CHAPTER I

PRELIMINARIES AND SOME REMARKS

Throughout this research, let $\mathbf{Z}$ and $\mathbf{Z}_m$ for a positive integer m denote the ring of integers and the ring of integers modulo m, respectively, and let each element of $\mathbf{Z}_m$ be donoted by $\bar{a}$ where $a \in \mathbf{Z}$.

For a ring R and a positive integer n, let $M_n(R)$ denote the full $n \times n$ matrix ring over R, that is, $M_n(R)$ is the ring of all $n \times n$ matrices over R under the usual addition and multiplication of matrices, and for $A \in M_n(R)$ and $i, j \in \{1, 2, \dots, n\}$, let A_{ij} denote the element of A in $i^{\underline{th}}$ row and $j^{\underline{th}}$ column. Let the zero matrix in $M_n(R)$ be denoted by $[0]_{n \times n}$.

A ring R is said to be *regular* if for every $a \in R$, there exists $x \in R$ such that $a = axa$.

Let R be a ring.

For subsets A and B of R, let AB denote the set of all finite sums of the form $\sum_{i=1}^{n} a_i b_i$ where $a_i \in A$ and $b_i \in B$. An additive subgroup Q of R is said to be a *quasi-ideal* of R if $RQ \cap QR \subseteq Q$. A quasi-ideal Q of R is said to *have the intersection property* if there exist a left ideal H and a right ideal K of R such that $Q = H \cap K$. If each quasi-ideal of R has the intersection property, we say that R *has the intersection property of quasi-ideals.*

We observe that the following statements hold.

(1) If Q is an additive subgroup of R, then RQ and QR are a left ideal and a right ideal of R, respectively.

(2) If Q is an additive subgroup of R such that $Q = RQ \cap QR$, then Q is a quasi-ideal of R which has the intersection property.

(3) The intersection of a left and a right ideal of R is a quasi-ideal of R.

(4) Every left ideal and every right ideal of R is a quasi-ideal of R having the intersection property.

(5) If R is commutative, then every quasi-ideal of R is an ideal of R, so R has the intersection property of quasi-ideals. In particular, these facts hold in any zero ring.

(6) If R has an identity, then every quasi-ideal Q of R, $Q = RQ \cap QR$, so R has the intersection property of quasi-ideals.

(7) Let R be a regular ring and Q a quasi-ideal of R. Then for every $a \in Q$, there exists $x \in R$ such that $a = axa$ which implies that $a \in RQ \cap QR$. Thus $Q = RQ \cap QR$. This fact is proved in [4]. Hence R has the intersection property of quasi-ideals.

Let n be a positive integer. For $i, j \in \{1, 2, \dots, n\}$, let $Q(i,j)$ be the set of all matrices $A \in M_n(R)$ such that $A_{k\ell} = 0$ if $k \neq i$ or $\ell \neq j$, that is, $Q(i,j)$ is the set of all matrices in $M_n(R)$ of the form

$$\begin{array}{cc} & \begin{matrix} & & & \underline{j^{th}} & & & \end{matrix} \\ \begin{matrix} \\ \\ \\ \underline{i^{th}} \\ \\ \\ \\ \end{matrix} & \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & a & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \end{array}$$

where $a \in R$. Then $Q(i,j)$ is an additive subgroup of $M_n(R)$ for all $i, j \in \{1, 2, \dots, n\}$.

Let $k, \ell \in \{1, 2, \dots, n\}$. For $A \in M_n(R)$ and $B \in Q(k, \ell)$, we have

$$AB = \begin{array}{c} \underline{\ell^{th}} \\ \begin{bmatrix} 0 & \cdots & 0 & A_{1k}B_{k\ell} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & A_{2k}B_{k\ell} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & A_{nk}B_{k\ell} & 0 & \cdots & 0 \end{bmatrix} \end{array}$$

and

$$BA = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \\ B_{k\ell}A_{\ell 1} & B_{k\ell}A_{\ell 2} & \cdots & B_{k\ell}A_{\ell n} \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{array}{c} \\ \\ \\ \underline{k^{th}} \\ \\ \\ \\ \end{array}.$$

Then each element of $M_n(R)Q(k, \ell)$ and each element of $Q(k, \ell)M_n(R)$ are of the forms

$$\begin{array}{c} \underline{\ell^{th}} \\ \begin{bmatrix} 0 & \cdots & 0 & x_{1\ell} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & x_{2\ell} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & x_{n\ell} & 0 & \cdots & 0 \end{bmatrix} \end{array} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \\ y_{k1} & y_{k2} & \cdots & y_{kn} \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{array}{c} \\ \\ \\ \underline{k^{th}} \\ \\ \\ \\ \end{array},$$

respectively where $x_{1\ell}, x_{2\ell}, \dots, x_{n\ell}, y_{k1}, y_{k2}, \dots, y_{k\ell} \in R$. It follows that every element of $M_n(R)Q(k, \ell) \cap Q(k, \ell)M_n(R)$ is of the form

$$\begin{array}{cc} & \begin{array}{ccccccc} & & & \ell^{th} & & & \end{array} \\ \begin{array}{c} \\ \\ \\ k^{th} \\ \\ \\ \\ \end{array} & \left[\begin{array}{ccccccc} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & z_{k\ell} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{array}\right] \end{array}$$

where $z_{k\ell} \in R$. Hence $M_n(R)Q(k, \ell) \cap Q(k, \ell)M_n(R) \subseteq Q(k, \ell)$.

Next, we shall show that $Q(k, \ell)$ is neither a left ideal nor a right ideal if R is not a zero ring and $n \geq 2$. Assume that R is not a zero ring and $n \geq 2$. Then there exist $a, b \in R$ such that $ab \neq 0$. Let $A, B, C, D \in M_n(R)$ be defined by

$$A = \begin{array}{c} \begin{array}{ccccccc} & & & k^{th} & & & \end{array} \\ \left[\begin{array}{ccccccc} 0 & \cdots & 0 & a & 0 & \cdots & 0 \\ 0 & \cdots & 0 & a & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & a & 0 & \cdots & 0 \end{array}\right] \end{array}, \quad B = \begin{array}{cc} \begin{array}{ccccccc} & & & \ell^{th} & & & \end{array} & \\ \left[\begin{array}{ccccccc} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & b & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{array}\right] & \begin{array}{c} \\ \\ \\ k^{th} \\ \\ \\ \\ \end{array} \end{array},$$

$$C = \begin{array}{cc} \begin{array}{ccccccc} & & & \ell^{th} & & & \end{array} & \\ \left[\begin{array}{ccccccc} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & a & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{array}\right] & \begin{array}{c} \\ \\ \\ k^{th} \\ \\ \\ \\ \end{array} \end{array} \text{ and } D = \begin{array}{cc} \left[\begin{array}{cccc} 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \\ b & b & \cdots & b \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{array}\right] & \begin{array}{c} \\ \\ \\ \ell^{th} \\ \\ \\ \\ \end{array} \end{array}.$$

Then $B, C \in Q(k, \ell)$ and

$$AB = \begin{array}{c} \begin{array}{ccccccc} & & & \ell^{th} & & & \end{array} \\ \begin{bmatrix} 0 & \cdots & 0 & ab & 0 & \cdots & 0 \\ 0 & \cdots & 0 & ab & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & ab & 0 & \cdots & 0 \end{bmatrix} \end{array} \quad \text{and} \quad CD = \begin{array}{cc} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \\ ab & ab & \cdots & ab \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} & \begin{array}{c} \\ \\ \\ k^{th} \\ \\ \\ \\ \end{array} \end{array}$$

which do not belong to $Q(k, \ell)$ since $ab \neq 0$. Hence $Q(k, \ell)$ is neither a left ideal nor a right ideal of $M_n(R)$.

From the above proof and the fact that a matrix ring over a zero ring is a zero ring, the following proposition is obtained.

Proposition 1.1. *Let R be a ring, n a positive integer and $n \geq 2$. Then the following statements are equivalent.*

(1) *R is not a zero ring.*

(2) *There exists a quasi-ideal of $M_n(R)$ which is neither a left ideal nor a right ideal.*

An example which shows that not every quasi-ideal of an arbitrary ring has the intersection property can be seen in [2].

The following theorem is well-known.

Theorem 1.2. ([1]) *For every ring R and a positive integer n, $M_n(R)$ is regular if and only if R is regular.*

The following theorem gives necessary and sufficient conditions for a quasi-ideal of a ring to have the intersection property. It was given in [5]. This theorem is a main tool of the next theorem.

Theorem 1.3. ([5]) *Let Q be a quasi-ideal of a ring R. Then the following statements are equivalent.*

(1) *Q has the intersection property.*

(2) $(RQ + Q) \cap (QR + Q) = Q$.

(3) $RQ \cap (QR + Q) \subseteq Q$.

(4) $QR \cap (RQ + Q) \subseteq Q$.

The following result given in [2] is a main tool of our research.

Theorem 1.4. ([2]) *Let R be a ring. Then R has the intersection property of quasi-ideals if and only if for any finite subset X of R,*

$$RX \cap (\mathbb{Z}X + XR) \subseteq \mathbb{Z}X + (RX \cap XR).$$

We know that in a semigroup S, if e is a left identity of S and f is a right identity of S, then $e = f$. This implies that if a ring R has more than one left [right] identity, then R has no right [left] identity.

The following theorem is obtained from [4].

Theorem 1.5. ([4]) *If a ring R has a left identity or a right identity, then R has the intersection property of quasi-ideals.*

We give the next proposition as an application of Theorem 1.5.

Proposition 1.6. *Let R be a ring with identity 1, $|R| > 1$, n a positive integer and $n > 1$. For $i \in \{1, 2, \dots, n\}$, let $R(i)$ denote the subring of $M_n(R)$ consisting of all matrices $A \in M_n(R)$ such that $A_{kj} = 0$ for all $k, j \in \{1, 2, \dots, n\}$ and $k \neq i$, that is, $R(i)$ is the subring of all matrices in $M_n(R)$ of the form*

$$\begin{array}{cc} \begin{array}{c} \\ \\ \\ i^{\underline{th}} \\ \\ \\ \\ \end{array} & \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \end{array}$$

where $a_{i1}, a_{i2}, \ldots, a_{in} \in R$. *For* $j \in \{1, 2, \ldots, n\}$, *let* $C(j)$ *denote the subring of* $M_n(R)$ *consisting of all matrices* $A \in M_n(R)$ *such that* $A_{ik} = 0$ *for all* $i, k \in \{1, 2, \ldots, n\}$ *and* $k \neq j$, *that is,* $C(j)$ *is the subring of all matrices in* $M_n(R)$ *of the form*

$$\begin{array}{ccccccc} & & & j^{\underline{th}} & & & \\ \end{array}$$
$$\begin{bmatrix} 0 & \cdots & 0 & a_{1j} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & a_{2j} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & a_{nj} & 0 & \cdots & 0 \end{bmatrix}$$

where $a_{1j}, a_{2j}, \ldots, a_{nj} \in R$.

Then the following statements hold.

(1) *For* $i \in \{1, 2, \ldots, n\}$, (1.1) *if* $A \in R(i)$ *is such that* $A_{ii} = 1$, *then* A *is a left identity of* $R(i)$, (1.2) $R(i)$ *has no right identity and* (1.3) $R(i)$ *has the intersection property of quasi-ideals.*

(2) *For* $j \in \{1, 2, \ldots, n\}$, (2.1) *if* $A \in C(j)$ *is such that* $A_{jj} = 1$, *then* A *is a right identity of* $C(j)$, (2.2) $C(j)$ *has no left identity and* (2.3) $C(j)$ *has the intersection property of quasi-ideals.*

Proof. (1) Let $k \in \{1, 2, \ldots, n\}$. If $A \in R(k)$ is such that $A_{kk} = 1$, then A is a left identity of $R(k)$ since for $B \in R(k)$,

$$AB = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ A_{k1} & \cdots & A_{k,k-1} & 1 & A_{k,k+1} & \cdots & A_{kn} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{array}{c} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \\ B_{k1} & B_{k2} & \cdots & B_{kn} \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \\ k^{\underline{th}} \\ \\ \\ \\ \end{array}$$

$$= \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \\ B_{k1} & B_{k2} & \cdots & B_{kn} \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{array}{l} \\ \\ \\ k^{\underline{th}} \\ \\ \\ \\ \end{array} = B.$$

Then (1.1) is proved. Since $|R| > 1$ and $n \geq 2$, $R(k)$ has more than one left identity which implies that $R(k)$ has no right identity. Then (1.2) holds. By Theorem 1.5, $R(k)$ has the intersection property of quasi-ideals. Then we have (1.3).

(2) Let $\ell \in \{1, 2, \ldots, n\}$. If $A \in C(\ell)$ is such that $A_{\ell\ell} = 1$, then A is a right identity of $C(\ell)$ since for $B \in C(\ell)$,

$$BA = \begin{array}{c} \begin{array}{ccccccc} & & & \ell^{\underline{th}} & & & \end{array} \\ \begin{bmatrix} 0 & \cdots & 0 & B_{1\ell} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & B_{2\ell} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & B_{n\ell} & 0 & \cdots & 0 \end{bmatrix} \end{array} \begin{array}{c} \begin{array}{ccccccc} & & & \ell^{\underline{th}} & & & \end{array} \\ \begin{bmatrix} 0 & \cdots & 0 & A_{1\ell} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & A_{\ell-1,\ell} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & A_{\ell+1,\ell} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & A_{n\ell} & 0 & \cdots & 0 \end{bmatrix} \end{array}$$

$$
= \begin{array}{c}
\begin{array}{ccccccc} & & & \ell th & & & \end{array} \\
\begin{bmatrix} 0 & \cdots & 0 & B_{1\ell} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & B_{2\ell} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & B_{n\ell} & 0 & \cdots & 0 \end{bmatrix}
\end{array} = B.
$$

Then (2.1) is proved. Since $|R| > 1$ and $n \geq 2$, $C(\ell)$ has more than one right identity which implies that $C(\ell)$ has no left identity. Then (2.2) holds. By Theorem 1.5, $C(\ell)$ has the intersection property of quasi-ideals. Then we have (2.3). □