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RESIDUAL TYPE A POSTERIORI ERROR ESTIMATES FOR SEMI-LINEAR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

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การวิเคราะห์ค่าความผิดพลาดภายหลังเป็นหลักการที่สำคัญสำหรับวิธีสมาชิกจำกัดแบบปรับตัวได้สำหรับการประมาณค่าผลเฉลยของสมการเชิงอนุพันธ์ย่อยแบบต่างๆ ในวิทยานิพนธ์นี้เราให้ความสนใจในการวิเคราะห์ความผิดพลาดภายหลังสำหรับสมการเชิงอนุพันธ์ย่อยเชิงพาราโบลาแบบกึ่งเชิงเส้นบนโดเมนรูปหลายเหลี่ยมใน 2 มิติภายใต้เงื่อนไขค่าขอบแบบดีรีเคล โดยแสดงประสิทธิภาพและความน่าเชื่อถือของค่าประมาณความผิดพลาดภายหลัง โดยหาขอบเขตบนและขอบเขตล่างด้วยตามวิธีการประมาณชนิดตกค้างมาตรฐานภายใต้เงื่อนไขที่ว่าฟังก์ชันไม่เชิงเส้น f เป็นฟังก์ชันลิพชิทซ์ของตัวแปร u สุดท้ายเราได้สร้างขั้นตอนวิธีสมาชิกจำกัดแบบปรับตัวได้โดยอาศัยข้อมูลจากการประมาณค่าความผิดพลาดภายหลัง

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A posteriori error analysis is the key idea for adaptive finite element methods for solving partial differential equations(PDEs). In this thesis, we are interested in a posteriori error analysis for semi-linear parabolic PDEs over polygonal domain in 2-D with Dirichlet boundary condition. We showed the efficiency and reliability of a posteriori error estimator by deriving the upper and local lower bounds based on the standard residual estimator under the assumption that the nonlinear function f is Lipschitz with respect to the variable u. We also constructed an algorithm for adaptive finite element method based on a posterior error estimations.

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CONTENTS

CHAPTER I

INTRODUCTION

Finite element method is a standard numerical technique for obtaining approximate solutions which are based on variational formulation of partial differential equations(PDEs). The finite element method is widely used in many applications in science and engineering, for example, mechanical engineering, structural simulation, aeronautical, biomechanical, automotive industries, etc.

Adaptivity is one of the key idea for improving accuracy and performance for finite element methods in an efficient way. Adaptive finite element method was first introduced in late 70's by I. Babuska[2]. Adaptive finite element method is more efficient and less work than finite element method if high accuracy is required especially in the presence of singularities or boundary layers, for examples.

A posteriori error analysis is the main idea for designing adaptive algorithm for finite element methods. In the adaptive algorithm, we use a posteriori error estimates as indicators, which are computable quantities of known data. The adaptive algorithm solves for finite element solutions and selects some elements for refinement and some elements for coarsening depending on the error indicators on each element.

An adaptive finite element method will loop the following procedure

$$\ldots \;\rightarrow\; Solve \rightarrow Estimate \rightarrow Refine/Coarsen \rightarrow \ldots$$

With a given initial mesh,

Solve finds finite element solution based on current mesh.

Estimate computes the error indicators on each element based on known data and solution.

Refine/Coarsen repartitions the current mesh to maintain the accuracy and performance in the system based on the error indicators.

The analysis and convergence results about adaptive finite element method is begun by the work of W. Dorfler[8] in 1996 for Poisson's equation. In 2002, P. Morin et al[11] extended [8] to elliptic PDEs with piecewise constant coefficient A. They also introduced the concept of oscillators. K. Mekchay and R. H. Nochetto[10] worked on general second order linear elliptic PDEs in 2005.

For parabolic PDE, Z. Chen and F. Jia[5] derived a posteriori error estimates for linear parabolic PDEs in 2004. Here, the considered the model problem,

$$\frac{\partial u}{\partial t} - \nabla \cdot (a(x)\nabla u) = f(x,t) \qquad \text{in } \Omega \times (0,T)$$
$$u = 0 \quad \text{on } \partial\Omega \times (0,T), \quad u(x,0) = u_0(x) \quad \text{in } \Omega,$$

where $u \in L^2(\Omega)$, $a(x)$ is a piecewise constant function and $f \in L^2(0,T;L^2(\Omega))$, i.e., $f : (0,T) \to L^2(\Omega)$.

In this thesis, we extended the work from Z. Chen and F. Jia by considering a semi-linear parabolic problem:

$$\frac{\partial u}{\partial t} - \nabla \cdot (a(x)\nabla u) = f(u) \qquad \text{in } \Omega \times (0,T)$$
$$u = 0 \quad \text{on } \partial\Omega \times (0,T), \quad u(x,0) = u_0(x) \quad \text{in } \Omega,$$

where $a(x)$ is now a positive function in $L^\infty(\Omega)$ and f is non-linear Lipschitz function of u.

We derived the upper and local lower bounds based on the standard residual technique to show that a posteriori error estimators are reliable and efficiency, and also constructed an adaptive algorithm for the finite element methods.

CHAPTER II

PRELIMINARY

In this Chapter, we provided some basic knowledge of finite element analysis including definitions and theorems used in the proof of the main results. The proofs of theorems in this Chapter are omitted but can be found in the provided references. This Chapter consists of 3 parts: the Sobolev spaces, the construction of the finite element space, and some approximation results.

2.1 Sobolev Spaces

This section provides some basic knowledge about Sobolev spaces required later in this thesis. To obtain the variational problem from the given PDE problem one need to use functions in some Sobolev spaces. More details about Sobolev spaces can be found in Chapter 2 of [3].

Let Ω be an open subset of $\mathbb{R}^d$ with piecewise smooth boundary. $L^2(\Omega)$ is a set of function $u(x)$ which is square-integrable in the Lebesgue sense over Ω. It is known that $L^2(\Omega)$ is a Hilbert space with inner product [13]

$$(u,v)_0 = \int_\Omega uv\,dx \qquad \forall\, u,v \in L^2(\Omega),$$

with the norm defined by

$$||u||_0 = \sqrt{(u,u)_0}.$$

Definition 2.1. *Given an integer $m \geq 0$, let $H^m(\Omega)$ be the set of all functions u in $L^2(\Omega)$ which possess weak derivatives $\partial^\alpha u$ for all $|\alpha| \leq m$. We can define a scalar product on $H^m(\Omega)$ by*

$$(u, v)_m = \sum_{|\alpha| \leq m} (\partial^\alpha u, \partial^\alpha v)_0,$$

with the norm

$$||u||_m = \sqrt{(u, u)_m} = \sqrt{\sum_{|\alpha| \leq m} ||\partial^\alpha u||_0^2}.$$

And the semi-norm

$$|u|_m = \sqrt{\sum_{|\alpha| = m} ||\partial^\alpha u||_0^2}.$$

In this thesis, we are interested in functions in $H^1(\Omega)$.

Definition 2.2. *The completion of $C_0^\infty(\Omega)$ with respect to the Sobolev norm $||\cdot||_m$ is denoted by $H_0^m(\Omega)$.*

Note 2.3. $H^m(\Omega)$ and $H_0^m(\Omega)$ are Hilbert spaces.

Note 2.4. In this thesis, we only use the spaces $H^1(\Omega)$ and $H_0^1(\Omega)$.

Theorem 2.5. *Suppose Ω is bounded and contained in a d-dimensional cube with side length s. Then*

$$||v||_0 \leq s|v|_1 \qquad \forall\, v \in H_0^1(\Omega).$$

Proof. The proof can be found in the book by D. Braess [3]. □

Theorem 2.6. *If Ω is bounded, then $|\cdot|_m$ is a norm on $H_0^m(\Omega)$ which is equivalent to $||\cdot||_m$. In addition, if Ω is contained in a cube with side length s, then*

$$|v|_m \leq ||v||_m \leq (1+s)^m |v|_m \qquad \forall\, v \in H_0^1(\Omega).$$

Proof. The proof can be found in the book by D. Braess [3]. □

Definition 2.7. *Let H be a Hilbert space with norm $||\cdot||_H$.*

A bilinear form $b : H \times H \to \mathbb{R}$ is called continuous provided there exists $c > 0$ such that

$$|b(u,v)| \leq c||u||_H \, ||v||_H \qquad \forall\, u, v \in H.$$

A bilinear form $b(\cdot,\cdot)$ is called coercive for a subspace V in H, provided for some $\alpha > 0$,

$$b(v,v) \geq \alpha ||v||_V^2 \qquad \forall\, v \in V$$

Remark 2.8. *We can define an energy norm on V with coercive bilinear form $b(\cdot,\cdot)$ by $\|v\|_b = \sqrt{b(v,v)}$. The norm $\|\cdot\|_b$ is equivalent to the norm of the Hilbert space $||\cdot||_H$, namely, there exist a constant $C_e > 0$ such that*

$$\frac{1}{C_e}||\cdot||_H \leq \|\cdot\|_b \leq C_e||\cdot||_H.$$

2.2 Standard Finite Element

The goal for this section is to build a finite element space V, a finite dimensional subspace of $H_0^1(\Omega)$, and to introduce some approximation results.

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$.

Definition 2.9. *A partition $\mathcal{M} = \{K_1, K_2, \ldots, K_N\}$ of Ω into triangular subdomains K_i is called a* ***triangulation*** *of Ω if the following properties holds:*

1. $\bar{\Omega} = \bigcup_{i=1}^{N} K_i$.

2. If $K_i \cap K_j$ consists of exactly one point, then it is a common vertex of K_i and K_j.

3. If for $i \neq j$, $K_i \cap K_j$ consists of more than one point, then $K_i \cap K_j$ is a common edge of K_i and K_j.

Definition 2.10. *A family of triangulation $\{\mathcal{M}_k\}_{k\geq 0}$ is called **shape regular** provided that there exists a number $\kappa > 0$ such that every K in $\mathcal{M}_k$ and for every k contains a circle of radius ρ_K with*

$$\rho_K \geq \frac{h_K}{\kappa},$$

where h_K is the diameter of element K.

To define a finite element space V, for fixed a non-negative integer h, let $\mathcal{M}_h$ be a shape-regular triangulation of $\Omega \subset \mathbb{R}^2$ and $\mathbb{P}_l$ denote the set of polynomials of degree $\leq l$. Let V be a finite element spaces consisting of continuous piecewise linear functions, defined by

$$V = \{v \in H^1(\Omega) \mid \; v|_K \in \mathbb{P}_1, \;\; \forall K \in \mathcal{M}_h\}.$$

Here, we use linear Lagrange elements with nodal basis functions, i.e., for each node x_j of element K, the nodal basis for node x_j is $\phi_j(x_i) = \delta_{ij}$. For each $v \in V$, $v(x) = \sum_{i=1}^{N} v(x_i)\phi_i(x)$ where N is the total number of node.

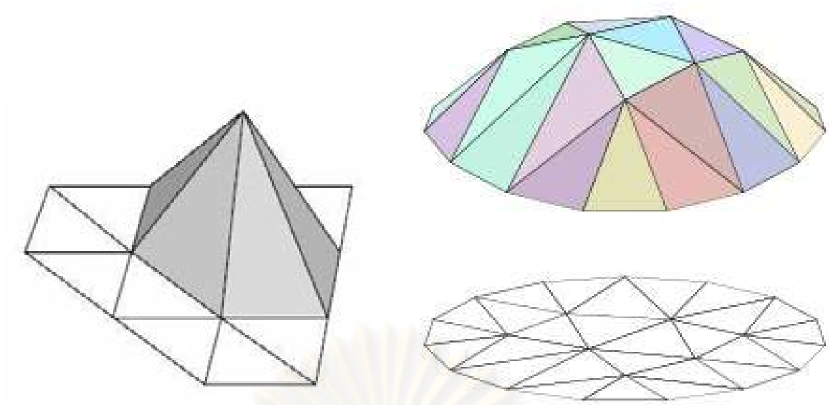

Figure 2.1: Example of nodal basis and a continuous piecewise linear function

2.3 Approximation Results

Let $\mathcal{B}$ be the set of all inter-element boundaries (interior sides) of all elements $K \in \mathcal{M}_h$. We denoted patches as follows:

$$\begin{aligned} \omega_e &= \bigcup\{K \in \mathcal{M}_h \,|\, e \subset \partial K\} && \forall\, e \in \mathcal{B}, \\ \omega_K &= \bigcup_{e \subset \partial K} \omega_e && \forall\, K \in \mathcal{M}_h \\ \tilde{\omega}_K &= \bigcup\{K' \in \mathcal{M}_h \,|\, K \cap K' \neq \varnothing\} && \forall\, K \in \mathcal{M}_h \end{aligned}$$

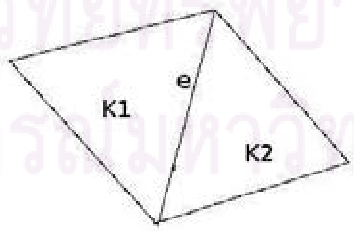

Figure 2.2: The example of the patch ω_e for the edge e

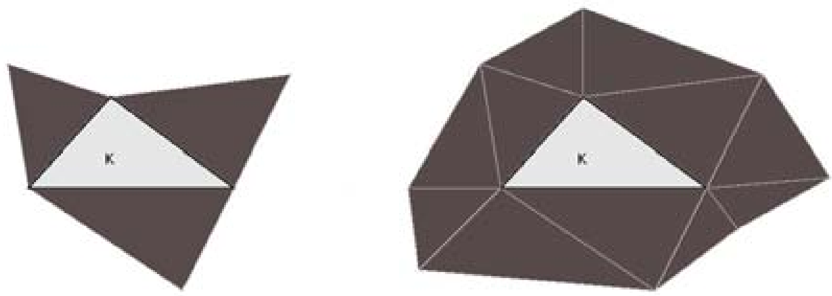

Figure 2.3: The left picture is the patch ω_K and the right picture is the patch $\widetilde{\omega}_K$

We state some important theorems and properties used in the proof of the main results as follows.

Theorem 2.11. *(Clement Interpolation Approximation) Let $\mathcal{M}_h$ be a shape-regular triangulation of Ω. Then there exists a linear mapping $\mathcal{I}_h : H^1(\Omega) \to V$ such that*

$$||v - \mathcal{I}_h v||_{0,K} \leq c h_K ||v||_{1,\tilde{\omega}_K} \qquad \forall\, v \in H^1(\Omega), K \in \mathcal{M}_h, \tag{2.1}$$

$$||v - \mathcal{I}_h v||_{0,e} \leq c h_K^{\frac{1}{2}} ||v||_{1,\tilde{\omega}_K} \qquad \forall\, v \in H^1(\Omega), e \subset \partial K, K \in \mathcal{M}_h. \tag{2.2}$$

Proof. The proof can be found in [6] by Ph. Clement . □

The Clement's interpolation approximations are the main ingredients for obtaining the upper bound in the error estimates. To obtain the local lower bound, we used the ideas of bubble functions. There are 2 types of bubble functions, element bubble functions and edge bubble functions. The definitions and properties are given below.

Definition 2.12. *Let $K \in \mathcal{M}_h$ and $e \in \mathcal{B}$. The functions ψ_K, ψ_e are the bubble functions corresponding to K and e, respectively, with properties:*

$$\psi_K \in \mathbb{P}_3,\ supp\ \psi_K = K,\ 0 \leq \psi_K \leq 1,\ \max \psi_K = 1,$$

and

$$\psi_e \in \mathbb{P}_2,\ supp\ \psi_e = \omega_e,\ 0 \leq \psi_e \leq 1,\ \max \psi_e = 1.$$

Proposition 2.13. *Let $\mathcal{M}_h$ be a shape-regular triangulation. Then there exists a constant c which depends only on the shape parameter κ such that*

$$\begin{array}{rcll}
||\psi_K v||_{0,K} & \leq & ||v||_{0,K} & \forall\, v \in L^2(K), \\
||\psi_K^{\frac{1}{2}} p||_{0,K} & \geq & c||p||_{0,K} & \forall\, p \in \mathbb{P}_l, \\
||\nabla(\psi_K p)||_{0,K} & \leq & ch_K^{-1}||\psi_K p||_{0,K} & \forall\, p \in \mathbb{P}_l, \\
||\psi_e^{\frac{1}{2}} \sigma||_{0,e} & \geq & c||\sigma||_{0,e} & \forall\, \sigma \in \mathbb{P}_l, \\
ch_e^{\frac{1}{2}}||\sigma||_{0,e} \leq ||\psi_e E\sigma||_{0,K} & \leq & ch_e^{\frac{1}{2}}||\sigma||_{0,K} & \forall\, \sigma \in \mathbb{P}_l, \\
||\nabla(\psi_e E\sigma)||_{0,K} & \leq & ch_K^{-1}||\psi_e E\sigma||_{0,K} & \forall\, \sigma \in \mathbb{P}_l,
\end{array}$$

where $E :\ L^2(e) \rightarrow L^2(\omega_e)$ is an extension function on an edge e and h_e is the length of the edge e.

Proof. The proof can be found in [14] by R. Verfurth and [1] by M. Ainsworth and J.T. Oden. □

CHAPTER III

MODEL PROBLEM

In this Chapter, we introduced the model problem, a semi-linear parabolic PDE with some assumptions used in this thesis. Thereafter, we formulated the variational problem and discretized the problem in order to use a finite element method.

3.1 Model Problem

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$ with boundary denoted by $\Gamma = \partial\Omega$ and a final time $T > 0$. We consider a semi-linear parabolic PDE

$$\begin{aligned} \tfrac{\partial u}{\partial t} - \nabla \cdot (a(x)\nabla u) &= f(u(x,t)) && \text{in } \Omega \times (0,T) \\ u &= 0 && \text{on } \Gamma \times (0,T) \\ u &= u_0 && \text{on } \Omega \times \{t=0\}, \end{aligned}$$

where $u_0 \in L^2(\Omega)$, $a(x) \in L^\infty(\Omega)$ is a positive function ($a(x) \geq \gamma$ for some $\gamma > 0$) and the function $f(u(\cdot,t)) \in L^2(\Omega)$ satisfying the Lipschitz condition, i.e., there exists a constant $L > 0$ such that for each fixed t,

$$||f(u_1) - f(u_2)||_0 \leq L||u_1 - u_2||_0 \qquad \forall u_1(\cdot,t), u_2(\cdot,t) \in L^2(\Omega). \tag{3.1}$$

To obtain the weak form, we multiply the PDE by $\varphi \in H_0^1(\Omega)$ and apply Green's theorem (see [9] page 459) to get

$$(\frac{\partial u}{\partial t}, \varphi)_0 + (a\nabla u, \nabla\varphi)_0 = (f, \varphi)_0, \tag{3.2}$$

where $(v,w)_0 = \int_\Omega vw\,dx.$

We define bilinear form $b(\cdot,\cdot)$ by

$$b(v,w) = (a(x)\nabla v, \nabla w)_0 = \int_\Omega a(x)\nabla v\cdot\nabla w\,dx \qquad \forall\, v,w\in H_0^1(\Omega).$$

Lemma 3.1. *The bilinear form $b(\cdot,\cdot)$ is a continuous symmetric and coercive on $H_0^1(\Omega)$.*

Proof. First, we will show that a bilinear $b(\cdot,\cdot)$ is a continuous. We need to show that there exists $c>0$ such that $|b(u,v)|\le c||u||_1\,||v||_1$ for any $u,v\in H_0^1(\Omega)$.

Let $u,v\in H_0^1(\Omega)$. Since $a(x)\in L^\infty(\Omega)$ in bounded domain Ω, so

$$\begin{aligned} |b(u,v)| &= \left|\int_\Omega a(x)\nabla u\cdot\nabla v\,dx\right| \\ &\le \int_\Omega ||a||_{L^\infty(\Omega)}\,|\nabla u\cdot\nabla v|\,dx \\ &\le ||a||_{L^\infty(\Omega)}\int_\Omega |\nabla u|\,|\nabla v|\,dx \\ &\le ||a||_{L^\infty(\Omega)}|u|_1|v|_1 \\ &\le c||u||_1||v||_1, \end{aligned}$$

where the last 2 steps follow from the Cauchy-Schwarz inequality and the norm equivalent of $||\cdot||_1$ and $|\cdot|_1$ on $H_0^1(\Omega)$, (see Theorem 2.6.) Note that the constant $c := ||a||_{L^\infty(\Omega)}$ depends only on function $a(x)$.

Next, we will show that a bilinear form $b(\cdot,\cdot)$ is a symmetric and coercive in $H_0^1(\Omega)$. It easy to see that $b(\cdot,\cdot)$ is a symmetric by the definition. To show that $b(\cdot,\cdot)$ is coercive in $H_0^1(\Omega)$. Let $v\in H_0^1(\Omega)$. Since $a(x)$ is a positive function in $L^\infty(\Omega)$, so

$a(x) \geq \gamma > 0$ a.e. $x \in \Omega$ and

$$\begin{aligned} b(v,v) &= \int_\Omega a(x)\nabla v \cdot \nabla v \, dx \\ &\geq \gamma \int_\Omega \nabla v \cdot \nabla v \, dx \\ &= \gamma \int_\Omega |\nabla v|^2 \, dx \\ &= \gamma ||\nabla v||_0^2 \\ &= \gamma |v|_1^2 \end{aligned}$$

Since semi-norm $|\cdot|_1$ and norm $||\cdot||_1$ on $H_0^1(\Omega)$ are equivalent, so by Theorem 2.6 with $m = 1$, $|v|_1 \geq \frac{1}{(1+s)}||v||_1$ and

$$b(v,v) \geq \frac{\gamma}{(1+s)^2}||v||_1^2.$$

Hence, $b(\cdot,\cdot)$ is coercive in $H_0^1(\Omega)$ with coercive constant $\alpha = \frac{\gamma}{(1+s)^2}$. □

Since $b(\cdot,\cdot)$ is coercive and continuous in $H_0^1(\Omega)$, the energy norm

$$|||\varphi||| := \sqrt{b(\varphi,\varphi)} \qquad \forall\, \varphi \in H_0^1(\Omega),$$

is equivalent to $||\cdot||_1$-norm by Remark 2.8.

Lemma 3.2. *For any $\varphi \in H_0^1(\Omega)$, there exists a constant $C_p > 0$ such that*

$$||\varphi||_0 \leq C_p |||\varphi|||.$$

Proof. Let $\varphi \in H_0^1(\Omega)$. By Theorem 2.5 and Remark 2.8,

$$||\varphi||_0 \leq s|\varphi|_1 \leq s||\varphi||_1 \leq sC_e|||\varphi|||$$

where $C_p := sC_e$. □

In order to approximate weak solution, we assume the uniqueness and existence of weak solution in (3.2). To obtain the discrete problem, we divided this procedure into 2 steps.

1. Discretization on time $(0,T)$.

   First, we partition $(0,T)$ into N subintervals (t^{n-1},t^n), $n = 1,2,...,N$ where $t^0 = 0$ and $t^N = T$.

   We define the n-th time-step size by

$$\tau_n = t^n - t^{n-1}, \qquad \text{for } n = 1,2,...,N.$$

   It follows that

$$t^m = \sum_{n=1}^{m} \tau_n, \qquad \text{for } m = 1,2,...,N,$$

   Consider at the time $t = t^n$, from the weak form in (3.2)

$$(\frac{\partial u}{\partial t}(t^n),\varphi)_0 + b(u^n,\varphi) = (f^n,\varphi)_0 \quad \forall \varphi \in H_0^1(\Omega),$$

   where $u^n := u(x,t^n)$ and $f^n := f(u^n)$.

   Next, we approximate $\frac{\partial u}{\partial t}$ by the backward Euler, namely, $\frac{\partial u}{\partial t}\big|_{t=t^n} \approx \frac{u^n - u^{n-1}}{\tau_n}$, so

$$(\frac{u^n - u^{n-1}}{\tau_n},\varphi)_0 + b(u^n,\varphi) \approx (f^n,\varphi)_0 \quad \forall \varphi \in H_0^1(\Omega). \tag{3.3}$$

   This approximation is used in the finite element scheme.

2. Discretization on space Ω.

   With a given initial triangulation $\mathcal{M}^0$ of Ω, for $n \geq 1$, let $\{\mathcal{M}^n\}$ be a conforming and shape-regular family of triangulations where $\mathcal{M}^n$ is obtained from $\mathcal{M}^{n-1}$, does not need to be nested. Let V^n be a Lagrange finite element space of a continuous piecewise linear functions over the triangulation $\mathcal{M}^n$ and $V_0^n = V^n \cap H_0^1(\Omega)$. Let $\mathcal{P}_n : H_0^1(\Omega) \to V_0^n$ be a projection operator for mesh $\mathcal{M}^n$ and define $U_h^0 = \mathcal{P}_0 u_0$. With initial information $U_h^{n-1} \in V_0^{n-1}$, we seek an approximation $U_h^n \in V_0^n$ satisfying the discrete weak form

$$(\frac{U_h^n - U_h^{n-1}}{\tau_n}, v)_0 + b(U_h^n, v) = (f_h^n, v)_0 \quad \forall v \in V_0^n \tag{3.4}$$

where $f_h^n := f(U_h^n)$.

Note 3.3. We take $U_h^{n-1} \in V_0^{n-1} \subset H_0^1(\Omega)$, a solution from the $(n-1)$-th step to be an initial information for the n-th step in (3.4).

To approximate $u(x,t)$ for $t \in (t^{n-1}, t^n)$, we interpolate linearly between U_h^{n-1} and U_h^n, namely, for each $x \in \Omega$

$$U_h(x,t) := \frac{t^n - t}{\tau_n} U_h^{n-1}(x) + \frac{t - t^{n-1}}{\tau_n} U_h^n(x).$$

Note 3.4. $U_h(x,t^n) = U_h^n(x)$, $U_h(x,t^{n-1}) = U_h^{n-1}$ and $\frac{\partial U_h}{\partial t} = \frac{U_h^n - U_h^{n-1}}{\tau_n}$ for $t \in (t^{n-1}, t^n)$ which use in the next Chapter.

CHAPTER IV

A POSTERIORI ERROR ESTIMATES

In this Chapter, we derived the upper and local lower bounds for the errors using the standard residual technique. The upper bound gives the bound of the global error in term of the estimator to ensure that the finite element solution is acceptable. The local lower bound gives the relation between the local errors and their estimators with some other quantities.

To obtain a posteriori error estimates, we employed the standard residual technique. We used area-based residual on element $K \in \mathcal{M}^n$ and edge-based residual on edge e on the element K to estimate the error on the element K.

We defined the area-based residual for element $K \in \mathcal{M}^n$ at fixed $t = t^n$ by

$$R^n := f_h^n - \frac{U_h^n - U_h^{n-1}}{\tau_n} + \nabla \cdot (a \nabla U_h^n)$$

and the edge-based residual for interior side $e \in \mathcal{B}^n$ by

$$J_e^n = (a\nabla U_h^n|_{K_1} - a\nabla U_h^n|_{K_2}) \cdot \vec{n}_e, \quad \text{where } e = \partial K_1 \cap \partial K_2.$$

Note 4.1. Since U_h^n is a piecewise linear function, so $\triangle U_h^n = 0$ and

$$\nabla \cdot (a\nabla U_h^n) = \nabla a \cdot \nabla U_h^n + a\Delta U_h^n = \nabla a \cdot \nabla U_h^n.$$

Note that, we need $\nabla a(x)$ to be well defined, i.e., $a(x)$ is differentiable in K, for each

$K \in \mathcal{M}_h$, thus we need to assume in additional that $a(x)$ is piecewise differentiable on Ω, i.e., $a|_K$ is differentiable for all $K \in \mathcal{M}_h$.

We define the *local error indicator* η_K^n for any $K \in \mathcal{M}^n$ by

$$\eta_K^n = \left(h_K^2 ||R^n||_{0,K}^2 + \sum_{e \subset \partial K} h_e ||J_e^n||_{0,e}^2 \right)^{\frac{1}{2}}. \tag{4.1}$$

For each element $K \in \mathcal{M}^n$, we use η_K^n as an indicator for refinement or coarsening. To check the error of the approximation on Ω to ensure that the finite element solution is acceptable, we defined the global error estimator on the space for fixed $t = t^n$ by

$$\eta_{space}^n = \sqrt{\sum_{K \in \mathcal{M}^n} (\eta_K^n)^2}.$$

We use η_{space}^n as a stopping criteria of the current loop of discrete system at time $t = t^n$. To start the next discrete system at time $t = t^{n+1}$, we need to find the suitable time step size that is not too large or too small. So we defined error estimators η_{time}^n to control time step size by

$$(\eta_{time}^n)^2 = \frac{1}{3} |||U_h^n - U_h^{n-1}|||^2.$$

This η_{time}^n is used for finding a suitable τ_n.

4.1 Upper Bound

To analyze the upper bound, we measured the error by the energy norm in space and L^2-norm in time. First, we estimated the error at a fixed time $t = t^n$, and then combined for all time in $(0, T)$.

Note 4.2. Since a constant C in each inequalities can change from line to line, we will use the same C to indicate a constant for convenience.

Lemma 4.3. *For any integer $n \geq 1$,*

$$\begin{aligned}(\tfrac{\partial(u-U_h)}{\partial t},\varphi)_0 &+ b(u-U_h^n,\varphi)\\ &= (f-f_h^n,\varphi)_0 + \sum_{K\in\mathcal{M}^n}\int_K R^n(\varphi-v)\,dx + \sum_{e\in\mathcal{B}^n}\int_e J_e^n(\varphi-v)\,ds\end{aligned}$$

for all $\varphi \in H_0^1(\Omega)$, $v \in V_0^n$.

Proof. Let $\varphi \in H_0^1(\Omega)$ and $v \in V_0^n$. From the discrete weak form (3.4)

$$\begin{aligned}(\tfrac{U^n-U^{n-1}}{\tau_n},\varphi)_0 &+ b(U_h^n,\varphi)\\ &= (\tfrac{U_h^n-U_h^{n-1}}{\tau_n},\varphi)_0 + b(U_h^n,\varphi) + (f_h^n,v)_0 - (\tfrac{U_h^n-U_h^{n-1}}{\tau_n},v)_0 - b(U_h^n,v)\\ &= (f_h^n,\varphi)_0 - (f_h^n,\varphi-v)_0 + (\tfrac{U_h^n-U_h^{n-1}}{\tau_n},\varphi-v)_0 + b(U_h^n,\varphi-v)\end{aligned}$$

We apply Green's theorem to term $b(U_h^n,\varphi-v)$, on each element $K \in \mathcal{M}^n$,

$$b(U_h^n,\varphi-v) = -\sum_{K\in\mathcal{M}^n}\int_K \nabla\cdot(a\nabla U_h^n)(\varphi-v)dx - \sum_{e\in\mathcal{B}^n}\int_e J_e^n(\varphi-v)\,ds.$$

Substituting the above equality to get

$$\begin{aligned}(\tfrac{U^n-U^{n-1}}{\tau_n},\varphi)_0 &+ b(U_h^n,\varphi)\\ &= (f_h^n,\varphi)_0 - (f_h^n,\varphi-v)_0 + (\tfrac{U_h^n-U_h^{n-1}}{\tau_n},\varphi-v)_0\\ &\quad - \sum_{K\in\mathcal{M}^n}\int_K \nabla\cdot(a\nabla U_h^n)(\varphi-v)\,dx - \sum_{e\in\mathcal{B}^n}\int_e J_e^n(\varphi-v)\,ds\\ &= (f_h^n,\varphi)_0 - \sum_{K\in\mathcal{M}^n}\int_K R^n(\varphi-v)\,dx - \sum_{e\in\mathcal{B}^n}\int_e J_e^n(\varphi-v)\,ds\end{aligned}$$

We subtract the weak form (3.2) by the above equation to complete the proof. □

Lemma 4.4. *For any $n \geq 1$,*

$$b(u-U_h^n,u-U_h) = \tfrac{1}{2}\left(|||u-U_h^n|||^2 + |||u-U_h|||^2 - |||U_h-U_h^n|||^2\right).$$

Proof.

$$
\begin{aligned}
& b(u-U_h^n, u-U_h) \\
& \quad = b(u-U_h^n, u-U_h^n) - b(u-U_h^n, U_h-U_h^n) \\
& \quad = \|u-U_h^n\|^2 - b(U_h-U_h^n, U_h-U_h^n) + b(U_h-u, U_h-U_h^n) \\
& \quad = \|u-U_h^n\|^2 - \|U_h-U_h^n\|^2 + b(U_h-u, U_h-u) \\
& \qquad -b(U_h-u, U_h^n-u) \\
& \quad = \|u-U_h^n\|^2 + \|u-U_h\|^2 - \|U_h-U_h^n\|^2 - b(u-U_h^n, u-U_h)
\end{aligned}
$$

Thus, $b(u-U_h^n, u-U_h) = \frac{1}{2}\left(\|u-U_h^n\|^2 + \|u-U_h\|^2 - \|U_h-U_h^n\|^2\right).$ □

Now, we use 2 above Lemmas to bound the error at time $t = t^n$ in the following Lemma.

Lemma 4.5. *For fixed time $t = t^n$, if $e^{-2(C_pL)^2t}||u-U_h||_0^2$ is an increasing function of t then there exists a constant $C_1 > 0$ such that*

$$\frac{d}{dt}(e^{-2(C_pL)^2t}||u-U_h||_0^2) + \frac{1}{2}\|u-U_h^n\|^2 \ \leq \ C_1(\eta^n_{space})^2 + \|U_h-U_h^n\|^2,$$

where L is the Lipschitz constant of the function $f(u)$ in (3.1).

Proof. By Clement's approximations, there exists the interpolation function $\mathcal{I}^n$: $H^1_0(\Omega) \to V^n_0$ satisfying Clement's inequalities (2.1) and (2.2).

Applying the Cauchy-Schwarz inequality to Lemma 4.3 and set $v = \mathcal{I}^n\varphi$, we get $(\frac{\partial(u-U_h)}{\partial t}, \varphi)_0 + b(u-U_h^n, \varphi)$

$$
\begin{aligned}
\leq \ & ||f-f_h^n||_0||\varphi||_0 + \sum_{K\in\mathcal{M}^n} ||R^n||_{0,K}||\varphi-\mathcal{I}^n\varphi||_{0,K} \\
& + \sum_{e\in\mathcal{B}^n} ||J_e^n||_{0,e}||\varphi-\mathcal{I}^n\varphi||_{0,e}.
\end{aligned}
$$

By the Lipschitz continuity of f and Clement's approximations (2.1) and (2.2),

$$\begin{aligned}
&(\tfrac{\partial(u-U_h)}{\partial t},\varphi)_0 + b(u-U_h^n,\varphi)\\
&\quad\leq\ L||u-U_h^n||_0||\varphi||_0\\
&\quad\quad + \sum_{K\in\mathcal{M}^n} Ch_K||R^n||_{0,K}||\nabla\varphi||_{0,\widetilde{\omega}_K} + \sum_{e\in\mathcal{B}^n} Ch_e^{\frac{1}{2}}||J_e^n||_{0,e}||\nabla\varphi||_{0,\widetilde{\omega}_K}\\
&\quad\leq\ L||u-U_h^n||_0||\varphi||_0 + C\left(\sum_{K\in\mathcal{M}^n}(\eta_K^n)^2\right)^{\frac{1}{2}}||\nabla\varphi||_0\\
&\quad\leq\ L||u-U_h^n||_0||\varphi||_0 + C\eta_{space}^n|||\varphi|||
\end{aligned}$$

where the second inequality follows from Cauchy-Schwarz inequality.

Set $\varphi = u - U_h$, then use the Lemma 4.4, we get

$$\begin{aligned}
&\tfrac{d}{dt}||u-U_h||_0^2 + (|||u-U_h^n|||^2 + |||u-U_h|||^2)\\
&\quad\leq 2L||u-U_h^n||_0\cdot||u-U_h||_0 + \ C\eta_{space}^n|||u-U_h||| + |||U_h-U_h^n|||^2
\end{aligned} \tag{4.2}$$

By the Young's inequality, namely, for any $a, b > 0$

$$ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2} \qquad \forall\,\varepsilon > 0,$$

we separate terms $2L||u-U_h||_0$ from $||u-U_h^n||_0$ and the terms $|||u-U_h|||$ from η_{space}^n, by

$$C\eta_{space}^n|||u-U_h||| \leq \frac{C^2}{4}(\eta_{space}^n)^2 + |||u-U_h|||^2, \tag{4.3}$$

$$2L||u-U_h^n||_0\cdot||u-U_h||_0 \leq 2(C_pL)^2||u-U_h||_0^2 + \frac{||u-U_h^n||_0^2}{2C_p^2}. \tag{4.4}$$

Note, in (4.3), we used $\varepsilon = 2$ and in (4.4) we used $\varepsilon = \frac{1}{C_p^2}$.

By Lemma 3.2, so $\frac{||u-U_h^n||_0^2}{2C_p^2} \leq \frac{|||u-U_h^n|||^2}{2}$.

Substituting them in main inequality and cancelling the term $|||u-U_h|||^2$ in both sides, we get

$$\frac{d}{dt}||u-U_h||_0^2+\frac{1}{2}|||u-U_h^n|||^2 \ \leq \ 2(C_pL)^2||u-U_h||_0^2+C_1(\eta_{space}^n)^2+|||U_h-U_h^n|||^2.$$

Since $0 \leq \frac{d}{dt}(e^{-2(C_pL)^2t}||u-U_h||_0^2)$ and

$$\begin{aligned}\tfrac{d}{dt}(e^{-2(C_pL)^2t}||u-U_h||_0^2) &\leq e^{2(C_pL)^2t}\tfrac{d}{dt}(e^{-2(C_pL)^2t}||u-U_h||_0^2)\\ &= \tfrac{d}{dt}||u-U_h||_0^2-2(C_pL)^2||u-U_h||_0^2,\end{aligned}$$

then we obtain the result

$$\frac{d}{dt}(e^{-2(C_pL)^2t}||u-U_h||_0^2)+\frac{1}{2}|||u-U_h^n|||^2 \ \leq \ C_1(\eta_{space}^n)^2+|||U_h-U_h^n|||^2.$$

□

Corollary 4.6. *If* $L<\frac{1}{\sqrt{2}C_p^2}$, *then*

$$\frac{d}{dt}||u-U_h||_0^2+\frac{1}{2}|||u-U_h^n|||^2 \leq C_2(\eta_{space}^n)^2+|||U_h-U_h^n|||^2$$

Proof. From the inequality (4.2) in the proof of Lemma (4.5), we have

$$\begin{aligned}&\tfrac{d}{dt}||u-U_h||_0^2+(|||u-U_h^n|||^2+|||u-U_h|||^2)\\ &\qquad \leq \ 2L||u-U_h^n||_0\cdot||u-U_h||_0+C\eta_{space}^n|||u-U_h||| \ +|||U_h-U_h^n|||^2.\end{aligned}$$

We apply Young's inequality to the first 2 terms on the right side by

$$2L||u-U_h^n||_0\cdot||u-U_h||_0 \leq \frac{L^2}{2\varepsilon_1}||u-U_h||_0^2+2\varepsilon_1||u-U_h^n||_0^2, \tag{4.5}$$

$$C\eta_{space}^n|||u-U_h||| \leq \frac{C}{2\varepsilon_2}(\eta_{space}^n)^2+\frac{\varepsilon_2 C}{2}|||u-U_h|||^2 \tag{4.6}$$

where we choose ε_1 and ε_2 such that $\frac{(C_pL)^2}{2\varepsilon_1}+\frac{C\varepsilon_2}{2} \leq 1$. This implies that we have to choose

$$\varepsilon_1 > \frac{(C_pL)^2}{2}.$$

By (4.5), (4.6) and Lemma 3.2 we get,

$$\frac{d}{dt}||u-U_h||_0^2 + \|u-U_h^n\|^2 \leq 2C_p^2\varepsilon_1\|u-U_h^n\|^2 + \frac{C}{2\varepsilon_2}(\eta_{space}^n)^2 + \|U_h-U_h^n\|^2$$

Since $L < \frac{1}{\sqrt{2}C_p^2}$, so $\frac{(C_pL)^2}{2} < \frac{1}{4C_p^2}$, and by choosing $\varepsilon_1 = \frac{1}{4C_p^2}$, then

$$\frac{d}{dt}||u-U_h||_0^2 + \frac{1}{2}\|u-U_h^n\|^2 \leq C_2(\eta_{space}^n)^2 + \|U_h-U_h^n\|^2$$

□

Theorem 4.7. *(Upper Bound) For any integer $1 \leq m \leq N$, under the assumption of Lemma 4.5, there exists a constant $C_1 > 0$ depending only on the shape constant κ of meshes $\mathcal{M}^n$, the coefficient $a(x)$, Lipschitz constant L and domain Ω such that the following error estimate holds*

$$\begin{aligned} e^{-2(C_pL)^2t^m}||u^m-U_h^m||_0^2 &+ \tfrac{1}{2}\sum_{n=1}^{m}\int_{t^{n-1}}^{t^n}\|u-U_h^n\|^2dt \\ &\leq ||u_0-U_h^0||_0^2 + \sum_{n=1}^{m}\tau_n(\eta_{time}^n)^2 + C_1\sum_{n=1}^{m}\tau_n(\eta_{space}^n)^2. \end{aligned}$$

Proof. From the Lemma 4.5, we combined the errors from time $t=0$ to time $t=t^n$. Integrating to collect the error from $t=0$ to $t=t^m$, we get

$$\begin{aligned} e^{-2(C_pL)^2t^m}||u^m-U_h^m||_0^2 &+ \tfrac{1}{2}\sum_{n=1}^{m}\int_{t^{n-1}}^{t^n}\|u-U_h^n\|^2dt \\ &\leq ||u_0-U_h^0||_0^2 + \sum_{n=1}^{m}\int_{t^{n-1}}^{t^n}\|U_h-U_h^n\|^2dt + C_1\sum_{n=1}^{m}\tau_n(\eta_{space}^n)^2. \end{aligned}$$

Note that $\int_{t^{n-1}}^{t^n} \|U_h - U_h^n\|^2 dt = \int_{t^{n-1}}^{t^n} (\frac{t-t^n}{\tau_n})^2 \|U_h^n - U_h^{n-1}\|^2 dt = \tau_n(\eta_{time}^n)^2.$ □

Corollary 4.8. *If we assume $L < \frac{1}{\sqrt{2}C_p^2}$, we obtain a sharper estimate,(without the assumption of Lemma 4.5)*

$$||u^m - U_h^m||_0^2 + \frac{1}{2}\sum_{n=1}^{m}\int_{t^{n-1}}^{t^n} \|u - U_h^n\|^2 dt \le ||u_0 - U_h^0||_0^2 + \sum_{n=1}^{m}\tau_n(\eta_{time}^n)^2 + C_2\sum_{n=1}^{m}\tau_n(\eta_{space}^n)^2.$$

Proof. We integrate Corollary 4.6 from $t = 0$ to $t = t^m$ to get the result. □

4.2 Local Lower Bound

The local lower bound is used for improving the finite element solutions at the fixed time $t = t^n$, with the given initial data as the solution from the previous time step $U_h^{n-1} \in V_0^{n-1}$. To compare the error, we consider $U_*^n \in H_0^1(\Omega)$, a solution of the auxiliary problem

$$(\frac{U_*^n - U_h^{n-1}}{\tau_n}, \varphi)_0 + b(U_*^n, \varphi) = (f_*^n, \varphi)_0 \qquad \forall \varphi \in H_0^1(\Omega), \tag{4.7}$$

where $f_*^n := f(U_*^n)$.

Note 4.9. The equation in (4.7) is the corresponding weak form for the discrete problem (3.4) where $H_0^1(\Omega)$ is approximated by V_0^n.

Again, we measured the local error $U_*^n - U_h^n$ using the L^2-norm. Since error indicators η_K^n consist of 2 parts, the area-based and edge-based residuals, to bound the error indicators, we estimated the two residuals using the idea of element and edge bubble functions.

For convenience, we denote the square of error on element $K \in \mathcal{M}^n$ by

$$err_n^2(K) = \frac{h_K^2 ||U_*^n - U_h^n||_{0,K}^2}{\tau_n^2} + |U_*^n - U_h^n|_{1,K}^2.$$

Lemma 4.10. *(Error Representation) For any $\varphi \in H_0^1(\Omega)$,*

$$b(U_*^n - U_h^n, \varphi) = (f_*^n - f_h^n, \varphi)_0 - (\frac{U_*^n - U_h^n}{\tau_n}, \varphi)_0 + \sum_{K \in \mathcal{M}^n} \int_K R^n \varphi dx + \sum_{e \in \mathcal{B}^n} \int_e J_e^n \varphi \, ds$$

Proof. Let $\varphi \in H_0^1(\Omega)$.

$$\begin{aligned} b(U_*^n - U_h^n, \varphi) & \\ &= b(U_*^n, \varphi) - b(U_h^n, \varphi) \\ &= \left[(f_*^n, \varphi)_0 - (\tfrac{U_*^n - U_h^{n-1}}{\tau_n}, \varphi)_0 \right] + \sum_{K \in \mathcal{M}^n} \int_K \nabla \cdot (a \nabla U_h^n) \varphi \, dx \\ &\quad + \sum_{e \in \mathcal{B}^n} \int_e J_e^n \varphi \, ds \\ &= (f_*^n - f_h^n, \varphi)_0 - (\tfrac{U_*^n - U_h^n}{\tau_n}, \varphi)_0 + \sum_{K \in \mathcal{M}^n} \int_K R^n \varphi dx + \sum_{e \in \mathcal{B}^n} \int_e J_e^n \varphi \, ds \end{aligned}$$

□

4.2.1 Estimate of R^n

First, let $\mathcal{P}_K : L^2(K) \to \mathbb{P}_l(K)$ be a L^2-projection to a space of polynomials of degree $\leq l$ on K.

Lemma 4.11. *For $n \geq 1$ and $K \in \mathcal{M}^n$, we have the estimate, there exist constants $c_1, c_2 > 0$ such that*

$$\begin{aligned} h_K^2 ||\mathcal{P}_K R^n||_{0,K}^2 &\leq c_1 (h_K^2 ||\mathcal{P}_K R^n - R^n||_{0,K}^2 + h_K^2 ||f_h^n - f_*^n||_{0,K}^2) \\ &\quad + c_2 err_n^2(K). \end{aligned}$$

Proof. Let $K \in \mathcal{M}^n$ and ψ_K be the element bubble function for the element K. Define $w = \psi_K \cdot \mathcal{P}_K R^n$. Note that $w \in \mathbb{P}_l(K)$ since ψ_K and $\mathcal{P}_K R^n$ are polynomials.

By proposition of bubble function, so

$$C^{-1}||\mathcal{P}_K R^n||_{0,K}^2 \leq ||\psi_K^{\frac{1}{2}}\mathcal{P}_K R^n||_{0,K}^2 = (\mathcal{P}_K R^n, w)_{0,K}.$$

Since $w|_{\partial K} = 0$, we can extend w to the full domain Ω by letting $w = 0$ outside element K, so that $w \in H_0^1(\Omega)$.

Thus, $(\mathcal{P}_K R^n, w)_0 = (\mathcal{P}_K R^n, w)_{0,K}$.

By the Lemma 4.10, we set $\varphi = w$, so

$$(R^n, w)_{0,K} = (f_h^n - f_*^n, w)_{0,K} + (\frac{U_*^n - U_h^n}{\tau_n}, w)_{0,K} + (a\nabla(U_*^n - U_h^n), \nabla w)_{0,K}$$

and

$$\begin{aligned}(\mathcal{P}_K R^n, w)_{0,K} &\\ &= (\mathcal{P}_K R^n - R^n, w)_{0,K} + (R^n, w)_{0,K} \\ &= (\mathcal{P}_K R^n - R^n, w)_{0,K} + (f_h^n - f_*^n, w)_{0,K} \\ &\quad + (\tfrac{U_*^n - U_h^n}{\tau_n}, w)_{0,K} + (a\nabla(U_*^n - U_h^n), \nabla w)_{0,K}\end{aligned}$$

Thus, we get the inequality

$$\begin{aligned}C^{-1}||\mathcal{P}_K R^n||_{0,K}^2 &\leq (\mathcal{P}_K R^n - R^n, w)_{0,K} + (f_h^n - f_*^n, w)_{0,K} \\ &\quad +(\tfrac{U_*^n - U_h^n}{\tau_n}, w)_{0,K} + (a\nabla(U_*^n - U_h^n), \nabla w)_{0,K}\end{aligned}$$

Then we apply Cauchy-Schwarz inequality to the above inequality.

By proposition of the bubble function and $w \in \mathbb{P}_l$, so $||\nabla w||_{0,K} \leq ch_K^{-1}||w||_{0,K}$ and $||w||_{0,K} \leq ||\mathcal{P}_K R^n||_{0,K}$.

Apply Cauchy-Schwarz and get

$$\begin{aligned}||\mathcal{P}_K R^n||_{0,K} &\leq C\left(||\mathcal{P}_K R^n - R^n||_{0,K} + ||f_h^n - f_*^n||_{0,K}\right) \\ &\quad +C\left(||\tfrac{U_*^n - U_h^n}{\tau_n}||_{0,K} + h_K^{-1}|U_*^n - U_h^n|_{1,K}\right).\end{aligned}$$

We multiply the inequality by h_K and get

$$\begin{aligned} h_K||\mathcal{P}_K R^n||_{0,K} &\leq C\left(h_K||\mathcal{P}_K R^n - R^n||_{0,K} + h_K||f_h^n - f_*^n||_{0,K}\right) \\ &\quad + C\left(h_K||\frac{U_*^n - U_h^n}{\tau_n}||_{0,K} + |U_*^n - U_h^n|_{1,K}\right). \end{aligned}$$

From the fact, if $a, b, c \geq 0$ and $a \leq b + c$ then $a^2 \leq 2(b^2 + c^2)$. We square the both sides of the inequality to get

$$\begin{aligned} h_K^2||\mathcal{P}_K R^n||_{0,K}^2 &\leq C\left(h_K^2||\mathcal{P}_K R^n - R^n||_{0,K}^2 + h_K^2||f_h^n - f_*^n||_{0,K}^2\right) \\ &\quad + C\left(\frac{h_K^2}{\tau_n^2}||U_*^n - U_h^n||_{0,K}^2 + |U_*^n - U_h^n|_{1,K}^2\right). \end{aligned}$$

Now, by definition of $err_n^2(K)$, we complete the proof. □

Lemma 4.12. *For $n \geq 1$ and $K \in \mathcal{M}^n$, we have the estimate, there exist constants $c_3, c_4 > 0$ such that*

$$\begin{aligned} h_K^2||R^n||_{0,K}^2 &\leq c_3(h_K^2||\mathcal{P}_K R^n - R^n||_{0,K}^2 + h_K^2||f_h^n - f_*^n||_{0,K}^2) \\ &\quad + c_4 err_n^2(K). \end{aligned}$$

Proof. By triangle inequality,

$$||R^n||_{0,K} = ||\mathcal{P}_K R^n + (R^n - \mathcal{P}_K R^n)||_{0,K} \leq ||\mathcal{P}_K R^n||_{0,K} + ||R^n - \mathcal{P}_K R^n||_{0,K}.$$

We multiply the inequality by h_K and square on both side to get

$$h_K^2||R^n||_{0,K}^2 \leq 2(h_K^2||\mathcal{P}_K R^n||_{0,K}^2 + h_K^2||R^n - \mathcal{P}_K R^n||_{0,K}^2).$$

Apply the Lemma 4.11 and complete the proof. □

4.2.2 Estimate of J_e^n

Let $\mathcal{P}_e : L^2(e) \to \mathbb{P}_l(e)$ be a L^2-projection onto the space of polynomials on e of degree $\leq l$.

Lemma 4.13. *For any $n \geq 1$ and $e \in \mathcal{B}^n$, the $h_e||\mathcal{P}_e J_e^n||_{0,e}^2$ can be bounded by*

$$\begin{aligned} h_e||\mathcal{P}_e J_e^n||_{0,e}^2 &\leq c_5 h_e||\mathcal{P}_e J_e^n - J_e^n||_{0,e} + c_6 \sum_{K' \subset \omega_e} err_n^2(K') \\ &\quad + c_7 \sum_{K' \subset \omega_e} \left(h_{K'}^2 ||R^n||_{0,K'}^2 + h_{K'}^2 ||f_h^n - f_*^n||_{0,K}^2 \right). \end{aligned}$$

Proof. Let $e \in \mathcal{B}^n$ and ψ_e be the bubble function for the edge e.

Since J_e^n is a function define on the edge e, we can extend J_e^n constantly along the normal of e to ω_e.

Define $w = \psi_e \cdot \mathcal{P}_e J_e^n$. Since supp $w = \omega_e$, we can extend w by $w = 0$ outside ω_e, so that $w \in H_0^1(\Omega)$. Note that $w \in \mathbb{P}_l(\omega_e)$ since ψ_e and $\mathcal{P}_e J_e^n$ are both polynomials.

By proposition of bubble function for the edge e,

$$C^{-1}||\mathcal{P}_e J_e^n||_{0,e}^2 \leq ||\psi_e^{\frac{1}{2}} \mathcal{P}_e J_e^n||_{0,e}^2 = (\mathcal{P}_e J_e^n, w)_{0,e}.$$

By the Lemma 4.10, we set $\varphi = w$, so

$$\begin{aligned} (J_e^n, w)_{0,e} &= (f_*^n - f_h^n, w)_{0,\omega_e} + (R^n, w)_{0,\omega_e} \\ &\quad -(\tfrac{U_*^n - U_h^n}{\tau_n}, w)_{0,\omega_e} - (a\nabla(U_*^n - U_h^n), \nabla w)_{0,\omega_e}, \end{aligned}$$

so

$$
\begin{aligned}
C^{-1}||\mathcal{P}_e J_e^n||_{0,e}^2 &\leq (\mathcal{P}_e J_e^n, w)_{0,e} \\
&= (\mathcal{P}_e J_e^n - J_e^n, w)_{0,e} + (J_e^n, w)_{0,e} \\
&= (\mathcal{P}_e J_e^n - J_e^n, w)_{0,e} + (f_*^n - f_h^n, w)_{0,\omega_e} + (R^n, w)_{0,\omega_e} \\
&\quad -(\tfrac{U_*^n - U_h^n}{\tau_n}, w)_{0,\omega_e} - (a\nabla(U_*^n - U_h^n), \nabla w)_{0,\omega_e}
\end{aligned}
$$

We apply Cauchy-Schwarz inequality and proposition of bubble function such that

$$
\begin{aligned}
||\nabla w||_{0,K} &\leq c h_K^{-1} ||w||_{0,K} \\
||w||_{0,K} &\leq c h_e^{\frac{1}{2}} ||\mathcal{P}_e J_e^n||_{0,e} \\
||w||_{0,e} &\leq ||\mathcal{P}_e J_e^n||_{0,e},
\end{aligned}
$$

then

$$
\begin{aligned}
||\mathcal{P}_e J_e^n||_{0,e} &\leq C||\mathcal{P}_e J_e^n - J_e^n||_{0,e} + C \sum_{K' \subset \omega_e} h_{K'}^{\frac{1}{2}} \left(||f_*^n - f_h^n||_{0,K'} + ||R^n||_{0,K'}\right) \\
&\quad + C \sum_{K' \subset \omega_e} \left(h_{K'}^{\frac{1}{2}} ||\tfrac{U_*^n - U_h^n}{\tau_n}||_{0,K'} + h_e^{-\frac{1}{2}} |U_*^n - U_h^n|_{1,K'} \right).
\end{aligned}
$$

We multiply the above inequality by $h_e^{\frac{1}{2}}$ and get

$$
\begin{aligned}
h_e^{\frac{1}{2}} ||\mathcal{P}_e J_e^n||_{0,e} &\leq C h_e^{\frac{1}{2}} ||\mathcal{P}_e J_e^n - J_e^n||_{0,e} + C \sum_{K' \subset \omega_e} h_{K'} \left(||f_*^n - f_h^n||_{0,K'} + ||R^n||_{0,K'}\right) \\
&\quad + C \sum_{K' \subset \omega_e} \left(h_{K'} ||\tfrac{U_*^n - U_h^n}{\tau_n}||_{0,K'} + |U_*^n - U_h^n|_{1,K'} \right).
\end{aligned}
$$

Square the both sides of inequality and get

$$
\begin{aligned}
h_e ||\mathcal{P}_e J_e^n||_{0,e}^2 &\leq C h_e ||\mathcal{P}_e J_e^n - J_e^n||_{0,e}^2 + C \sum_{K' \subset \omega_e} h_{K'}^2 \left(||f_*^n - f_h^n||_{0,K'}^2 + ||R^n||_{0,K'}^2\right) \\
&\quad + C \sum_{K' \subset \omega_e} \left(\tfrac{h_e^2}{\tau_n^2} ||U_*^n - U_h^n||_{0,K'}^2 + |U_*^n - U_h^n|_{1,K'}^2 \right).
\end{aligned}
$$

By definition of $err_n^2(K')$, we complete the proof. □

Lemma 4.14. *For any $n \geq 1$ and $e \in \mathcal{B}^n$, the $h_e||J_e^n||_{0,e}^2$ can be bounded by*

$$h_e||J_e^n||_{0,e}^2 \leq c_8 h_e||\mathcal{P}_e J_e^n - J_e^n||_{0,e} + c_9 \sum_{K' \subset \omega_e} err_n^2(K') + c_{10} \sum_{K' \subset \omega_e} \left(h_{K'}^2 ||R^n||_{0,K'}^2 + h_{K'}^2 ||f_h^n - f_*^n||_{0,K}^2 \right).$$

Proof. By triangle inequality, so

$$||J_e^n||_{0,e} = ||\mathcal{P}_e J_e^n + (J_e^n - \mathcal{P}_e J_e^n)||_{0,e} \leq ||\mathcal{P}_e J_e^n||_{0,e} + ||J_e^n - \mathcal{P}_e J_e^n||_{0,e}.$$

We multiply the inequality by $h_e^{\frac{1}{2}}$ and square on both sides to get

$$h_e||J_e^n||_{0,e}^2 \leq 2(h_e||\mathcal{P}_e J_e^n||_{0,K}^2 + h_e||J_e^n - \mathcal{P}_e J_e^n||_{0,K}^2).$$

Apply the Lemma 4.13 and complete the proof. □

4.2.3 Estimate of the error indicator η_K^n

Define an oscillation on $K \in \mathcal{M}^n$ by

$$osc^2(K) = h_K^2||\mathcal{P}_K R^n - R^n||_{0,K}^2 + \sum_{e \subset \partial K} h_e||\mathcal{P}_e J_e^n - J_e^n||_{0,e}^2$$

and

$$osc^2(\omega_K) = \sum_{K \subset \omega_K} osc^2(K).$$

Theorem 4.15. *(Local Lower Bound) There exist constants $\hat{C}_1, \hat{C}_2 > 0$ depending on Lipschitz constant L, such that for any $K \in \mathcal{M}^n$, the following estimate holds*

$$(\eta_K)^2 \leq \hat{C}_1 osc^2(\omega_K) + \hat{C}_2 \sum_{K' \subset \omega_K} err_n^2(K')$$

Proof. By definition of η_K^n in (4.1) and Lemma 4.14, we get

$$\begin{aligned}(\eta_K^n)^2 \;\leq\;& h_K^2||R^n||_{0,K}^2 + C\sum_{e\subset\partial K}\Big\{h_e||\mathcal{P}_eJ_e^n - J_e^n||_{0,e}^2 \\ &+ \sum_{K'\subset\omega_e}\left[h_{K'}^2\left(||f_*^n - f_h^n||_{0,K'}^2 + ||R^n||_{0,K'}^2\right) + err_n^2(K')\right]\Big\}\end{aligned}.$$

Since $\omega_K = \bigcup_{e\subset\partial K}\omega_e$,

$$\begin{aligned}(\eta_K^n)^2 \;\leq\;& C\sum_{K'\subset\omega_K} h_{K'}^2||R^n||_{0,K'}^2 + C\sum_{e\subset\partial K} h_e||\mathcal{P}_eJ_e^n - J_e^n||_{0,e}^2 \\ &+C\sum_{K'\subset\omega_K} h_K^2||f_*^n - f_h^n||_{0,K'}^2 + C\sum_{K'\subset\omega_K} err_n^2(K')\end{aligned}.$$

By Lemma 4.12 and Lipschitz condition of function f, we get

$$\begin{aligned}(\eta_K^n)^2 \;\leq\;& CL^2\sum_{K'\subset\omega_K} h_{K'}^2||U_*^n - U_h^n||_{0,K'}^2 \\ &+\hat{C}_1 osc^2(\omega_K) + \hat{C}_2\sum_{K'\subset\omega_K} err_n^2(K')\;. \\ \leq\;& \hat{C}_1 osc^2(\omega_K) + \hat{C}_2\sum_{K'\subset\omega_K} err_n^2(K')\end{aligned}$$

□

CHAPTER V

ADAPTIVE FINITE ELEMENT ALGORITHM

In this Chapter, we designed an adaptive algorithm for finite element methods for obtaining sequences of approximate solution $\{U_h^n\}$. The algorithm in this Chapter is designed based on the a posteriori error estimations obtained in the previous Chapter.

5.1 Time Error Control

Let TOL_{time} be the tolerance for the error control on time discretization. We control the error on time discretization according to

$$\sum_{n=1}^{m} \tau_n (\eta_{time}^n)^2 \leq TOL_{time}. \tag{5.1}$$

We used equi-distribute technique (equally distribution of errors on all elements) in order to control the time error indicators η_{time}^n by checking

$$(\eta_{time}^n)^2 < \frac{TOL_{time}}{T} \tag{5.2}$$

for all $n = 1, 2, ..., m$, to guarantee (5.1).

Given $\delta_1 \in (0, 1)$ and $\delta_2 > 1$, we use δ_1 to shorten the time-step size τ_n in order to reduce the time error indicator, and if the error indicator is too small, we expand the time-step size with δ_2 in order to improve the performance.

Typically, the smaller time-step size, the more accuracy we get. But if the time-step size is too small, it will reduce the performance of the program, namely, more loops in the program. So we may control the time error indicator in such a way that

$$\frac{\theta_{time} TOL_{time}}{T} \leq (\eta_{time}^n)^2 \leq \frac{TOL_{time}}{T} \tag{5.3}$$

where $\theta_{time} \in (0, 1)$ is a chosen parameter. (Typically, the value of θ_{time} is 0.5)

The following is an algorithm for obtaining a suitable time-step size τ_n with given parameters TOL_{time}, δ_1, δ_2 and τ_{n-1}, where $\delta_1\delta_2 < 1$.

Time Step Control Algorithm

1. Set $\tau_n = \tau_{n-1}$.

2. Solve for U_h^n and compute the error time indicators η_{time}^n.

3. If (5.3) is satisfy, then exit the loop,

   else go to the next step.

4. If $(\eta_{time}^n)^2 > \frac{TOL_{time}}{T}$ do $\tau_n = \delta_1 \tau_n$ and go to step 2,

   else $\tau_n = \delta_2 \tau_n$ and go to step 2.

Remark 5.1. *This algorithm guarantees that* (5.3) *is satisfied in finite steps.*

5.2 Space Error Control

We balance between accuracy and performance by the controlling parameter $\theta_{space} \in (0, 1)$ by checking the condition

$$\frac{\theta_{space} TOL_{space}}{T} \leq (\eta_{space}^n)^2 \leq \frac{TOL_{space}}{T}. \tag{5.4}$$

With mesh $\mathcal{M}^n$, we refine the mesh in order to increase accuracy and coarsen the mesh for maintaining performance. Let TOL_{space} be the tolerance for the space error control,

Adaptive Finite Element Algorithm

1. Set $\mathcal{M}^n = \mathcal{M}^{n-1}$.
2. Find the suitable τ_n using **Time Step Control Algorithm**.

   Compute the error indicators η_K^n and estimator η_{space}^n.
3. $t^n = t^{n-1} + \tau_n$.
4. While (5.4) is not satisfied do
   (a) Refine/Coarsen the mesh $\mathcal{M}^n$ to obtain a new $\mathcal{M}^n$.
   (b) Solve for U_h^n.
   (c) Compute the error indicators η_K^n for all $K \in \mathcal{M}^n$.
5. Check τ_n. If η_{time}^n is satisfied then exited this loop, else go to step 2.

From the algorithm, we will be looping in steps 2 to 4 until the error estimates are in the ranges we set, then exit the loops in step 5.

Refine/Coarsen Algorithm

With a given $\theta_{refine}, \theta_{coarsen} \in (0, 0.5]$, we compute error indicator for each element $K \in \mathcal{M}^n$, we sort the element by value of error indicator. We refine the first $\theta_{refine}N$ elements and coarsen the last $\theta_{coarsen}N$ elements to obtain a new mesh where N is the total number of element in $\mathcal{M}^n$.

CHAPTER VI

CONCLUSION

In this work, we used standard residual technique to derive a posteriori error estimate for semi-linear parabolic PDEs. From the results in upper and local lower bound, we see the true errors that occur in the system come from the approximation of the time derivative, from the finite element space and from the nonlinear function f. We control the errors by using adaptive technique on time and space. But the error from the nonlinear function, we use Lipschitz condition in order to absorb this error to the error in the system.

From the upper bound, it shows that we can control the total error by controlling the error indicators and estimators from time and space. The local lower bound shows that the local error can be controlled by controlling the local error indicators; we can reduce the local errors by refining elements with high error indicators, assuming that the oscillation and error from nonlinear function f are small. We finally use the result from the upper and local lower bounds to design an adaptive algorithm to control the error in the system based on time and space error indicators.

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