CHAPTER II

REAL ANALYTIC HOMOMORPHISMS

It has been shown that every nonzero real analytic homomorphism of the multiplicative semigroup $\mathbb{R}$ to $\mathbb{R}$ taking 0 to 0 is of the form x^m $(=(\det x)^m)$ for some $m \in \mathbb{N}$, see [5,p.60]. In fact, the proof is the same as the proof given at the beginning of chapter I.

In the previous chapter we have shown that all nonzero complex analytic homomorphisms of the multiplicative semigroup $M(n,\mathbb{C})$ to $\mathbb{C}$ taking 0 to 0 is of the form $(\det A)^m$ for some $m \in \mathbb{N}$, for all $A \in M(n,\mathbb{C})$. We shall now prove that the above result is true in the real case also. First we shall need two lemmas.

<u>Lemma 2.1</u> Let $A \in M(n,\mathbb{R})$. Then the characteristic polynomial of A is the product of its invariant factors.

<u>Proof</u>. Let us recall that the characteristic polynomial of $A = \det(XI-A)$ is the same as the characteristic polynomial of any matrix similar to A [4, theorem 1.2, p.279], also the invariant factors of A are identical with those of matrices similar to A [4, cor.to theorem 4.1, p.320].

Let $A \in M(n,\mathbb{R})$ have invariant factors $m_1(x),\dots,m_t(x)$. Then for all $j = 1,2,\dots,t$,

$$m_j(x) = P_{j1}(x)^{\ell_{j1}} P_{j2}(x)^{\ell_{j2}} \dots P_{jk_j}(x)^{\ell_{jk_j}}$$

where $p_{j1}(x),\ldots,p_{jk_j}(x)$ are distinct monic nonconstant irreducible polynomials in $\mathbb{R}[x]$ and $\ell_{j1},\ldots,\ell_{jk_j}$ are natural numbers. Since the only irreducible, nonconstant polynomials over the field of real numbers are either of degree 1 or 2 (by chapter 0, p. 3), $p_{ji}(x) = x-\lambda_{ji}$ or $x^2+b_{ji}x+c_{ji}$ where $b_{ji}^2-4c_{ji}<0$, $\lambda_{ji}, b_{ji}, c_{ji} \in \mathbb{R}$ for all $ji \in \{11,\ldots,1k_1,\ldots,t1,\ldots,tk_t\}$.

Suppose that

$$m_1(x) = (x^2+b_{11}x+c_{11})^{\ell_{11}}\ldots(x^2+b_{1q_1}x+c_{1q_1})^{\ell_{1q_1}}(x-\lambda)_{1(q_1+1)}^{\ell_{1(q_1+1)}}\ldots(x-\lambda_{1k_1})^{\ell_{1k_1}},$$

$$\vdots$$

$$m_t(x) = (x^2+b_{t1}x+c_{t1})^{\ell_{t1}}\ldots(x^2+b_{tq_t}x+c_{tq_t})^{\ell_{tq_t}}(x-\lambda)_{t(q_t+1)}^{\ell_{t(q_t+1)}}\ldots(x-\lambda_{tk_t})^{\ell_{tk_t}},$$

where $0 \leq q_j \leq k_j$ for all $j \in \{1,2,\ldots,t\}$. From chapter 0 p. 10 we have that there exists an invertible matrix B such that

$$BAB^{-1} = \begin{bmatrix} D(p_{11}(x)^{\ell_{11}}) & 0 & \ldots & 0 \\ 0 & D(p_{12}(x)^{\ell_{12}}) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & D(p_{tk_t}(x)^{\ell_{tk_t}}) \end{bmatrix}.$$

The characteristic polynomial of $BAB^{-1} = \det(xI-BAB^{-1}) = \det(xI-D(p_{11}(x)^{\ell_{11}}))$. $\det(xI-D(p_{12}(x)^{\ell_{12}}))\ldots\det(xI-D(p_{tk_t}(x)^{\ell_{tk_t}}))$. Since we have that

$$D(p_{ji}(x)^{\ell_{ji}}) = \begin{bmatrix} 0 & 1 & 0 & 0 & & & & & & \\ -c_{ji} & -b_{ji} & 1 & 0 & & & & & \bar{0} & \\ & & 0 & 1 & 0 & 0 & & & & \\ & & -c_{ji} & -b_{ji} & 1 & 0 & & & & \\ & & & & & \ddots & & & & \\ & & & & & & 0 & 0 & & \\ & & & & & & 1 & 0 & & \\ & \bar{0} & & & & & 0 & 1 & & \\ & & & & & & -c_{ji} & -b_{ji} & & \end{bmatrix}$$

in which the submatrix $\begin{bmatrix} 0 & 1 \\ -c_{ji} & -b_{ji} \end{bmatrix}$ occurs ℓ_{ji} times on the diagonal for all $ji \in \{11,\ldots,1q_1,\ 21,\ldots,2q_2,\ldots,\ t1,\ldots,tq_t\}$,

$$\det(xI-D(p_{ji}(x)^{\ell_{ji}}))=\det \begin{bmatrix} x & -1 & 0 & 0 & & & & \\ c_{ji} & x+b_{ji} & -1 & 0 & & & \bar{0} & \\ & & x & -1 & 0 & 0 & & \\ & & c_{ji} & x+b_{ji} & -1 & 0 & & \\ & & & & \ddots & & & \\ & & & & & & 0 & 0 \\ & & & & & & -1 & 0 \\ & & \bar{0} & & & & x & -1 \\ & & & & & & c_{ji} & x+b_{ji} \end{bmatrix}$$

$$= (x^2 + b_{ji}x + c_{ji})^{\ell_{ji}} \qquad \text{(from chapter 0 page 6)}$$

Since we also have that

$$D(p_{ji}(x)^{\ell_{ji}}) = \begin{bmatrix} \lambda_{ji} & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ & & \ddots & 1 \\ 0 & \cdots & \cdots & \lambda_{ji} \end{bmatrix}$$

in which λ_{ji} occurs ℓ_{ji} times on the diagonal, 1's on the super diagonal and 0's elsewhere for all $ji \in \{1(q_1+1),\ldots,1k_1,2(q_2+1),\ldots,2k_2,\ldots,$ $\ldots,t(q_t+1),\ldots,tk_t\}$,

$$\det(xI-D(p_{ji}(x)^{\ell_{ji}})) = \det \begin{bmatrix} x-\lambda_{ji} & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ & & \ddots & -1 \\ 0 & \cdots & \cdots & x-\lambda_{ji} \end{bmatrix}$$

$$= (x-\lambda_{ji})^{\ell_{ji}} \quad \text{(from chapter 0 page 6)} .$$

Therefore,

$$\det(xI-BAB^{-1}) = [(x^2+b_{11}x+c_{11})^{\ell_{11}}\cdots(x^2+b_{1q_1}x+c_{1q_1})^{\ell_{1q_1}}(x-\lambda_{1(q_1+1)})^{\ell_{1(q_1+1)}}\cdots$$

$$\cdots(x-\lambda_{1k_1})^{\ell_{1k_1}}]\cdots[(x^2+b_{t1}x+c_{t1})^{\ell_{t1}}\cdots$$

$$\cdots(x^2+b_{tq_t}x+c_{tq_t})^{\ell_{tq_t}}(x-\lambda_{t(q_t+1)})^{\ell_{t(q_t+1)}}\cdots$$

$$\cdots(x-\lambda_{tk_t})^{\ell_{tk_t}}]$$

$$= m_1(x)\cdots m_t(x).$$

Hence, the characteristic polynomial of $BAB^{-1} = m_1(x)\ldots m_t(x)$, the product of invariant factors of A. Since A and BAB^{-1} have the some characteristic polynomial, the characteristic polynomial of A is then the product of its invariant factors. #.

Lemma 2.2 Let $\Phi : M(2,\mathbb{R}) \to \mathbb{R}$ be such that ϕ is a real analytic multiplicative homomorphism and $\phi(0) = 0$. Then $\phi \equiv 0$ or there is an $m \in \mathbb{N}$ such that $\phi(A) = (\det A)^m$ for all $A \in M(2,\mathbb{R})$.

Proof. Since Φ is analytic and $\phi(0) = 0$,

$$\phi\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right) = \sum_0^\infty c_{ijk\ell}x^i y^j z^k w^\ell \quad \text{for all } x,y,z,w \text{ in some}$$

neighborhood U of 0 and $c_{0000} = 0$.

We have that $\begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} s & t \\ u & v \end{bmatrix} = \begin{bmatrix} xs+yu & xt+yv \\ zs+wu & zt+wv \end{bmatrix}$, so

$$\Phi\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} s & t \\ u & v \end{bmatrix}\right) = \Phi\left(\begin{bmatrix} xs+yu & xt+yv \\ zs+wu & zt+wv \end{bmatrix}\right)$$

$$= \sum_0^\infty c_{ijk\ell}(xs+yu)^i(xt+yv)^j(zs+wu)^k(zt+wv)^\ell.$$

Now
$$\Phi\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} s & t \\ u & v \end{bmatrix}\right) = \Phi\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right)\Phi\left(\begin{bmatrix} s & t \\ u & v \end{bmatrix}\right)$$

$$= \left(\sum_0^\infty c_{ijk\ell}x^i y^j z^k w^\ell\right)\left(\sum_0^\infty c_{ijk\ell}s^i t^j u^k v^\ell\right)$$

$$= \sum_0^\infty c_{ijk\ell}\cdot c_{mnpq}x^i y^j z^k w^\ell s^m t^n u^p v^q .$$

Thus, $\sum_{0}^{\infty} c_{ijk\ell}\,(xs+yu)^i(xt+yv)^j(zs+wu)^k(zt+wv)^\ell$

(2.1)

$$= \sum_{0}^{\infty} c_{ijk\ell} \cdot c_{mnpq} x^i y^j z^k w^\ell s^m t^n u^p v^q .$$

By considering the LHS. of (2.1) we have that for each term in the power series $\deg(x) + \deg(z) = \deg(s) + \deg(t)$, so $c_{ijk\ell} \cdot c_{mnpq} = 0$ if $i+k \neq m+n$. This implies that for all $i,j,k,\ell \in \mathbb{N} \cup \{0\}$ $c_{ijk\ell} \cdot c_{ijk\ell} = 0$ if $j \neq k$. Hence

(2.2) $$c_{ijk\ell} = 0 \qquad \text{for all } j \neq k.$$

On the LHS of (2.1) the coefficient of the term $(yu)^n(zt)^\ell$ is $c_{n00\ell}$. On the RHS of (2.1) the corresponding coefficient of this term is $c_{0n\ell 0} \cdot c_{0\ell n0}$, hence $c_{n00\ell} = c_{0n\ell 0} \cdot c_{0\ell n0}$ $\forall n, \ell \in \mathbb{N} \cup \{0\}$. If $n \neq \ell$, then $c_{0n\ell 0} = 0$ by (2.2). Therefore,

(2.3) $$c_{n00\ell} = 0, \qquad \forall n \neq \ell .$$

On the LHS of (2.1) the coefficient of the term $(xs)^n(wv)^n$ is c_{n00n}. On the RHS of (2.1) the corresponding coefficient of this term is $c_{n00n} \cdot c_{n00n}$. Therefore,

(2.4) $$c_{n00n} = c_{n00n}^2 \qquad \forall n \in \mathbb{N} \cup \{0\}.$$

Let $m \neq n$. On the LHS of (2.1) the coefficient of the term $x^n w^n s^m v^m$ is zero since there is no z and t factors in this term. On the RHS the corresponding coefficient of this term is $c_{n00n} \cdot c_{m00m}$. Therefore,

(2.5) $$c_{n00n} \cdot c_{m00m} = 0, \qquad \forall m \neq n.$$

On the LHS of (2.1) the coefficient of $(xs)^n(yv)^k(zs)^k(wv)^\ell$ is $c_{nkk\ell}$.

On the RHS of (2.1) the corresponding coefficient of this term is

$c_{nkk\ell} \cdot c_{n+k,0,0,k+\ell}$.

Therefore,

(2.6) $\quad c_{nkk\ell} = c_{nkk\ell} \cdot c_{n+k,0,0,k+\ell} \qquad \forall\, n,k,\ell \in \mathbb{N} \cup \{0\}$, and if $n \neq \ell$,

then $n+k \neq \ell+k$, hence $c_{n+k,0,0,k+\ell} = 0$ by (2.3). So

(2.7) $\quad c_{nkk\ell} = 0 \qquad \forall\, n,k,\ell \in \mathbb{N} \cup \{0\}$ such that $n \neq \ell$.

Now we shall only consider c_{nkkn} for $n,k \in \mathbb{N} \cup \{0\}$ for otherwise they are all zero. By (2.4), $c_{n00n} = 0$ or 1 $\forall n \in \mathbb{N} \cup \{0\}$. If $c_{n00n} = 0$ $\forall n$, then by (2.6) $c_{nkkn} = 0$ $\forall\, n,k \in \mathbb{N} \cup \{0\}$. Therefore $\Phi \equiv 0$. Suppose that there is an $n \in \mathbb{N} \cup \{0\}$ such that $c_{n00n} \neq 0$. Since $c_{0000} = 0$, $n \in \mathbb{N}$; i.e., there is $n \in \mathbb{N}$ such that $c_{n00n} \neq 0$. Then $c_{n00n} = 1$. By (2.5), $c_{m00m} = 0$ $\forall\, m \neq n$. Because $c_{mkkm} = c_{mkkm} \cdot c_{m+k,0,0,m+k}$, then $c_{mkkm} = 0$ for all $m, k \in \mathbb{N} \cup \{0\}$ such that $m+k \neq n$. So

$$\Phi\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right) = c_{n00n}x^n w^n + c_{n-1,1,1,n-1}x^{n-1}yzw^{n-1} + \ldots$$

$$\ldots + c_{1,n-1,n-1,1}xy^{n-1}z^{n-1}w + c_{0nn0}y^n z^n .$$

(Note that now we have a finite sum so there is no convergence problem. Thus the above equation is true for all matrices in $M(2,\mathbb{R})$).

Claim that $c_{n-k,k,k,n-k} = (-1)^k\binom{n}{k}$ for all $k = 0,1,2,\ldots,n$.

Suppose that $k = 0$, then $c_{n00n} = 1$ and $(-1)^0\binom{n}{0} = \frac{1.n!}{(n-0)!0!} = 1$, so done

Assume that $c_{n-\ell,\ell,\ell,n-\ell} = (-1)^\ell\binom{n}{\ell}$ $\forall \ell < k$ and $k \geqslant 1$. We must show that $c_{n-k,k,k,n-k} = (-1)^k\binom{n}{k}$. Claim that on the LHS. of (2.1), the coefficient of $x^k s^k y^{n-k} u^{n-k} z^n t^n$ is $\binom{n}{n-k} c_{n00n} + \binom{n-1}{n-k} c_{n-1,1,1,n-1}$ $+\binom{n-2}{n-k} c_{n-2,2,2,n-2} + \ldots + \binom{n-k+1}{n-k} c_{n-k+1,k-1,k-1,n-k+1} + c_{n-k,k,k,n-k}$.

To see this, note that on the LHS. of (2.1) it is possible to get the expression of $x^k s^k y^{n-k} u^{n-k} z^n t^n$ from the terms $(x^k s^k y^{n-k} u^{n-k})(\ )(\ )(z^n t^n)$, $(x^{k-1} s^{k-1} y^{n-k} u^{n-k})(xt)(zs)(z^{n-1} t^{n-1})$, $(x^{k-2} s^{k-2} y^{n-k} u^{n-k})(x^2 t^2)(z^2 s^2)$ $(z^{n-2} t^{n-2}), \ldots, (xs\, y^{n-k} u^{n-k})(x^{k-1} t^{k-1})(z^{k-1} s^{k-1})(z^{n-(k-1)} s^{n-(k-1)})$, $(y^{n-k} u^{n-k})(x^k t^k)(z^k s^k)(z^{n-k} s^{n-k})$ and from no other terms since the other terms would involve v or w which do not appear. On the LHS. of (2.1) the corresponding coefficients of the above terms are $\binom{n}{n-k} c_{n00n}$, $\binom{n-1}{n-k} c_{n-1,1,1,n-1}$, $\binom{n-2}{n-k} c_{n-2,2,2,n-2}, \ldots, \binom{n-k+1}{n-k} c_{n-k+1,k-1,k-1,n-k+1}$ and $c_{n-k,k,k,n-k}$; respectively. Therefore the coefficient of $x^k s^k y^{n-k} u^{n-k} z^n t^n$ on the LHS. of (2.1) is $\binom{n}{n-k} c_{n00n} + \binom{n-1}{n-k} c_{n-1,1,1,n-1} + \binom{n-2}{n-k} c_{n-2,2,2,n-2} + \ldots$ $\ldots + \binom{n-k+1}{n-k} c_{n-k+1,k-1,k-1,n-k+1} + c_{n-k,k,k,n-k}$. On the RHS. of (2.1) the corresponding coefficient of this term is $c_{k,n-k,n,0} \cdot c_{k,n,n-k,0}$.

Hence,

$$\binom{n}{n-k} c_{n00n} + \binom{n-1}{n-k} c_{n-1,1,1,n-1} + \binom{n-2}{n-k} c_{n-2,2,2,n-2} + \ldots + \binom{n-k+1}{n-k} c_{n-k+1,k-1,k-1,n-k+1}$$

$+ c_{n-k,k,k,n-k} = c_{k,n-k,n,0} \cdot c_{k,n,n-k,0}$. Since $k \neq 0$, $n-k \neq n$, so by (2.2)

$c_{k,n-k,n,0} = 0$. Hence,

$$\binom{n}{n-k}c_{n00n} + \binom{n-1}{n-k}c_{n-1,1,1,n-1} + \binom{n-2}{n-k}c_{n-2,2,2,n-2} + \cdots$$

$$\cdots + \binom{n-k+1}{n-k}c_{n-k+1,k-1,k-1,n-k+1} + c_{n-k,k,k,n-k} = 0.$$

Therefore,

$$c_{n-k,k,k,n-k} = -\binom{n}{n-k}c_{n00n} - \binom{n-1}{n-k}c_{n-1,1,1,n-1} - \binom{n-2}{n-k}c_{n-2,2,2,n-2} - \cdots$$

$$\cdots - \binom{n-k+2}{n-k}c_{n-(k-2),k-2,k-2,n-(k-2)}$$

$$-\binom{n-k+1}{n-k}c_{n-(k-1),k-1,k-1,n-(k-1)}$$

$$= -\binom{n}{n-k} - (-1)^{1}\binom{n}{1}\binom{n-1}{n-k} - (-1)^{2}\binom{n}{2}\binom{n-2}{n-k} - \cdots - (-1)^{k-2}\binom{n}{k-2}\binom{n-k+2}{n-k} -$$

$$(-1)^{k-1}\binom{n}{k-1}\binom{n-k+1}{n-k}$$

$$= \frac{1}{(n-k)!}\Big[-\frac{n!}{k!} + \frac{n(n-1)!}{(k-1)!} - \frac{n!}{(n-2)!2!}\cdot\frac{(n-2)!}{(k-2)!} + \cdots -$$

$$(-1)^{k-2}\frac{n!}{(n-k+2)!(k-2)!}\cdot\frac{(n-k+2)!}{2!}$$

$$- (-1)^{k-1}\frac{n!}{(n-k+1)!(k-1)!}\cdot\frac{(n-k+1)!}{1!}\Big]$$

$$= \frac{n!}{(n-k)!}\Big[-\frac{1}{k!} + \frac{1}{(k-1)!} - \frac{1}{(k-2)!2!} + \frac{1}{(k-3)!3!} - \cdots$$

$$\cdots - (-1)^{k-2}\frac{1}{(k-2)!2!} - (-1)^{k-1}\frac{1}{(k-1)!}\Big]$$

$$= \frac{n!}{(n-k)!k!}\left[-1+k-\frac{k(k-1)}{2!}+\frac{k(k-1)(k-2)}{3!}-\ldots\right.$$

$$\left.\ldots-(-1)^{k-2}\frac{k(k-1)}{2!}-(-1)^{k-1}k\right]$$

$$= \frac{n!}{(n-k)!k!}\left[-1+\binom{k}{1}-\binom{k}{2}+\binom{k}{3}-\ldots-(-1)^{k-2}\binom{k}{2}-(-1)^{k-1}\binom{k}{1}\right]$$

Case 1. k is odd. Then

$$c_{n-k,k,k,n-k} = \frac{n!}{(n-k)!k!}\left[-1+\binom{k}{1}-\binom{k}{2}+\binom{k}{3}-\ldots+\binom{k}{2}-\binom{k}{1}\right]$$

$$= \frac{n!}{(n-k)!k!}\;(-1+0)$$

$$= \frac{-n!}{(n-k)!k!}$$

$$= (-1)^k\,\frac{n!}{(n-k)!k!}$$

$$= (-1)^k\binom{n}{k}\;.$$

Case 2. k is even. Then

$$c_{n-k,k,k,n-k} = \frac{n!}{(n-k)!k!}\left[-1+\binom{k}{1}-\binom{k}{2}+\binom{k}{3}-\ldots+\binom{k}{3}-\binom{k}{2}+\binom{k}{1}\right].$$

By the binomial theorem we have that for any x, $y \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$(x-y)^n = \binom{n}{0}x^n+(-1)\binom{n}{1}x^{n-1}y+\binom{n}{2}x^{n-2}y+\ldots+(-1)^{n-1}\binom{n}{1}xy^{n-1}+(-1)^n\binom{n}{0}y^n.$$

So, $0=(1-1)^n = \binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\ldots+(-1)^{n-1}\binom{n}{1}+(-1)^n\binom{n}{0}$. If n is even, then

$$0 = \binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\ldots+\binom{n}{2}-\binom{n}{1}+\binom{n}{0}$$

$$= -\binom{n}{0}+\binom{n}{1}-\binom{n}{2}+\ldots-\binom{n}{2}+\binom{n}{1}-\binom{n}{0}$$

$$= -1+\binom{n}{1}-\binom{n}{2}+\ldots-\binom{n}{2}+\binom{n}{1}-1.$$

Therefore, $$2 = \binom{n}{1}-\binom{n}{2}+\ldots-\binom{n}{2}+\binom{n}{1}.$$

Hence
$$\begin{aligned} c_{n-k,k,k,n-k} &= \frac{n!}{(n-k)!k!}\,(-1+2) \\ &= \frac{n!}{(n-k)!k!} \\ &= (-1)^k \frac{n!}{(n-k)!k!} \\ &= (-1)^k\binom{n}{k}. \end{aligned}$$

Consequently, we have the claim; i.e., $c_{n-k,k,k,n-k} = (-1)^k\binom{n}{k}$ $\forall\, k=1,2,\ldots,n$.

This shows that

$$\begin{aligned} \Phi\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right) &= x^n w^n+(-1)\binom{n}{1}x^{n-1}w^{n-1}yz+(-1)^2\binom{n}{2}x^{n-2}w^{n-2}y^2z^2+\ldots \\ &\quad \ldots+(-1)^{n-1}\binom{n}{n-1}xwy^{n-1}z^{n-1}+(-1)^n\binom{n}{n}y^nz^n \\ &= (xw-yz)^n \\ &= \left(\det \begin{bmatrix} x & y \\ z & w \end{bmatrix}\right)^n. \end{aligned}$$

This proves the lemma. #

Now we are ready for the theorem.

<u>Theorem 2.3</u> Let $\Phi : M(n,\mathbb{R}) \to \mathbb{R}$ such that Φ is a real analytic multiplicative homomorphism and $\Phi(0) = 0$. Then $\Phi \equiv 0$ or there is an $m \in \mathbb{N}$ such that $\Phi(A) = (\det A)^m$ for all $A \in M(n,\mathbb{R})$.

Proof. We proceed by induction over $n \in \mathbb{N}$. If $n = 1$, then $\Phi \equiv 0$ or there exists $m \in \mathbb{N}$ such that $\Phi(x) = x^m$ $(=(\det x)^m)$ for all $x \in \mathbb{R}$ [5,p.60]. Let $n > 1$ and assume that the assertion is true for all natural numbers $k < n$. We must show that it is true for n. But lemma 2.2 implies that it is true for $n = 2$. Thus we need only prove the assertion for $n > 2$.

Let $A \in M(n,\mathbb{R})$ have $m_1(x),\ldots,m_t(x)$ as invariant factors. Therefore,

$$m_j(x) \quad = \quad q_{j1}(x)^{\ell_{j1}} \cdots q_{jk_j}(x)^{\ell_{jk_j}}$$

where $q_{j1}(x),\ldots,q_{jk_j}(x)$ are distinct monic, nonconstant irreducible polynomials in $\mathbb{R}[x]$ and $\ell_{j1},\ldots,\ell_{jk_j}$ are natural numbers for all $j = 1,2,\ldots,t$. Since the only irreducible, nonconstant polynomials over the field of real numbers are either of degree 1 or 2, $q_{ji}(x) = x-\lambda_{ji}$ or $x^2+b_{ji}x+c_{ji}$ where $b_{ji}^2-4c_{ji} < 0$ and λ_{ji}, b_{ji} and $c_{ji} \in \mathbb{R}$ for all $ji \in \{n,\ldots,1k_1,21,\ldots,2k_2,\ldots\ldots,t1,\ldots,tk_t\}$.

Case 1. If $q_{ji}(x) = x-\lambda_{ji}$ for all ji, then all eigenvalues of A lie in $\mathbb{R} \subseteq \mathbb{C}$. By the same argument as in the proof of the theorem 1.3 we have that $\Phi \equiv 0$ or $\Phi(A) = (\det A)^m$ for some $m \in \mathbb{N}$.

Case 2. There exist j, i such that $q_{ji}(x) = x^2+b_{ji}x+c_{ji}$ with order $\ell_{ji} = 1$. Reorder the indices so that $j,i = 1$. From chapter 0 page 10 we have that there is an invertible matrix B such that

$$B^{-1}AB = \left[\begin{array}{cc|c} 0 & 1 & \bar{0} \\ -c_{11} & -b_{11} & \\ \hline \multicolumn{2}{c|}{\bar{0}} & A'_{(n-2)\times(n-2)} \end{array}\right].$$

Let $S = \left\{ \left[\begin{array}{cc|c} x_{11} & x_{12} & \bar{0} \\ x_{21} & x_{22} & \\ \hline \bar{0} & & T_{(n-2)\times(n-2)} \end{array} \right] ; x_{ij} \in \mathbb{R},\ T \text{ is an } (n-2)\times(n-2) \text{ matrix over } \mathbb{R} \right\}$.

Clearly, S is a semigroup under matrix multiplication and $S \cong M(2,\mathbb{R})\times M(n-2,\mathbb{R})$. From equation (1.10) we have that $\Phi((\bar{0},I_{n-2})) = 0$ and from equation (1.8) we have that $\Phi((I_2,\bar{0})) = 0$ (the proof is similar to lemma 1.2). Then by lemma 1.1 there exist multiplicative homomorphisms $\alpha : M(2,\mathbb{R}) \to \mathbb{R}$ with $\alpha(0) = 0$ and $\beta : M(n-2,\mathbb{R}) \to \mathbb{R}$ with $\beta(0) = 0$ such that $\Phi((x,y)) = \alpha(x)\beta(y)$ for all $x \in M(2,\mathbb{R})$, for all $y \in M(n-2,\mathbb{R})$. Since Φ is real analytic, α and β are real analytic by the corollary to lemma 1.1. Suppose that $\Phi \not\equiv 0$. Then $\alpha(x) = (\det x)^m$ for some $m \in \mathbb{N}$, for all $x \in M(2,\mathbb{R})$ by lemma 2.2, and by the induction hypothesis $\beta(x) = (\det x)^t$ for some $t \in \mathbb{N}$, for all $x \in M(n-2,\mathbb{R})$. Therefore, $\Phi(A) = \Phi(B^{-1}AB) = \left(\det \begin{bmatrix} 0 & 1 \\ -c_{11} & -b_{11} \end{bmatrix}\right)^m (\det A'_{(n-2)\times(n-2)})^t$

Claim that $m = t$. Suppose that $m \neq t$. Choose $B \in M(n,\mathbb{R})$ so that its invariant factors after factorization into irreducible polynomials are $(x-\lambda_1)$, $(x-\lambda_2),(x-\lambda_3),p_1(x)^{s_1}\cdots p_u(x)^{s_u}$, where $\lambda_1,\lambda_2,\lambda_3$ are all distinct nonzero real numbers and $\lambda_1^{m-t} \neq \lambda_3^{m-t}$ and $p_i(x)$, $i = 1,\ldots,n$, are non constant polynomials having nonzero roots. Then from chapter 0 page 10, there exist invertible matrices $C, D \in M(n,\mathbb{R})$ such that

$$C^{-1}BC = \left[\begin{array}{cc|c|c} \lambda_1 & 0 & & \bar{0} \\ 0 & \lambda_2 & & \\ \hline & & \lambda_3 & \\ \hline \bar{0} & & & E_{(n-3)\times(n-3)} \end{array} \right], \text{ and}$$

$$D^{-1}BD = \left[\begin{array}{cc|c|c} \lambda_3 & 0 & & \bar{0} \\ 0 & \lambda_2 & & \\ \hline & & \lambda_1 & \\ \hline & & & E_{(n-3)\times(n-3)} \\ \bar{0} & & & \end{array}\right]$$

Since $\Phi(B) = \Phi(C^{-1}BC) = (\lambda_1\lambda_2)^m \left(\det \left[\begin{array}{c|c} \lambda_3 & 0 \\ \hline 0 & E_{(n-3)\times(n-3)} \end{array}\right]\right)^t$

$$= \lambda_1^m \lambda_2^m \lambda_3^t (\det E_{(n-3)\times(n-3)})^t$$

and $\Phi(B) = \Phi(D^{-1}BD) = (\lambda_3\lambda_2)^m \left(\det \left[\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & E_{(n-3)\times(n-3)} \end{array}\right]\right)^t$

$$= \lambda_3^m \lambda_2^m \lambda_1^t (\det E_{(n-3)\times(n-3)})^t ,$$

we get that $\lambda_1^m \lambda_2^m \lambda_3^t (\det E_{(n-3)\times(n-3)})^t = \lambda_3^m \lambda_2^m \lambda_1^t (\det E_{(n-3)\times(n-3)})^t$.

So $\lambda_1^{m-t} = \lambda_3^{m-t}$, a contradiction. Therefore $m = t$. Hence $\Phi(A)$ = $\left(\det \begin{bmatrix} 0 & 1 \\ -c_{11} & -b_{11} \end{bmatrix}\right)^m (\det A'_{(n-2)\times(n-2)})^m = (\det B^{-1}AB)^m = ((\det B^{-1})(\det A)(\det B))^m = ((\det B^{-1})(\det B)(\det A))^m = (\det (I) \det A)^m = (\det A)^m$. Thus $\Phi(A) = (\det A)^m$.

<ins>Case 3</ins>. All irreducible polynomials of degree 2 are of order > 1.

Suppose that for all $j = 1,2,\ldots,t$,

$$m_j(x) = (x^2+b_{j1}x+c_{j1})^{\ell_{j1}} \ldots (x^2+b_{jp_j}x+c_{jp_j})^{\ell_{jp_j}} (x-\lambda_{j(p_j+1)})^{\ell_{j(p_j+1)}} \ldots$$
$$\ldots (x-\lambda_{jk_j})^{\ell_{jk_j}},$$

where $x^2+b_{j1}x+c_{j1} = q_{j1}(x),\ldots,x^2+b_{jp_j}x+c_{jp_j} = q_{jp_j}(x)$, $x-\lambda_{j(p_j+1)} = q_{j(p_j+1)}(x),\ldots,x-\lambda_{jk_j} = q_{jk_j}(x)$. Choose $ji \in \{11,\ldots,1p_1,21,\ldots,2p_2,\ldots \ldots,t1,\ldots,tp_t\}$ so that $|b_{ji}^2 - 4c_{ji}|$ is maximum and reorder the indices so that $j = i = 1$. Then $|b_{11}^2 - 4c_{11}|$ is maximum. Since $b_{ji}^2 - 4_{ji} < 0$, $b_{11}^2 - 4c_{11} \leq b_{ji}^2 - 4c_{ji} < 0$ for all $ji \in \{11,\ldots,1p_1,21,\ldots,2p_2,\ldots,t1,\ldots,tp_t\}$.

Then from chapter 0 page 10 , there exists an invertible matrix B such that

$$B^{-1}AB = \begin{bmatrix} D(q_{11}(x)^{\ell_{11}}) & 0 & \cdots & 0 \\ 0 & D(q_{12}(x)^{\ell_{12}}) & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D(q_{tk_t}(x)^{\ell_{tk_t}}) \end{bmatrix}$$

$$= \left[\begin{array}{cc|cc|cc|cc|c} 0 & 1 & 0 & 0 & & & & & \bar{0} \\ -c_{11} & -b_{11} & 1 & 0 & & & & & \\ \hline & & 0 & 1 & 0 & 0 & & & \\ & & -c_{11} & -b_{11} & 1 & 0 & & & \\ \hline & & & & \ddots & & 0 & 0 & \\ & & & & & & 1 & 0 & \\ \hline & & & & & & 0 & 1 & \\ & \bar{0} & & & & & -c_{11} & -b_{11} & \\ \hline & & & & & & & & F \end{array}\right] = Y,$$

where

$$F = \begin{bmatrix} D(q_{12}(x)^{\ell_{12}}) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & D(q_{tk_t}(x)^{\ell_{tk}}) \end{bmatrix}.$$

let $m \in \mathbb{N}$. So $b_{11}^2 - 4(c_{11} + \frac{1}{m}) < b_{11}^2 - 4c_{11} \leq b_{ji}^2 - 4c_{ji} < 0 \quad \forall ji.$

For each m, let $P_m(x) = (x^2+b_{11}x+c_{11}+\frac{1}{m})(x^2+b_{11}x+c_{11})^{\ell_{11}-1}(x^2+b_{12}x+c_{12})^{\ell_{12}}\ldots$

$$\ldots(x^2+b_{1p_1}x+c_{1p_1})^{\ell_{1p_1}}(x-\lambda_{1(p_1+1)})^{\ell_{1(p_1+1)}}\ldots(x-\lambda_{1k_1})^{\ell_{1k_1}}.$$

Therefore the polynomial $x^2+b_{11}x+c_{11}+\frac{1}{m}$ is not a factor of $m_2(x),\ldots,m_t(x)$. Hence the order of $x^2+b_{11}x+c_{11}+\frac{1}{m}$ is exactly one in the product of

$p_m(x)m_2(x)\ldots m_t(x)$.

Let $\gamma_m =$

$$\left[\begin{array}{cc|cc|ccc}
0 & 1 & 0 & 0 & & & \bar{0} \\
-(c_{11}+\frac{1}{m}) & -b_{11} & 1 & 0 & & & \\
\hline
 & & 0 & 1 & 0 & 0 & \\
 & & -c_{11} & -b_{11} & 1 & 0 & \\
\hline
 & & & & \ddots & & \\
 & & & & & \begin{array}{cc} 0 & 0 \\ 1 & 0 \\ \hline 0 & 1 \\ -c_{11} & -b_{11} \end{array} & \\
\bar{0} & & & & & & F
\end{array}\right]$$

Then the characteristic polynomial of $\gamma_m = p_m(x)m_2(x)\ldots m_t(x)$ and $\gamma_m \to \gamma$ as $m \to \infty$. Since $\gamma = B^{-1}AB$, $A = B\gamma B^{-1}$. Let $A_m = B\gamma_m B^{-1}$ for all $m \in N$. Because $\gamma_m \to \gamma$ as $m \to \infty$, then $B\gamma_m B^{-1} \to B\gamma B^{-1}$ as $m \to \infty$; i.e. $A_m \to A$ as $m \to \infty$. By lemma 2.1, the product of the invariant factors of $\gamma_m = p_m(x)m_2(x)\ldots m_t(x)$. Since A_m is similar to γ_m, $p_m(x)m_2(x)\ldots m_t(x)$ is the product of the invariant factors of A_m. So the irreducible factor, $x^2+b_{11}x+c_{11}+\frac{1}{m}$, must be a factor of one of the invariant factors of A_m. Because it has order 1, then by case 2 we have that $\Phi(A_m) = (\det A_m)^m$. Since $A = \lim_{m\to\infty} A_m$,

$$\Phi(A) = \Phi(\lim_{m\to\infty} A_m) = \lim_{m\to\infty} \Phi(A_m) = \lim_{k\to\infty} (\det A_k)^m = (\lim_{k\to\infty} (\det A_k))^m$$

$$= (\det(\lim_{k\to\infty} A_k))^m = (\det A)^m;$$ i.e., $\Phi(A) = (\det A)^m$ for all $A \in M(n,\mathbb{R})$.

So the theorem is proved. #