the concepts of approximation, and the formulas are expressed. Note that this is only a conjecture. However, when we compare the CDO pricing performance of the approximation approach to the traditional one, the results in terms of mean absolute pricing error (MAPE) look promising. The verification in terms of proof is on progress.

Instead of using the correlated simulation approach as mentioned above, the Bernoulli mixture framework can be applied when some restrictions are imposed. If we assume that the path of the market factor $\vec{M}$ is given so that the firm values are conditionally independent, the conditional default probability of a firm before time t under first hitting time approach is

$$\begin{aligned} p(t|\vec{M}) &= \left\{\tau \leq t \mid \inf_{\tau} X(\tau) \leq K^*(\tau)\right\} \\ &= \left\{\tau \leq t \mid \inf_{\tau} \sqrt{\rho}M(\tau) + \sqrt{1-\rho}Z(\tau) \leq a + b\tau\right\} \\ &= \left\{\tau \leq t \mid \inf_{\tau} Z(\tau) \leq \frac{a + b\tau - \sqrt{\rho}M(\tau)}{\sqrt{1-\rho}}\right\} \end{aligned}$$

Therefore, the probability that k out of N firms default before and at time t is as follows

$$Pr(N,k,t|\vec{M}) = \begin{pmatrix} N \\ k \end{pmatrix} \left[p(t|\vec{M})\right]^k \left[1 - p(t|\vec{M})\right]^{N-k}$$

Hence, the portfolio loss distribution function is

$$F_t(z) = E_{\vec{M}}\left[\sum_{k=1}^{\lfloor zN/(1-R)\rfloor} \begin{pmatrix} N \\ k \end{pmatrix} \left[p(t|\vec{M})\right]^k \left[1 - p(t|\vec{M})\right]^{N-k}\right]$$

where $E_{\vec{M}}$ represents the expectation over all possible paths of the market factor.

Recall that $\vec{M}$ follows a Lévy process. One can find the portfolio loss distribution function by first sampling a number of paths of the market factor and then calculate that expected portfolio loss distribution function under all possible paths. Alternatively, if the end point of the market factor is given, the conditional portfolio loss distribution can be calculated in the expectation over all possible paths starting from 0 to that given end point. Then, we find the unconditional one by integrating over all the possible end points. Therefore, we can reformulate the portfolio loss distribution function as follows

$$F_t(z) = \int_{-\infty}^{+\infty} E_{B(\vec{M}_t)} \left[\sum_{k=1}^{\lfloor zN/(1-R) \rfloor} \begin{pmatrix} N \\ k \end{pmatrix} \left[p(t|B(\vec{M}_t)) \right]^k \left[1 - p(t|B(\vec{M}_t)) \right]^{N-k} \right] \phi(M_t) dM_t$$

where M_t is an arbitrary point at the end of the path, $B(\vec{M}_t)$ is a path starting from 0 and ending at M_t, and the distribution of M_t depends on t i.e. $M_t \sim N(0,t)$ in case of Brownian motion and $M_t \sim \phi_{Meixner}(1, \beta, \delta t)$ in case of Meixner.

To *approximate*, we replace the paths starting from 0 to any given points in time t with its expectation (as a line). Specifically, for time horizon t, the market factor at the time s before t is $M(s) = ms\,;\, s \leq t$. Hence, the proxy of asset return becomes.

$$X_j(s) = \sqrt{\rho} ms + \sqrt{1-\rho} Z_j(s) \quad ; \quad s \leq t$$

and the conditional default probability under time t is

$$\begin{aligned} p(t|m) &= \left\{ \tau \leq t \mid \inf_{\tau} X(\tau) \leq K(\tau)^* \right\} \\ &= \left\{ \tau \leq t \mid \inf_{\tau} \sqrt{\rho} m\tau + \sqrt{1-\rho} Z(\tau) \leq a + b\tau \right\} \\ &= \left\{ \tau \leq t \mid \inf_{\tau} Z(\tau) \leq \frac{a + \left(b - \sqrt{\rho} m\right)\tau}{\sqrt{1-\rho}} \right\} \\ &= \left\{ \tau \leq t \mid \inf_{\tau} Z(\tau) \leq \frac{a}{\sqrt{1-\rho}} + \frac{\left(b - \sqrt{\rho} m\right)}{\sqrt{1-\rho}} \tau \right\} \end{aligned}$$

As a consequence, the portfolio loss distribution under time t is

$$\begin{aligned} F_{t_i}(z) &= \int_{m=-\infty}^{m=+\infty} \sum_{k=1}^{\lfloor zN/(1-R) \rfloor} \begin{pmatrix} N \\ k \end{pmatrix} p_{t_i|m}^k (1 - p_{t_i|m})^{N-k} \phi_{t_i}(m)\, dm \\ p_{t_i|m} &= Pr \left\{ \tau \leq t \mid \inf_{\tau} Z(\tau) \leq \frac{a}{\sqrt{1-\rho}} + \frac{\left(b - \sqrt{\rho} m\right)}{\sqrt{1-\rho}} \tau \right\} \end{aligned} \quad \text{(D.1)}$$

where the probability distribution of m depends on $\frac{1}{t}$, i.e. $m \sim N(0, \frac{1}{t})$ in case of Brownian motion and $m \sim \phi_{Meixner}\left(\frac{1}{t}, \beta, \delta t\right)$.

Interestingly, if there is a function that inverts a default probability to a set of default barrier parameters i.e. $g^{-1}(p_t) = \{a, b, t\}$, then there will be a copula that represent this simplified correlated structural model. Assuming further that $g_l^{-1}(p_t)$ gives the l-th parameter e.g.$g_1^{-1}(p_t) = a$, one can the portfolio loss distribution as follows

$$F_{t_i}(z) = C(p_{t_i}; \rho)$$

where C is a copula that has the following expressions

$$\begin{aligned}
C(p_{t_i}) &= \int_{m=-\infty}^{m=+\infty} \sum_{k=1}^{\lfloor zN/(1-R) \rfloor} \begin{pmatrix} N \\ k \end{pmatrix} p_{t_i|m}^k (1-p_{t_i|m})^{N-k} \phi_{t_i}(m)\, dm \\
p_{t_i|m} &= Pr\left\{ \tau \leq t \mid \inf_{\tau} Z(\tau) \leq \frac{a}{\sqrt{1-\rho}} + \frac{\left(b-\sqrt{\rho}m\right)}{\sqrt{1-\rho}} \tau \right\} \\
a &= g_1^{-1}(p_t) \\
b &= g_2^{-1}(p_t) \\
t &= g_3^{-1}(p_t)
\end{aligned}$$

Note also that there are some ways, under some copulas such as Brownian motion,to normalize the time scale to $t \in [0,1]$ in the copula function so that the copula is standardized. One may call it First Passage Time Copula, or FPT Copula. Moreover, unlike copula models that join the marginals with an arbitrary dependence structure, It has economic underpinnings on the choice of dependence structure. However, this is only our conjecture; the proof is on progress.

D.1 Correlated-Brownian Motion Structural Model

Under the approximation correlated structure approach, the model has some similarities to copula models as follows

$$\begin{aligned}
F_{t_i}(z) &= \int_{m=-\infty}^{m=+\infty} \sum_{k=1}^{\lfloor zN/(1-R) \rfloor} \begin{pmatrix} N \\ k \end{pmatrix} p_{t_i|m}^k (1-p_{t_i|m})^{N-k} \phi^{Normal}\left(m; 0, \frac{1}{t_i}\right) dm \\
p_{t_i|m} &= Pr\left\{ \tau \leq t \mid \inf_{\tau} Z(\tau) \leq a' + b'\tau \right\} \\
&= \Phi\left(\frac{a'+b't}{\sqrt{t}}\right) + exp\left(-2a'b'\right) \Phi\left(\frac{a'-b't}{\sqrt{t}}\right) \qquad \text{(D.2)}
\end{aligned}$$

where $a' = \frac{a}{\sqrt{1-\rho}}$ and $b' = \frac{b-\sqrt{\rho}m}{\sqrt{1-\rho}}$. Since Harrison (1990) shows that the probability of first hitting the barrier has analytical formula as above, our simplified correlated structural model becomes semi-analytic as in copula models. This approach is much faster than the correlated simulation approach.

Table D.1 shows the MAPEs under the correlated simulation and the approximation approach for the case of Brownian motion. It reveals that our simplified approach is a good approximation for the traditional approach in pricing CDOs.

D.2 Correlated-Meixner Structural Model

Under the approximation correlated structural approach, the Meixner model becomes

$$\begin{aligned} F_{t_i}(z) &= \int_{m=-\infty}^{m=+\infty} \sum_{k=1}^{\lfloor zN/(1-R) \rfloor} \begin{pmatrix} N \\ k \end{pmatrix} p_{t_i|m}^k (1-p_{t_i|m})^{N-k} \phi^{Meixner}\left(m; \frac{1}{t_i}, \beta, \delta t_i\right) dm \\ p_{t_i|m} &= Pr\left\{\tau \leq t \mid \inf_{\tau} Z(\tau) \leq a' + b'\tau\right\} \end{aligned} \tag{D.3}$$

where Z is a Meixner process with parameters of $(1, \beta, \delta)$, $a' = \frac{a}{\sqrt{1-\rho}}$ and $b' = \frac{b-\sqrt{\rho}m}{\sqrt{1-\rho}}$.

Table D.1: Traditional BS vs Simplified BS

		5YS5	5YS6	7YS5	7YS6	10YS5	10YS6
Traditional BS	MAPE	84.68	78.84	156.72	146.60	255.14	235.55
	S.D.	(40.73)	(37.36)	(47.89)	(73.78)	(73.07)	(44.50)
Simplified BS	MAPE	63.32	50.62	141.93	121.21	231.50	206.82
	S.D.	(33.80)	(13.41)	(52.32)	(21.39)	(79.45)	(50.69)

Note: The number of sample paths in simulation is 20,000 per entity.

BIOGRAPHY

Mr. Kridsda Nimmanunta was born on September 3, 1983 in Bangkok. During 1989-2001, He attended Assumption College where he demonstrated not only competence in mathematics, physics and informatics but also substance in public contributions. In 2000, he was the first president of Assumption College Mathematical Olympiad Club (ACMO Club) and awarded a silver medal prize in Thailand Olympiad in Informatics. At the undergraduate level, he received a 1^{st} class honours degree in Computer Engineering from Chulalongkorn University in 2005. Subsequently, he won a honourable scholarship Chulalongkorn University Graduate Scholarship to Commemorate the 72^{nd} Anniversary of His Majesty King Bhumibol Adulyadej, and joined the Master of Science Program in Finance. As a graduate student, he worked as a research assistant, participating in many studies ranging from an investigation of stock returns to credit risk modeling. Thereafter, he was engrossed in investment science, management science, and the applications of mathematical and computational methods to the business realm.

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