ทฤษฎีเชิงสเปกตรัมบนพีชคณิตซีสตาร์ไครน์สมมาตรเต็มสลับที่ซึ่งมีเอกลักษณ์

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SPECTRAL THEORY ON UNITAL COMMUTATIVE

FULL SYMMETRIC KREĬN C*-ALGEBRAS

Mr. Pichkitti Bannangkoon

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By	Mr. Pichkitti Bannangkoon
Field of Study	Mathematics
Thesis Advisor	Associate Professor Wicharn Lewkeeratiyutkul, Ph.D.
Thesis Co-Advisor	Paolo Bertozzini, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree

S. Hannongbua .... Dean of the Faculty of Science

(Professor Supot Hannongbua, Dr.rer.nat.)

THESIS COMMITTEE

Yupaporn Kemprasit .... Chairperson

(Professor Yupaporn Kemprasit, Ph.D.)

.... Advisor

(Associate Professor Wicharn Lewkeeratiyutkul, Ph.D.)

.... Co-Advisor

(Paolo Bertozzini, Ph.D.)

Zen Harper .... Examiner

(Zen Harper, Ph.D.)

.... External Examiner

(Associate Professor Pachara Chaisuriya, Ph.D.)

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เราเรียกพีชคณิตอาวัต $(A,*)$ ว่า พีชคณิตซีสตาร์ไครน์ ถ้า $(A,*)$ มีค่าประจำพีชคณิตบานาคอย่างน้อยหนึ่งค่าประจำ และมีสมมาตรหลักมูลของ $(A,*)$ กล่าวคือ $\alpha \in \mathrm{Aut}(A,*)$ ซึ่ง $\alpha^2 = 1_A$ และ $\|\alpha(x)^* x\| = \|x\|^2$ สำหรับทุก $x \in A$ เป้าหมายสูงสุดคือการพัฒนาทฤษฎีเชิงสเปกตรัมบนพีชคณิตซีสตาร์ไครน์สมมาตรเต็มสลับที่ซึ่งมีเอกลักษณ์ เมื่อส่วนคี่เป็นทวิมอดูลสมมาตรอิมพริมิทิวิทีบนส่วนคู่และมี "สมมาตรแลกเปลี่ยน" ε ที่เหมาะสมระหว่าง A_+ และ A_- ผลลัพธ์ที่ได้เป็นนัยทั่วไปของทฤษฎีเชิงสเปกตรัมบนพีชคณิตซีสตาร์สลับที่ซึ่งมีเอกลักษณ์

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PICHKITTI BANNANGKOON : SPECTRAL THEORY ON UNITAL COMMUTATIVE FULL SYMMETRIC KREĬN C*-ALGEBRAS. ADVISOR : ASSOC. PROF. WICHARN LEWKEERATIYUTKUL, Ph.D. CO-ADVISOR : PAOLO BERTOZZINI, Ph.D. 31 pp.

An involutive algebra $(A, *)$ is called a Kreĭn C*-algebra if it admits at least one Banach algebra norm and one fundamental symmetry of $(A, *)$, i.e., $\alpha \in \mathrm{Aut}(A, *)$ such that $\alpha^2 = 1_A$ and $\|\alpha(x^*)x\| = \|x\|^2$ for all $x \in A$. The ultimate goal is to develop a spectral theory on unital commutative Kreĭn C*-algebras when the odd part is a symmetric imprimitivity bimodule over the even part and there exists a suitable "exchange symmetry" ε between A_+ and A_-. The result we obtained is a generalization of spectral theory on unital commutative C*-algebras.

Department : ....Mathematics....	Student's Signature : พิชญ์กิตติ บรรณางกูร
Field of Study : ....Mathematics....	Principal Advisor's Signature : ....
Academic Year : ....2008...........	Co-Advisor's Signature : ....

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Finally, I would like to dedicate this accomplishment to my beloved parents- Terdgiad and Wanida Bannangkoon- the most important persons in my life who have always been there.

CONTENTS

CHAPTER I
INTRODUCTION

The study of vector spaces equipped with "inner products" that are not necessarily positive-definite has always been a theme of extreme importance in relativistic physics starting probably with the work of H.Minkowski. Complex vector spaces with indefinite sesquilinear forms have been introduced in relativistic quantum field theory by P.Dirac [5], W.Pauli [14] and then used by S.Gupta [9] K.Bleuler [3], although their mathematical definition has been given later by L.Pontrjagin [15]. Complete "indefinite" inner product spaces, called Kreĭn spaces, have been introduced by Ju.Ginzburg [8] and E.Scheibe [16] and the study of their properties has been undertaken by several Russian mathematicians.

Although algebras of continuous operators on Kreĭn spaces have been around for some time, the first definition of an abstract Kreĭn C*-algebra has been provided only recently by K.Kawamura [10, 11]. Kreĭn C*-algebras are somehow expected to play some role in a "semi-Riemannian" version of A.Connes noncommutative geometry [4] (see A.Strohmaier [17], M.Pasche-A.Rennie-R.Verch [13] for details) and for this reason it should be of some interest to develop a spectral theory that is suitable for them.

It is the purpose of this paper to introduce a simple spectral theory for the special class of Kreĭn C*-algebras that decompose, via a fundamental symmetry, in the direct sum $A = A_+ \oplus A_-$ with A_+ a commutative C*-algebra and A_- a symmetric imprimitivity (anti-)Hilbert C*-bimodule over A_+ and that are equipped with an odd symmetry exchanging A_+ and A_-. Our main result (Theorem 3.28) is that every such Kreĭn C*-algebra turns out to be isomorphic (via Gel'fand tranform) to an algebra of continuous functions with values in a very elementary Kreĭn C*-algebra defined in detail in Theorem 3.10 .

The main result presented here actually can be obtained in at least a few other ways that we briefly describe here below:

- For a symmetric imprimitivity commutative unital Kreĭn C*-algebra A, the even part A_+, as a commutative unital C*-algebra, is isomorphic to the algebra of sections of a trivial complex line bundle over the Gel'fand spectrum $\Omega(A_+)$. The odd part A_- is a symmetric imprimitivity Hilbert C*-bimodule over A_+ and, making use of the spectral theorem for imprimitivity Hilbert C*-bimodules developed in [2], it is isomorphic to the bimodule of sections of a complex line bundle over the same $\Omega(A_+)$. Under the existence of an odd symmetry on A, the Witney sum of the previous two line bundles turns out to be a bundle of rank-one Krein C*-algebras isomorphic to $\mathbb{K}$ over $\Omega(A_+)$.

- Although we are not aware now of a specific reference to a Gel'fand theorem, once a specific fundamental symmetry/odd symmetry α, ε has been chosen on A, the Kreĭn C*-algebra becomes completely equivalent to a $\mathbb{Z}_2$-graded C*-algebra with a commutative even part and for such elementary C*-algebras spectral results are, for sure, obtainable as special cases from the general theory of C*-dynamical systems.

- To every unital Kreĭn C*-algebra equipped with a given fundamental symmetry, we can always associate a C*-category with two objects. In the case of imprimitivity commutative unital Kreĭn C*-algebras, such C*-category will be commutative and full according to the definition provided in [1] and our spectral theorem can be recovered as a trivial application of the general spectral theory for commutative full C*-categories developed in [1].

The techniques utilized here in the proof of this result are essentially an adaptation of those developed in [1] for a spectral theory of commutative full C*-categories. Our choice to develop a completely independent proof of the result can be justified from the desire to test and caliber some of the general techniques introduced in [1] in a simple situation that in the near future might be used as a "laboratory" for non-commutative extensions of the spectral theorem. It is expected that more powerful spectral theories for wider classes of Kreĭn C*-algebras might be developed using a Kreĭn version of the spaceoid Fell bundle introduced in [1].

CHAPTER II
PRELIMINARIES

For the convenience and usefulness of the reader, we provide here some background material and beneficial theorems on the theory of C*-algebras and Hilbert C*-modules (see [12], [2] for all the details).

Definition 2.1. *An **algebra** over the complex numbers is a complex vector space A equipped with a bilinear map (called product) $\cdot : A \times A \to A, \quad \cdot : (x, y) \mapsto xy$ and satisfies an associative law, that is, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in A$. The algebra is called **unital** if $\exists 1_A \in A \ \forall x \in A, \quad x \cdot 1_A = 1_A \cdot x = x$.*

Definition 2.2. *An algebra is called **involutive** or also a **$*$-algebra** if it is equipped with a function $* : A \to A$, such that:*

$$(x^*)^* = x \quad \forall x \in A,$$

$$(x \cdot y)^* = y^* \cdot x^* \quad \forall x, y \in A,$$

$$(\alpha x + \beta y)^* = \bar{\alpha} x^* + \bar{\beta} y^* \quad \forall \alpha, \beta \in \mathbb{C},\ \forall x, y \in A.$$

Definition 2.3. *A **normed algebra** is an algebra A that is also a normed space and that satisfies the property: $\|x \cdot y\| \leq \|x\| \cdot \|y\| \quad \forall x, y \in A$. A **Banach algebra** is a normed algebra that, as a normed space, is complete.*

Definition 2.4. *A **pre-C*-algebra** is an involutive algebra A that is also a normed algebra that satisfies the property $\|x^* x\| = \|x\|^2 \quad \forall x \in A$. A **C*-algebra** is a pre-C*-algebra that is also Banach algebra.*

Example 2.5. *The followings are examples of C*-algebras:*

1. *Let X be a compact Hausdorff space. Then the space $C(X)$ of all continuous complex-valued functions on X is a unital commutative C*-algebra with the*

following operations and norm: $\forall x \in X$,

$$(f+g)(x) = f(x)+g(x), \qquad (cf)(x) = cf(x),$$
$$(fg)(x) = f(x)g(x), \qquad (f^*)(x) = \overline{f(x)},$$
$$\|f\| = \sup\{|f(x)| : x \in X\}.$$

2. *The space* $B(H)$ *of bounded linear maps on a Hilbert space* H *is a unital C*-algebra that is non-commutative if* $\dim(H) > 1$, *with the following operations and norm:*

$$(T+S)(x) = T(x)+S(x), \qquad (cT)(x) = cT(x),$$
$$(T \circ S)(x) = T(S(x)), \qquad \|T\| = \sup_{x \neq 0} \frac{\|T(x)\|}{\|x\|},$$

the involution T^* *of* T *is the adjoint of* T: $\langle Tx \mid y \rangle = \langle x \mid T^*y \rangle$, $\forall x, y \in H$.

Theorem 2.6. *Let* B *be a unital subalgebra of a unital C*-algebra* A. *If* $x \in B$ *and* $x \in A^+ := \{z^*z \mid z \in A\}$, *then* $x \in B^+$.

Definition 2.7. *A* ***homomorphism*** *from an algebra* A *to an algebra* B *is a linear map* $\varphi : A \to B$ *such that*

$$\varphi(ab) = \varphi(a)\varphi(b) \quad \forall a, b \in A.$$

A $*$***-homomorphism*** $\varphi : A \to B$ *of* $*$*-algebras* A *and* B *is a homomorphism of algebras such that* $\varphi(a^*) = \varphi(a)^* \quad \forall a \in A$. *If in addition* φ *is a bijection, it is a* $*$***-isomorphism***.

Theorem 2.8. *A* $*$*-homomorphism* $\varphi : A \to B$ *from a Banach* $*$*-algebra* A *to a C*-algebra* B *is necessarily norm-decreasing.*

Definition 2.9. *A* ***character*** *on a commutative algebra* A *is a unital homomorphism* $\tau : A \to \mathbb{C}$. *We denote by* $\Omega(A)$ *the set of characters on* A.

For each $a \in A$, define $\hat{a} : \Omega(A) \to \mathbb{C}$ by $\hat{a}(\tau) = \tau(a)$ for all $\tau \in \Omega(A)$. Equip $\Omega(A)$ with the smallest topology on $\Omega(A)$ which makes each $\hat{a}$ continuous.

Theorem 2.10. *If A is a unital commutative Banach algebra, then $\Omega(A)$ is a compact Hausdorff space with respect to the topology defined above.*

Theorem 2.11 (Gel'fand-Mazur). *If A is a unital Banach algebra in which every non-zero element is invertible, then $A = \mathbb{C}1$.*

Theorem 2.12 (Spectral theorem). *If A is a non-zero unital commutative C^*-algebra, then the Gelfand transform*

$$\varphi : A \to C(\Omega(A)),\ a \mapsto \hat{a}, \quad (\hat{a} : \Omega(A) \to \mathbb{C}, \tau \mapsto \tau(a))$$

is an isometric $*$-isomorphism.

Definition 2.13. *A **right R-module** E over a ring R is an Abelian group $(E, +)$ equipped with an operation $\cdot : E \times R \to E$, $\cdot : (x, a) \mapsto xa$, of right multiplication by elements of the ring R, that satisfies the following properties:*

$$x \cdot (a + b) = (x \cdot a) + (x \cdot b), \quad \forall x \in E\, \forall a, b \in R,$$

$$(x + y) \cdot a = (x \cdot a) + (y \cdot a), \quad \forall x, y \in E\, \forall a \in R,$$

$$x \cdot (ab) = (x \cdot a) \cdot b, \quad \forall x \in E\, \forall a, b \in R.$$

*If the ring R is unital, we say that E is a **unital right R-module** if the additional property here is satisfied:*

$$x \cdot 1_R = x, \quad \forall x \in E.$$

*Analogously, we can define a **(unital) left R-module** E over a ring R.*

Note that a unital module over a unital complex algebra is naturally a complex vector space.

Definition 2.14. *A **unital right pre-Hilbert C^*-module** M_B over a unital C^*-algebra B is a unital right module over the unital ring B that is equipped with* *a B-valued inner product* $(x, y) \mapsto \langle x|y\rangle$ *such that*

$$\langle z \mid x + y\rangle_B = \langle z \mid x\rangle_B + \langle z \mid y\rangle_B \quad \forall x, y, z \in M,$$

$$\langle z \mid x \cdot b\rangle_B = \langle z \mid x\rangle_B b \quad \forall b \in B \quad \forall x, z \in M,$$

$$\langle y \mid x\rangle_B = \langle x \mid y\rangle_B^* \quad \forall x, y \in M,$$

$$\langle x \mid x\rangle_B \in B^+ \quad \forall x \in M,$$

$$\langle x \mid x\rangle_B = 0_B \Rightarrow x = 0_M.$$

Analogously, a ***left pre-Hilbert C*-module*** ${}_AM$ *over a unital C*-algebra A is a unital left module M over the unital ring A, that is equipped with an A-valued inner product* $M \times M \to A$ *denoted by* $(x, y) \mapsto_A \langle x \mid y\rangle$*. Here the A-linearity is on the first variable.*

Remark 2.15. *A right (respectively left) pre-Hilbert C*-module* M_B *over the C*-algebra B is naturally equipped with a norm*

$$\|x\|_M := \sqrt{\|\langle x \mid x\rangle_B\|_B} \quad \forall x \in M.$$

Definition 2.16. *A right (resp. left)* ***Hilbert C*-module*** *is a right (resp. left) pre-Hilbert C*-module over a C*-algebra B that is a Banach space with respect to the previous norm* $\|\cdot\|_M$*(resp.* ${}_M\|\cdot\|$*).*

Definition 2.17. *A right Hilbert C*-module* M_B *is said to be* ***full*** *if*

$$\langle M_B \mid M_B\rangle_B := \overline{\mathrm{span}\{\langle x \mid y\rangle_B \mid x, y \in M_B\}} = B,$$

where the closure is in the norm topology of the C-algebra B. A similar definition holds for a left Hilbert C*-module.*

We recall here the following well-known result, see for example in [7, Page 65]:

Lemma 2.18. *Let* M_B *be a right Hilbert C*-module over a unital C*-algebra B. Then* M_B *is full if and only if* $\mathrm{span}\{\langle x \mid y\rangle_B \mid x, y \in M_B\} = B$*.*

We also recall a few definitions and results on bimodules (see for example [2]).

Theorem 2.19. *([2, Proposition 2.6]) Let M_A be a right Hilbert C*-module over a unital C*-algebra A and $J \subset A$ an involutive ideal in A. Then the set $MJ := \{\sum_{j=1}^{N} x_j a_j \mid x_j \in M, a_j \in J, N \in \mathbb{N}_0\}$ is a submodule of M. The quotient module $M/(MJ)$ has a natural structure as a right Hilbert C*-module over the quotient C*-algebra A/J. If M is full over A, also $M/(MJ)$ is full over A/J. A similar statement holds for a left Hilbert C*-module.*

Recall that a unital bimodule ${}_AM_B$ over two unital rings A and B is a left unital A-module and a right unital B-module such that $(a \cdot x) \cdot b = a \cdot (x \cdot b)$, for all $a \in A, b \in B$ and $x \in M$.

Definition 2.20. *A **pre-Hilbert C*-bimodule** ${}_AM_B$ over a pair of unital C*-algebras A, B is a left pre-Hilbert C*-module over A and a right pre-Hilbert C*-module over B such that:*

$$(a \cdot x) \cdot b = a \cdot (x \cdot b) \quad \forall a \in A \forall x \in M \forall b \in B,$$

$$\langle x \mid ay \rangle_B = \langle a^* x | y \rangle_B \quad \forall x, y \in M \forall a \in A,$$

$$_A\langle xb \mid y \rangle =_A \langle x \mid yb^* \rangle \quad \forall x, y \in M \forall b \in B.$$

*A **Hilbert C*-bimodule** ${}_AM_B$ is a pre-Hilbert C*-bimodule over A and B that is simultaneously a left Hilbert C*-module over A and a right Hilbert C*-module over B. A full Hilbert C*-bimodule ${}_AM_B$ over the C*-algebras A and B is said to be an **imprimitivity bimodule** or an **equivalence bimodule** if:*

$$_A\langle x \mid y \rangle \cdot z = x \cdot \langle y \mid z \rangle_B \quad \forall x, y, z \in M.$$

Remark 2.21. *(see for example [2, Remark 2.14]) In an A-B pre-Hilbert C*-bimodule there are two, usually different, norms, one as a left-C*-module over A and one as a right-C*-bimodule over B:*

$$_M\|x\| := \sqrt{\|_A\langle x \mid x \rangle\|_A}, \quad \|x\|_M := \sqrt{\|\langle x \mid x \rangle_B\|_B}, \quad \forall x \in M,$$

but this two norms coincide for an imprimitivity bimodule. In fact,

$$\begin{aligned} _M\|x\|^4 &= \|_A\langle x \mid x \rangle\|_A^2 = \|_A\langle x \mid x \rangle _A\langle x \mid x \rangle\|_A = \|_A\langle x \langle x \mid x \rangle_B \mid x \rangle\|_A \\ &\leq \|\langle x \mid x \rangle_B\|_B \cdot_A \|\langle x \mid x \rangle\|_A = \|x\|_M^2 \cdot_M \|x\|^2. \end{aligned}$$

Definition 2.22. *We say that a bimodule* ${}_AM_A$ *is a **symmetric bimodule** if* $ax = xa$*, for all* $x \in M$ *and all* $a \in A$*. If* ${}_AM_A$ *is a Hilbert C*-bimodule, we say that it is a **symmetric C*-bimodule** if it is symmetric as a bimodule and* ${}_A\langle x \mid y \rangle = \langle y \mid x \rangle_A$ *for all* $x, y \in M$*.*

CHAPTER III
MAIN RESULTS

First of all, we review the definition of a Kreĭn C*-algebra that has been introduced by K.Kawamura [11] and explore some elementary properties of a Kreĭn C*-algebra.

Definition 3.1. *(K.Kawamura [11, Definition 2]) A* ***Kreĭn C*-algebra*** *is a $*$-algebra A admitting at least one Banach algebra norm and one* ***fundamental symmetry*** *, i.e. a $*$-automorphism $\phi : A \to A$ with $\phi \circ \phi = i_A$, such that $\|\phi(a^*)a\| = \|a\|^2$ for all $a \in A$.*

Note that, in general, a Kreĭn C*-algebra can admit several different fundamental symmetries.

Remark 3.2. *A C*-algebra A is a Kreĭn C*-algebra with fundamental symmetry 1_A and so a Kreĭn C*-algebra is a generalization of a C*-algebra. Let (A, α) be a Kreĭn C*-algebra with a given fundamental symmetry α. Then we always have the decomposition*

$$A = A_+ \oplus A_-, \quad \text{where} \quad A_+ = \{x \in A \mid \alpha(x) = x\},\ A_- = \{x \in A \mid \alpha(x) = -x\}.$$

Indeed, if $a \in A_+ \cap A_-$, then $a = \phi(a) = -a$ and so $a = 0$. Moreover for all $a \in A$, $a = \frac{a+\alpha(a)}{2} + \frac{a-\alpha(a)}{2}$ where $\frac{a+\alpha(a)}{2} \in A_+$ and $\frac{a-\alpha(a)}{2} \in A_-$.

Every Kreĭn C*-algebra A with a given fundamental symmetry α becomes naturally a C*-algebra, denoted here by $(A, \dagger_\alpha)$ when equipped with a new involution $\dagger_\alpha : A \to A$ as described in the following theorem.

Theorem 3.3. *Let (A, α) be a Kreĭn C*-algebra with fundamental symmetry α. Then A becomes a C*-algebra when equipped with the new involution $\dagger_\alpha$ defined by*

$\dagger_\alpha : x \mapsto \alpha(x)^*$. *Furthermore, the fundamental symmetry* α *is a* $*$*-automorphism of this C*-algebra and hence continuous in norm.*

Proof. Firstly, we compute

$$(xy)^{\dagger_\alpha} = \alpha(xy)^* = (\alpha(x)\alpha(y))^* = \alpha(y)^*\alpha(x)^* = y^{\dagger_\alpha}x^{\dagger_\alpha}$$

$$x^{\dagger_\alpha\dagger_\alpha} = \alpha(\alpha(x)^*)^* = \alpha(\alpha(x)^{**}) = \alpha(\alpha(x)) = x$$

$$\|x^{\dagger_\alpha}x\| = \|\alpha(x)^*x\| = \|x\|^2$$

for all $x, y \in A$. Then the first assertion is verified. To show that α is continuous, by applying Theorem 2.8, it suffices to show that α is a $*$-homomorphism from $(A, \dagger)$ into itself. Since $\alpha(x^{\dagger_\alpha}) = \alpha(\alpha(x)^*) = \alpha(\alpha(x))^* = \alpha(x)^{\dagger_\alpha}$ for all $x \in A$, we finish the proof of the theorem. $\square$

Remark 3.4. *Note that given a* $\dagger$*-homomorphism* $\alpha : A \to A$ *such that* $\alpha \circ \alpha = 1_A$ *of a C*-algebra* A *(with involution denoted by* $\dagger$*), we can naturally construct a Kreĭn C*-algebra* $(A, *_\alpha)$ *with involution* $x^{*_\alpha} := \alpha(x^\dagger)$*, for* $x \in A$ *and that* α *becomes a fundamental symmetry for this Kreĭn C*-algebra.*

Now we observe the algebraic structure of the even and odd part of (A, α).

Theorem 3.5. *Let* (A, α) *be a unital commutative Kreĭn C*-algebra with a given fundamental symmetry* α*. Then* A_+ *is a unital commutative C*-algebra and* A_- *is a unital Hilbert C*-bimodule over* A_+*.*

Proof. Note first that on A_+ the two involutions $*$ and $\dagger_\alpha$ coincide and since A_+ is closed under multiplication and involution, it is clearly a unital C*-algebra.

Next we will show that A_- is a Hilbert C*-bimodule over A_+. We define right and left multiplications from $A_- \times A_+$ and $A_+ \times A_-$ into A_- as usual multiplications in A and we define a pair of A_+-valued inner products from $A_- \times A_-$ into A_+ by

$$_{A_+}\langle x \mid y \rangle = xy^{\dagger_\alpha} \quad \text{and} \quad \langle x \mid y \rangle_{A_+} = x^{\dagger_\alpha}y \tag{3.1}$$

for all $x, y \in A_-$. With this definitions, we have

$${}_{A_+}\langle x+y \mid z\rangle = (x+y)z^{\dagger_\alpha} = xz^{\dagger_\alpha} + yz^{\dagger_\alpha} =_{A_+} \langle x \mid z\rangle +_{A_+} \langle y \mid z\rangle \quad \forall x, y, z \in A_-$$

$${}_{A_+}\langle ax \mid z\rangle = (ax)z^{\dagger_\alpha} = a(xz^{\dagger_\alpha}) = a_{A_+}\langle x \mid z\rangle \quad \forall x, y \in A_-, \forall a \in A_+$$

$${}_{A_+}\langle y \mid x\rangle = yx^{\dagger_\alpha} = (xy^{\dagger_\alpha})^{\dagger_\alpha} = ({}_{A_+}\langle x \mid y\rangle)^{\dagger_\alpha} \quad \forall x, y \in A_-$$

$${}_{A_+}\langle x \mid x\rangle = xx^{\dagger_\alpha} \in (A_+)^+ \quad \forall x \in A_- \text{ by Theorem 2.6}$$

$${}_{A_+}\langle x \mid x\rangle = 0 \Rightarrow xx^{\dagger_\alpha} = 0 \Rightarrow \|x\|^2 = \|\alpha(x)^*x\| = \|x^{\dagger_\alpha}x\| = 0 \Rightarrow x = 0 \quad \forall x \in A_-$$

$$\langle z \mid x+y\rangle_{A_+} = z^{\dagger_\alpha}(x+y) = z^{\dagger_\alpha}x + z^{\dagger_\alpha}y = \langle z \mid x\rangle_{A_+} + \langle z \mid y\rangle_{A_+} \quad \forall x, y, z \in A_-$$

$$\langle z \mid xb\rangle_{A_+} = z^{\dagger_\alpha}(xb) = (zx^{\dagger_\alpha})b = \langle z \mid x\rangle_{A_+}b \quad \forall x, y \in A_-, \forall b \in A_+$$

$$\langle y \mid x\rangle_{A_+} = y^{\dagger_\alpha}x = (x^{\dagger_\alpha}y)^{\dagger_\alpha} = a(\langle x \mid y\rangle_{A_+})^{\dagger_\alpha} \quad \forall x, y \in A_-$$

$$\langle x \mid x\rangle_{A_+} = x^{\dagger_\alpha}x \in (A_+)^+ \quad \forall x \in A_- \text{ by Theorem 2.6}$$

$$\langle x \mid x\rangle_{A_+} = 0 \Rightarrow x^{\dagger_\alpha}x = 0 \Rightarrow \|x\|^2 = \|\alpha(x)^*x\| = \|x^{\dagger_\alpha}x\| = 0 \Rightarrow x = 0 \quad \forall x \in A_-$$

$$(ax)b = a(xb) \quad \forall x \in A_-, \forall a, b \in A_+$$

$$\langle x \mid ay\rangle_{A_+} = x^{\dagger_\alpha}(ay) = (x^{\dagger_\alpha}a)y = (a^{\dagger_\alpha}x)^{\dagger_\alpha}y = \langle a^{\dagger_\alpha}x \mid y\rangle_{A_+} \quad \forall x, y \in A_-, \forall a \in A_+$$

$${}_{A_+}\langle xb \mid y\rangle = (xb)y^{\dagger_\alpha} = x(by^{\dagger_\alpha}) = x(yb^{\dagger_\alpha})^{\dagger_\alpha} =_{A_+} \langle x \mid yb^{\dagger_\alpha}\rangle \quad \forall x, y \in A_-, \forall a \in A_+.$$

We know that

$$\|x\|_{A_-} = \sqrt{\|\langle x \mid x\rangle_{A_+}\|_{A_+}} = \sqrt{\|x^{\dagger_\alpha}x\|_{A_+}} = \sqrt{\|x\|^2_{A_+}} = \|x\|_{A_+}$$
$${}_{A_-}\|x\| = \sqrt{{}_{A_+}\|_{A_+}\langle x \mid x\rangle\|} = \sqrt{{}_{A_+}\|xx^{\dagger_\alpha}\|} = \sqrt{{}_{A_+}\|x^{\dagger_\alpha}x\|} = \sqrt{{}_{A_+}\|x\|^2} = \|x\|_{A_+}.$$

Since $\|\cdot\|_A$ is complete, $\|\cdot\|_{A_-}$ and ${}_{A_-}\|\cdot\|$ are also complete. Hence A_- is a unital Hilbert C*-bimodule over A_+, as desired. □

Additionally the Hilbert C*-bimodule A_- has the property,

$${}_{A_+}\langle x \mid y\rangle z = (xy^{\dagger_\alpha})z = x(y^{\dagger_\alpha}z) = x\langle y \mid z\rangle_{A_+}.$$

for all $x, y, z \in A$.

For any Kreĭn C*-algebra equipped with a fundamental symmetry α, the odd part A_- in the fundamental decomposition $A = A_+ \oplus A_-$ is a Hilbert C*-bimodule but it is not in general an imprimitivity bimodule because A_- might not be a full bimodule over A_+. In the following we will usually assume this further property.

Definition 3.6. *Let (A, α) be a Kreĭn C*-algebra with a given fundamental symmetry α. A Kreĭn C*-algebra (A, α) is said to be **imprimitive** or* full *if its odd part A_- is an imprimitivity bimodule over A_+. We say that (A, α) is **rank-one** if $\dim A_+ = \dim A_- = 1$ as complex vector spaces.*

Theorem 3.7. *Let (A, α) be a Kreĭn C*-algebra with a given fundamental symmetry α. The algebra A is commutative if and only if the even part A_+ is a commutative unital C*-algebra and the odd part is a symmetric Hilbert C*-bimodule over A_+. In particular a rank-one Kreĭn C*-algebra is always commutative.*

Proof. If A is commutative, clearly A_- is a symmetric bimodule over the commutative algebra A_+. Furthermore the inner products defined in (3.1) above satisfy ${}_{A_+}\langle x \mid y\rangle = xy^{\dagger_\alpha} = y^{\dagger_\alpha}x = \langle y \mid x\rangle_{A_+}$ and hence A_- is a symmetric C*-bimodule over the commutative C*-algebra A_+. Conversely, if A_- is a symmetric C*-bimodule, $a_-b_- = {}_{A_+}\langle a_- \mid b_-^{\dagger_\alpha}\rangle = \langle b_-^{\dagger_\alpha} \mid a_-\rangle_{A_+} = b_-a_-$, for all $a_-, b_- \in A_-$. Now, for all $a, b \in A$ with decompositions $a = a_+ + a_-$, $b = b_+ + b_-$ an easy computation shows: $ab = (a_+ + a_-)(b_+ + b_-) = a_+b_+ + a_+b_- + a_-b_+ + a_- + b_- = b_+a_+ + b_-a_+ + b_+a_- + b_-a_- = ba$. □

Definition 3.8. *We will say that a Kreĭn C*-algebra is **symmetric** if there exists an **odd symmetry** i.e. a linear map $\varepsilon : A \to A$ such that $\varepsilon \circ \varepsilon = i_A$, $\varepsilon(x^*) = -\varepsilon(x)^*$ for all $x \in A$, $\varepsilon(xy) = \varepsilon(x)y = x\varepsilon(y)$ for all $x, y \in A$ and if there exists a fundamental symmetry α such that $\varepsilon \circ \alpha = -\alpha \circ \varepsilon$ (in this case we say that the symmetry and the odd symmetry are compatible).*

By the properties of odd symmetry, we further have $\varepsilon(x)\varepsilon(y) = \varepsilon \circ \varepsilon(xy)$ and $\varepsilon(x)^{\dagger_\alpha} = \varepsilon(x^{\dagger_\alpha})$ which are useful in the proof of the next lemma.

Lemma 3.9. *If ε is an odd symmetry of a symmetric Kreĭn C*-algebra compatible with the symmetry α, then ε is always isometric.*

Proof. For all $x \in A$, $\|\varepsilon(x)\|^2 = \|\varepsilon(x)^{\dagger_\alpha}\varepsilon(x)\| = \|\varepsilon(x^{\dagger_\alpha})\varepsilon(x)\| = \|\varepsilon \circ \varepsilon(x^{\dagger_\alpha}x)\| = \|x^{\dagger_\alpha}x\| = \|x\|^2$. □

Next we give an example of a rank-one unital full symmetric Kreĭn C*-algebra, which will play an important role in the proof of our spectral theory. Since, as we will see, in this case, the fundamental symmetry is necessarily unique, this is just a well-known example of a commutative $\mathbb{Z}_2$-graded C*-algebra. Although all the properties described in the following three theorems are "standard" from the theory of $\mathbb{Z}_2$-graded C*-algebras, for the convenience of the reader we present here a direct proof of all of them in the "spirit" of Kreĭn C*-algebras.

Let A be a 2×2 complex matrix. We define left multiplication operator on the Hilbert space $\mathbb{C}^2$, $L_A : \mathbb{C}^2 \to \mathbb{C}^2$ by $L_A(\psi) := A\psi$ for all $\psi \in \mathbb{C}^2$. Note that $L : A \mapsto L_A$ is a unital $*$-homomorphism in particular $L_{AB} = L_A \circ L_B$ and $L_{A^\dagger} = {L_A}^\dagger$ where $A^\dagger := JAJ$ with $J = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ and ${L_A}^\dagger$ is the adjoint of L_A in the C*-algebra $B(\mathbb{C}^2)$.

For each a, $b \in \mathbb{C}$, let $T_{a,b} = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$. Let $\mathbb{K} = \big\{T_{a,b} : a, b \in \mathbb{C}\big\}$ with the usual matrix operations of addition and multiplication. We define the involution by $T^*_{a,b} = T_{\bar{a},-\bar{b}}$. Furthermore, we equip $\mathbb{K}$ with the operator norm $\|A\| = \|L_A\|$ (where the right-hand-side norm is the operator norm and choose a fundamental symmetry $\gamma : \mathbb{K} \to \mathbb{K}$ defined by $\gamma\left(\begin{bmatrix} a & b \\ b & a \end{bmatrix}\right) = \begin{bmatrix} a & -b \\ -b & a \end{bmatrix}$.

Theorem 3.10. *There is a rank-one unital symmetric Kreĭn C*-algebra.*

Proof. It is easy to verify that γ is a $*$-automorphism on $\mathbb{K}$ and $\gamma \circ \gamma = 1_{\mathbb{K}}$, where $1_{\mathbb{K}}$ is the identity map on $\mathbb{K}$.For all matrix A in $\mathbb{K}$, we obtain

$$\begin{aligned} \|\gamma(A^*)A\| = \|A^\dagger A\| &= \|L_{A^\dagger A}\| = \|L_{A^\dagger} \circ L_A\| \\ &= \|L_A^\dagger \circ L_A\| = \|L_A\|^2 = \|A\|^2. \end{aligned}$$

Then we can write $\mathbb{K} = \mathbb{K}_+ \oplus \mathbb{K}_-$ where

$$\mathbb{K}_+ = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in \mathbb{C} \right\} \quad \text{and} \quad \mathbb{K}_- = \left\{ \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} : b \in \mathbb{C} \right\}.$$

Since $\mathbb{K}_+ = \left\langle \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle$ and $\mathbb{K}_- = \left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle$, we see that $\dim \mathbb{K}_+ = \dim \mathbb{K}_- = 1$. The symmetry of the algebra $\mathbb{K}$ can be checked by defining $\varepsilon(T_{a,b}) := T_{b,a}$. □

Theorem 3.11. *The identity $\iota_{\mathbb{K}}$ and γ are the only two unital $*$-automorphisms from $\mathbb{K}$ onto itself. Furthermore, γ is the unique fundamental symmetry on $\mathbb{K}$.*

Proof. Let ϕ be a unital $*$-homomorphism from $\mathbb{K}$ into itself. Suppose that $\phi(e) = \phi(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}) = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ for some $a, b \in \mathbb{C}$. Since $e \cdot e = 1$ and $e^* = -e$, we have $\phi(e)\phi(e) = \phi(1) = 1$ and $\phi(e^*) = -\phi(e)$. From these conditions, we have

$$\begin{bmatrix} a^2+b^2 & 2ab \\ 2ab & a^2+b^2 \end{bmatrix} = \begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \bar{a} & -\bar{b} \\ -\bar{b} & \bar{a} \end{bmatrix} = \begin{bmatrix} a & b \\ b & a \end{bmatrix}^* = -\begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} -a & -b \\ -b & -a \end{bmatrix}.$$

Then we have four equations as follows: $a^2 + b^2 = 1$, $ab = 0$, $\bar{a} = -a$ and $\bar{b} = b$. By basic arithmetic, we have $(a, b) = (0, 1)$ or $(a, b) = (0, -1)$. Hence $\phi(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ or $\phi\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. By linearity of ϕ, we obtain that

$$\phi\left(\begin{bmatrix} a & b \\ b & a \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + b\phi\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \text{ or } \begin{bmatrix} a & -b \\ -b & a \end{bmatrix}$$

for all $a, b \in \mathbb{C}$. That is $\phi = 1_{\mathbb{K}}$ or γ, as desired.

For the second assertion, we let $\phi = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be the fundamental symmetry on the rank-one unital Kreĭn C*-algebra $\mathbb{K}$. By the property $\phi(1) = 1$, we have

$$\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and so ϕ has to be in of the form $\begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix}$. By the property that $\phi \circ \phi$ is the identity map,

$$\begin{bmatrix} x+by+bdy \\ d^2y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

By easy calculation, we have 2 solutions $(b,d) = (0,1)$ and $(b,d) = (k,-1)$ where k is arbitrary in $\mathbb{C}$. Hence ϕ has to be in the form $\begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Since the second possibility is the identity map making A not rank 1, ϕ must be in the form $\begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix}$ where $k \in \mathbb{C}$. Since

$$\phi(e \cdot e) = \begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \exp(i\theta) \\ 0 \end{bmatrix} = \begin{bmatrix} \exp(i\theta) \\ 0 \end{bmatrix}$$

and

$$\phi(e)^2 = \left(\begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)^2 = \left(\begin{bmatrix} k \\ -1 \end{bmatrix} \right)^2 = \begin{bmatrix} k^2 + \exp(i\theta) \\ -2k \end{bmatrix},$$

by the property $\phi(e \cdot e) = \phi(e)^2$, it follows that $k = 0$. Hence ϕ must be of the form $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ which is equal to γ, as claimed. □

Like Gel'fand-Mazur Theorem in C*-algebra, we can characterize the rank-one unital Kreĭn C*-algebras.

Theorem 3.12. *Every rank-one unital Kreĭn C*-algebra is isomorphic to* $\mathbb{K}$.

Proof. Since $\dim A_+ = 1$, there is a non-zero element a such that $\langle a \rangle = A_+$. Since the unit 1 is in A_+, there is $m \in \mathbb{C}$ such that $ma = 1$. Then $(m1)a = 1 = a(m1)$ and so a is invertible. Let $b \in A_+$ be a nonzero element. There is $n \in \mathbb{C}$ such that $b = na$. Then

$$b \cdot \frac{a^{-1}}{n} = na \cdot \frac{a^{-1}}{n} = 1 \qquad \text{and} \qquad \frac{a^{-1}}{n} \cdot b = \frac{a^{-1}}{n} \cdot na = 1.$$

Hence every non-zero element in A_+ is invertible. By Theorem 3.5, A_+ is a unital commutative C*-algebra over $\mathbb{C}$. By Gel'fand-Mazur Theorem (Theorem 2.11), $A_+ = \mathbb{C}1$.

Since $\dim A_- = 1$, there is a non-zero element e such that $\langle e \rangle = A_-$. Without loss of generality, we choose e such that $\|e\| = 1$. Since $e^* \in A_-$ and $e \cdot e \in A_+$, we have $e^* = \alpha e$ and $e \cdot e = \beta$ for some $\alpha, \beta \in \mathbb{C}$.

From the property $(xy)^* = y^* x^*$ for all $x, y \in A$, we have

$$\bar{\beta} = \alpha^2 \beta. \tag{3.2}$$

From the property $x^{**} = x$ for all x, we have

$$|\alpha| = 1. \tag{3.3}$$

From the property $\|\alpha(x)^* x\| = \|x\|^2$ for all x, we have

$$|\alpha\beta| = \|e\|^2 = 1. \tag{3.4}$$

From equations (3.3) and (3.4), we have $|\beta| = 1$. Since $|\alpha| = |\beta| = 1$, we can write $\alpha = \exp(i\delta)$ and $\beta = \exp(i\theta)$ for some $\delta, \theta \in \mathbb{C}$. From (3.2), $\alpha^2 = \frac{\bar{\beta}}{\beta} = \frac{(\bar{\beta})^2}{|\beta|^2} = (\bar{\beta})^2$ and then $\exp(2i\delta) = \exp(-2i\theta)$. Thus $\delta = -\theta + \pi k$ for all $k \in \mathbb{N}$ and so $\alpha = \exp(i\delta) = \exp(-i\theta + i\pi k)$ for all $k \in \mathbb{N}$, that is, $\alpha = \pm\exp(-i\theta)$ for some $\theta \in [0, 2\pi)$. Now we have that $e^* = \pm\exp(-i\theta)e$ and $e \cdot e = \exp(i\theta)$ for some $\theta \in [0, 2\pi)$. We use the notation $A_{\theta\pm}$ corresponding to the previous operations.

We claim that $\theta = 0$. Since a Kreĭn a C*-algebra is C*-algebra, by Theorem 3.3 and the property in C*-algebra that $\|x^\dagger\| = \|x\| = \|L_x\|$, we have

$$\|L_{\gamma(x^*)}\| = \|L_x\|. \tag{3.5}$$

Furthermore, we can see that

$$\sigma(L_x) = \left\{ \lambda \mid \det\left(\begin{bmatrix} a-\lambda & b \\ b & a-\lambda \end{bmatrix} \right) = 0 \right\} = \{a+b, a-b\}$$

and so

$$\|L_x\| = \sup_{\lambda \in \sigma(L_x)} |\lambda| = \max\{|a+b|, |a-b|\}. \tag{3.6}$$

Similarly we obtain

$$\begin{aligned} \sigma(L_{\gamma(x^*)}) &= \left\{ \lambda \mid \det\left(\begin{bmatrix} \bar{a}-\lambda & \pm\bar{b}\exp(-i\theta) \\ \pm\bar{b}\exp(-i\theta) & \bar{a}-\lambda \end{bmatrix} \right) = 0 \right\} \\ &= \{\bar{a} + \bar{b}\exp(-i\theta), \bar{a} - \bar{b}\exp(-i\theta)\} \end{aligned}$$

and so

$$\begin{aligned}\|L_{\gamma(x^*)}\| &= \max\{|\bar{a} + \bar{b}\exp(-i\theta)|, |\bar{a} - \bar{b}\exp(-i\theta)|\} \\ &= \max\{|a + b\exp(i\theta)|, |a - b\exp(i\theta)|\}. \end{aligned} \tag{3.7}$$

By (3.5), (3.6) and (3.7), we see that θ must be zero or π.

For the case $\theta = \pi$, we let $x = a + be$ and $y = a_1 + b_1 e$. Then $xy = (aa_1 - bb_1) + (ab_1 + a_1 b)e$. By the above procedure, we have

$$\begin{aligned}\|x\| &= \max\{|a + b|, |a - b|\} \\ \|y\| &= \max\{|a_1 + b_1|, |a_1 - b_1|\} \\ \|xy\| &= \max\{|aa_1 - bb_1 + ab_1 + a_1 b|, |aa_1 - bb_1 - ab_1 - a_1 b|\}.\end{aligned}$$

If $x = y = i + e$, then

$$\|xy\| = \max\{|-2 + 2i|, |-2 - 2i|\} = 2\sqrt{2} \not\leq 2 = \max\{|i + 1|, |i - 1|\}^2 = \|x\|\|y\|.$$

Hence when $\theta = \pi$, A is not a Kreĭn C*-algebra.

For $\theta = 0_-$, we have $e \cdot e = 1$ and $e^* = -e$. Since every element in A_{0_-} is of the form $m1 + ne$, where m and n are in $\mathbb{C}$, we define $f : A_{0_-} \to \mathbb{K}$ by $f(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $f(e) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then f is an isomorphism between the Kreĭn C*-algebras A_{0_-} and $\mathbb{K}$. Since $\mathbb{K}$ is Kreĭn C*-algebra, so is A_{0_-}.

For $\theta = 0_+$, we have $e \cdot e = 1$ and $e^* = e$. Suppose that γ is the fundamental symmetry on it. Note that $\gamma(x^{*_+}) = x^{*_-}$ for all $x \in A$. Indeed, let $x \in A$ then $x = a + be$ for some $a, b \in \mathbb{C}$ and so

$$\gamma(x^{*_+}) = \gamma(\bar{a} + \bar{b}e) = \bar{a} - \bar{b}e = (a + be)^{*_-} = x^{*_-}.$$

Then

$$\|x\|^2 = \|\gamma(x^{*_+})x\| = \|x^{*_-}x\|,$$

which contradicts the fact that A_- is a nontrivial Kreĭn C*-algebra. Since γ is the only possible fundamental symmetry on A, we can conclude that A_+ is not a Kreĭn C*-algebra. $\square$

Theorem 3.13. *The space $C(M,K)$ of all continuous functions from a compact Hausdorff space M into a Kreĭn C*-algebra K is a unital Kreĭn C*-algebra.*

Proof. Let γ be a fundamental symmetry of the Kreĭn C*-algebra K and M a compact Hausdorff space. We define all the operations and the norm as follows:

$$(f+g)(x) = f(x)+g(x) \qquad (fg)(x) = f(x)g(x)$$
$$(kf)(x) = kf(x) \qquad f^{*_C}(x) = f(x)^{*_K}$$
$$\|f\|_C = \sup_{x\in M}\|f(x)\|_K,$$

for all $k \in \mathbb{C}$, $x \in M$ and for all $f, g \in C(M,K)$. It is easy to check that $C(M,K)$ is a Banach algebra and a $*$-algebra with the operations and norm above.

To see that $C(M,K)$ is a Kreĭn C*-algebra, consider the map ϕ_C from $C(M,K)$ into itself defined by $\phi_C(f) = \gamma \circ f$. Since, for all $x \in M$,

$$\begin{aligned}
\phi_C(fg)(x) &= \gamma\circ(fg)(x) = \gamma(f(x)g(x)) = \gamma(f(x))\gamma(g(x)) = \phi_C(f)\phi_C(g)(x)\\
\phi_C(f^{*_C})(x) &= (\gamma\circ f^{*_C})(x) = \gamma(f(x)^{*_K}) = \gamma(f(x))^{*_K}\\
&= (\gamma\circ f)(x)^{*_K} = (\gamma\circ f)^{*_C}(x) = \phi_C(f)^{*_C}(x),\\
\phi_C(1_C)(x) &= \gamma(1_C(x)) = \gamma(1_K) = 1_K = 1_C(x),
\end{aligned}$$

ϕ_C is a unital $*$-homomorphism that is an involutive $*$-automorphism because

$$\phi_C\circ\phi_C(f) = \phi_C(\gamma\circ f) = \gamma\circ\gamma\circ f = f, \quad \forall f \in C(M,K),$$

and also a fundamental symmetry of a Kreĭn C*-algebra because

$$\begin{aligned}
\|\phi_C(f)^{*_C}f\|_C &= \sup_{x\in M}\|(\gamma\circ f)^{*_C}(x)f(x)\|_K = \sup_{x\in M}\|(\gamma\circ f)(x)^{*_K}f(x)\|_K\\
&= \sup_{x\in M}\|\gamma(f(x))^{*_K}f(x)\|_K = \sup_{x\in M}\|f(x)\|_K^2 = \|f\|_C^2,
\end{aligned}$$

for all $f \in C(M,K)$. □

When we take $K := \mathbb{K}$ in the previous theorem, we obtain another example of commutative full symmetric Kreĭn C*-algebra.

Corollary 3.14. *The Kreĭn C*-algebra $C(M, \mathbb{K})$ is a unital commutative full symmetric Kreĭn C*-algebra.*

Proof. Define the fundamental symmetry $\phi_C(f) := \gamma \circ f$ and $\varepsilon_C(f) := \varepsilon_{\mathbb{K}} \circ f$. $\square$

Theorem 3.15. *Let (A, α) be a (unital commutative) Kreĭn C*-algebra with a fundamental symmetry α. Let I be a closed ideal in A invariant under α, i.e. $\alpha(I) \subseteq I$, then A/I is also a (unital commutative) Kreĭn C*-algebra.*

Proof. Since I is a closed ideal in the C*-algebra $(A, \dagger_\alpha)$, the quotient $(A/I, \dagger_\alpha)$ is a C*-algebra with involution $(x + I)^{\dagger_\alpha} := x^{\dagger_\alpha} + I$. Since I is invariant under the $*$-automorphism α we can define $[\alpha] : A/I \to A/I$ by $x + I \mapsto \alpha(x) + I$ for all $x \in A$ and $[\alpha]$ is a $\dagger_\alpha$-automorphism of $(A/I, \dagger_\alpha)$. By Remark 3.4, $(A/I, *)$ is a Kreĭn C*-algebra with involution $(x + I)^* := \alpha(x^{\dagger_\alpha}) + I = x^* + I$. and $[\alpha]$ becomes the fundamental symmetry on $(A/I, *)$. $\square$

Theorem 3.16. *Let (A, α) and (B, β) be two Kreĭn C*-algebras with given fundamental symmetries α and β. A $*$-homomorphism $\phi : A \to B$ satisfying the property $\phi \circ \alpha = \beta \circ \phi$ is always continuous. Additionally, $\phi(A_+) \subseteq B_+$ and $\phi(A_-) \subseteq B_-$.*

Proof. Note that ϕ is a $\dagger$-homomorphism between the associated C*-algebras $(A, \dagger_\alpha)$ and $(B, \dagger_\beta)$. Indeed,

$$\phi(a^\dagger) = \phi(\alpha(a)^*) = \phi(\alpha(a))^* = \beta(\phi(a))^* = \beta(\phi(a)^*) = \phi(a)^\dagger$$

for all $a \in A$. The "invariance" of ϕ under α, β implies the last property. $\square$

If $\phi : A \to B$ is a unital $*$-homomorphism such that $\phi \circ \alpha = \beta \circ \phi$ for a given pair of fundamental symmetries of the Kreĭn C*-algebras A and B, in view of the last property in Theorem 3.16, we will denote by $\phi_+ : A_+ \to B_+$ and $\phi_- : A_- \to B_-$ the restrictions to the even and odd parts of the Kreĭn C*-algebras. Note that $\phi = \phi_+ \oplus \phi_-$. In particular, the quotient isomorphism $\pi : A \to A/I$ from a Kreĭn C*-algebra A to its quotient Kreĭn C*-algebra by an ideal I that is invariant under a fundamental symmetry α of A, can be written as a direct sum $\pi = \pi_+ \oplus \pi_-$ of the epimorphisms $\pi_+ : A_+ \to (A/I)_+$ and $\pi_- : A_- \to (A/I)_-$.

Corollary 3.17. *Let (A, α) be a unital Kreĭn C*-algebra with a fundamental symmetry α. Let w be a unital $*$-homomorphism from A into $\mathbb{K}$ with the property that $w \circ \alpha = \gamma \circ w$. Then $A/\ker(w)$ is a Kreĭn C*-algebra.*

Proof. By Theorem 3.3, $\ker(w)$ is a closed ideal in A. Moreover, $\alpha(\ker(w)) \subseteq \ker(w)$. To see this, we let $x \in \alpha(\ker(w))$. Then there is $y \in \ker(w)$ such that $\alpha(y) = x$. By the condition $w \circ \alpha = \gamma \circ w$,

$$w(x) = w \circ \alpha(y) = \gamma \circ w(y) = \gamma(0) = 0.$$

and so $x \in \ker(w)$. Consequently, $A/\ker(w)$ is a Kreĭn C*-algebra by Theorem 3.15. $\square$

Corollary 3.18. *Let (A, α) be a unital full Kreĭn C*-algebra with a fundamental symmetry α. Let w be a unital $*$-homomorphism from A into $\mathbb{K}$ with the property that $w \circ \alpha = \gamma \circ w$. Then $A/\ker(w) \cong \mathbb{K}$.*

Proof. By Corollary 3.17, $A/\ker(w)$ is a Kreĭn C*-algebra. Define $f : A/\ker(w) \to \mathbb{K}$ by $x + \ker(w) \mapsto w(x)$ for all $x \in A$. It is easily checked that f is a unital injective $*$-homomorphism. It remains to show the surjective property. Since ${A_-}_{A_+}$ is full, Without loss of generality, $1 = \sum_{j=1}^{n} x_j y_j$ for some $n \in \mathbb{N}$ and $x_j, y_j \in A_-$ for all $j = 1, \ldots, n$. Suppose that $w(x) = 0$ for all $x \in A_-$. Since w is a unital homomorphism, we have

$$1 = w(1) = w(\sum_{j=1}^{n} x_j y_j) = \sum_{j=1}^{n} w(x_j) w(y_j) = 0,$$

which leads to a contradiction. Thus there is an element x in A_- such that $w(x) \neq 0$. Since $w(A_-) \subseteq \mathbb{K}_-$ and $\dim \mathbb{K}_- = 1$, we have $w(A_-) = \mathbb{K}_-$. Similarly $w(A_+) = \mathbb{K}_+$. Thus we already have

$$f(A/\ker(w)) = w(A) = w(A_+ \oplus A_-) = w_+(A_+) \oplus w_-(A_-) = \mathbb{K}_+ \oplus \mathbb{K}_- = \mathbb{K},$$

and so f is surjective. $\square$

Definition 3.19. *A **character** on a unital (commutative symmetric full) Kreĭn C*-algebra A is a unital $*$-homomorphism $w : A \to \mathbb{K}$ such that there exists at least one fundamental symmetry α of the Kreĭn C*-algebra A satisfying the property $w \circ \alpha = \gamma \circ w$. In this case we will say that α and w are **compatible**.*

We denote by $\Omega(A)$ the set of characters on A, and by $\Omega(A, \alpha)$ the set of characters compatible with a given fundamental symmetry α that is

$$\Omega(A, \alpha) = \{w \mid w : A \to \mathbb{K} \text{ unital } *\text{-homomorphism compatible with } \alpha\}.$$

Note that for every C*-algebra A, $\Omega(A) \cong \Omega(A, \iota_A)$ where the left-hand-side is the set of characters in the C*-algebra.

For each $a \in A$, define $\hat{a} : \Omega(A) \to \mathbb{K}$ by $\hat{a}(w) = w(a)$ for all $w \in \Omega(A)$. Equip $\Omega(A)$ with the smallest topology which makes each $\hat{a}$ continuous.

Note that $\Omega(A, \alpha)$ with the induced subspace topology is clearly a closed set in $\Omega(A)$ since $\Omega(A, \alpha) = \{w \in \Omega(A) \mid \widehat{\alpha(x)}(w) = \gamma \circ \hat{x}(w), \forall x \in A\}$ and the functions $\widehat{\alpha(x)}, \gamma \circ \hat{x}$ are continuous on $\Omega(A)$.

We next define an equivalence relation between the characters as follows:

$$w_1 \sim w_2 \Longleftrightarrow w_2 = \phi \circ w_1 \text{ for some unital } *\text{-automorphism } \phi \text{ on } \mathbb{K}.$$

By Theorem 3.11, $[w] = \{w, \gamma \circ w\}$. Note that $w \in \Omega(A, \alpha)$ implies $[w] \subset \Omega(A, \alpha)$.

$$\text{We define} \quad \Omega_b(A) = \{[w] \mid w \in \Omega(A)\}, \quad \Omega_b(A, \alpha) = \{[w] \mid w \in \Omega(A, \alpha)\}.$$

Equip $\Omega_b(A)$ with the quotient topology induced by the quotient map

$$\mu : \Omega(A) \to \Omega_b(A) \text{ given by } \mu : w \mapsto [w].$$

Note that on $\Omega_b(A, \alpha)$ the quotient topology induced by $\Omega(A, \alpha)$ coincides with the subspace topology induced by $\Omega_b(A)$.

Lemma 3.20. *$\Omega(A)$ and $\Omega(A, \alpha)$ are nonempty compact Hausdorff spaces.*

Proof. Since a character $w : A \to \mathbb{K}$ becomes a unital $*$-homomorphism of the associated C*-algebras $(A, \dagger_\alpha)$ and $(\mathbb{K}, \dagger_\gamma)$, by the same techniques used in Banach-Alaoglu theorem, it is a standard matter to check that $\Omega(A)$ is a closed subset of compact set $\prod_{x\in A}\overline{B(0_{\mathbb{K}},\|x\|)}$ (Using Heine-Borel and Tychonoff Theorems) and hence $\Omega(A)$ is also a compact set. To show that $\Omega(A)$ is a Hausdorff space, we let w_1, w_2 be characters such that $w_1 \neq w_2$. Then there is $a \in A$ such that $w_1(a) \neq w_2(a)$, that is $\hat{a}(w_1) \neq \hat{a}(w_2)$. Since $\hat{a}$ is continuous from $\Omega(A)$ to the Hausdorff space $\mathbb{K}$, we obtain that $\Omega(A)$ is also Hausdorff. Since $\Omega(A,\alpha)$ is a closed subspace of $\Omega(A)$, the result follows. $\square$

Lemma 3.21. *Let w, w_1 and w_2 be characters. The following properties hold:*

a) If $w_1 \sim w_2$, then $\ker(w_1) = \ker(w_2)$.

b) If $w \in \Omega(A,\alpha)$ for all $x \in A$, we have $w(x) = 0 \Longleftrightarrow w(x^{\dagger_\alpha}x) = 0$.

c) If $w_1, w_2 \in \Omega(A,\alpha)$ and $w_{1+} = w_{2+}$, then $\ker(w_1) = \ker(w_2)$.

Proof. It is easy to show a).

For b), assume that $w(x^{\dagger_\alpha}x) = 0$. Then

$$\|w(x)\|^2 = \|\gamma(w(x))^*w(x)\| = \|w(\alpha(x))^*w(x)\| = \|w(\alpha(x)^*x)\| = \|w(x^{\dagger_\alpha}x)\| = 0,$$

and so $w(x) = 0$.

For c), by the assumption, we have $\ker(w_{1+}) = \ker(w_{2+})$. Let $x \in A_-$ be such that $w_1(x) = 0$. By b), $w_1(\alpha(x^*)x) = 0$ and also $w_2(\alpha(x^*)x) = 0$ because $\ker(w_{1+}) = \ker(w_{2+})$. Again by b), $w_2(x) = 0$ and hence $\ker(w_{1-}) \subseteq \ker(w_{2-})$. By the same argument, it is elementary to verify the inverse inclusion. $\square$

Lemma 3.22. *Let (A,α) be a Kreĭn C*-algebra equipped with a fundamental symmetry α and let $\omega \in \Omega(A_+)$ be a character defined on the even part of the Kreĭn C*-algebra $A = A_+ \oplus A_-$. Define*

$$I_+ := \ker(w), \quad I_- := A_- \ker(w) := \operatorname{span}\{xa \mid x \in A_-,\ a \in \ker w\}, \quad I := I_+ \oplus I_-.$$

Then I is an ideal in A invariant under α and the following properties hold:

a) A_+/I_+ is a C-algebra with dimension 1.*

b) A_-/I_- is a Hilbert C-bimodule over A_+/I_+.*

c) $A/I = (A/I)_+ \oplus (A/I)_- \cong A_+/I_+ \oplus A_-/I_-$.

d) A/I is a rank-one Kreĭn C-algebra.*

Proof. The bimodule A_-/I_- over A_+/I_+ is a Hilbert C*-bimodule with the inner products defined as in the proof of theorem 3.5 using the $(x+I_-)^{\dagger_\alpha} := x^{\dagger_\alpha} + I_-$ involution. The only other thing that is not completely straightforward is that A/I is rank-one. By b), since by Gel'fand-Mazur $A_+/I_+ \cong \mathbb{C}$, A_-/I_- is a Hilbert space over A_+/I_+. To show the rank-one property, suppose by contradiction that $x, y \in A_-/I_-$ is a pair of orthonormal vectors, then

$$y = \langle x \mid x\rangle y + \langle y \mid x\rangle y = \langle x+y \mid x\rangle y = (x+y)\langle x \mid y\rangle = 0,$$

which is impossible. $\square$

Theorem 3.23. *If (A,α) is a unital commutative imprimitivity Kreĭn C*-algebra with fundamental symmetry α, then $\Omega_b(A,\alpha)$ is a compact Hausdorff space.*

Proof. Since $\Omega(A,\alpha)$ is compact by Lemma 3.20 and μ is continuous, $\Omega_b(A,\alpha)$ is also compact.

We now consider the map ϕ from $\Omega_b(A,\alpha)$ to $\Omega(A_+)$ defined by $[w] \mapsto w_+$. If we can show that this map is a homeomorphism, we can conclude that $\Omega_b(A,\alpha)$ is a Hausdorff space since $\Omega(A_+)$ is a Hausdorff space by the spectral theorem for unital commutative C*-algebras. It is sufficient to show that ϕ is a continuous bijective map because $\Omega_b(A,\alpha)$ is a compact space and $\Omega(A)$ is a Hausdorff space.

To see that, ϕ is well-defined, let $[w_1] = [w_2]$. If $w_1 \neq w_2$, then $w_2 = \gamma \circ w_1$. For any $x \in A_+$, $w_1(x) = w_1(\alpha(x)) = \gamma(w_1(x)) = w_2(x)$, that is, $w_{1+} = w_{2+}$.

We next show that $R := \phi \circ \mu$ is a continuous map. Note that, by definition, we have $R(w) = w_+$. For easier consideration, we provide a diagram of all the functions involved.

$$\begin{tikzcd} & \Omega(A,\alpha) \arrow[dl, "\mu"'] \arrow[d, "R"] \arrow[dr, "\hat{a}"] & \\ \Omega_b(A,\alpha) \arrow[r, "\phi"'] & \Omega(A_+) \arrow[r, "\hat{a}_+"'] & \mathbb{C} \end{tikzcd}$$

Since for all $a \in A_+$, $\hat{a} = \hat{a}_+ \circ R$ is continuous on $\Omega(A, \alpha)$, R is also continuous. By the properties of quotient topology we also have that ϕ is a continuous map. Next, we will show that ϕ is a bijection.

To show that ϕ is injective, we suppose that w_1, w_2 are characters on (A, α) such that $w_{1+} = w_{2+}$. We next examine the following diagram

$$\begin{tikzcd}
A \arrow[rr, "w_1"] \arrow[d, "\pi_1"'] & & \mathbb{K} & & A \arrow[ll, "w_2"'] \arrow[d, "\pi_2"] \\
\mathbb{K} \cong A/\ker(w_1) \arrow[urr, "\beta_1"] & & & & A/\ker(w_2) \cong \mathbb{K} \arrow[ull, "\beta_2"']
\end{tikzcd}$$

By Lemma 3.21 (c) and Corollary 3.18 we have $A/\ker(w_1) \cong \mathbb{K} \cong A/\ker(w_2)$. Hence $\beta_2 \circ \beta_1^{-1}$ is a unital $*$-automorphism on $\mathbb{K}$. Since by Theorem 3.11 a unital $*$-automorphism on $\mathbb{K}$ is either the identity map or γ, we have to consider two cases.

1. $\beta_2 \circ \beta_1^{-1}$ is an identity on $\mathbb{K}$

   Then $\beta_1 = \beta_2$. Since $\ker(w_1) = \ker(w_2)$, we have

   $$w_1(a) = \beta_1(a + \ker w_1) = \beta_2(a + \ker w_2) = w_2(a),$$

   for all $a \in A$. Thus $w_1 = w_2$.

2. $\beta_2 \circ \beta_1^{-1} = \gamma$

   Then $\beta_2 = \gamma \circ \beta_1$. Again by the fact that $\ker(w_1) = \ker(w_2)$, we have

   $$w_2(a) = \beta_2(a + \ker w_2) = \gamma \circ \beta_1(a + \ker w_1) = \gamma \circ w_1(a),$$

   for all $a \in A$. Hence $w_2 = \gamma \circ w_1$.

From both cases, we can conclude that $[w_1] = [w_2]$ which implies the injection of ϕ, as desired. To prove the surjectivity of ϕ, let $w^o : A_+ \to \mathbb{K}_+$. Consider $I_+ := \ker(w^o)$ and define the ideal I in A as in Lemma 3.22.

Since A/I is a rank-one Kreĭn C*-algebra, A/I is isomorphic to $\mathbb{K}$ by an isomorphism f such that $f \circ [\alpha] = \gamma \circ f$, where $[\alpha]$ is the fundamental symmetry of A/I such that $\pi \circ \alpha = [\alpha] \circ \pi$. Note that $f = f_+ \oplus f_-$ where $f_+ : (A/I)_+ \to \mathbb{K}_+$ is the isomorphism $f_+(a + I) = w^o(a)$, for all $a \in A_+$.

$$A \xrightarrow{\pi} A/I \xrightarrow{f} \mathbb{K}.$$

Define $w := f \circ \pi$. Then $w \in \Omega(A, \alpha)$ because $w \circ \alpha = \gamma \circ w$. We claim that $w_+ = w^o$. To see this, let $a \in A_+$. Then

$$w(a) = f \circ \pi(a) = f(a + I) = w^o(a),$$

and the theorem is proved. □

Corollary 3.24. *If w_1 and w_2 are characters on $\Omega(A, \alpha)$, then*

$$[w_1] = [w_2] \Longleftrightarrow \ker w_1 = \ker w_2.$$

Definition 3.25. *If (A, α, ε) is a unital commutative full symmetric Kreĭn C*-algebra with a given fundamental symmetry α and a given odd symmetry ε, we define the* ***even spectrum*** *$\Omega(A, \alpha, \varepsilon) := \{w \in \Omega(A, \alpha) \mid \varepsilon_{\mathbb{K}} \circ w \circ \varepsilon = w\}$, the set of* ***even characters*** *of A. Similarly we call $w \in \Omega(A, \alpha)$ an* ***odd character*** *if it satisfies $\varepsilon_{\mathbb{K}} \circ w \circ \varepsilon = -w$.*

Theorem 3.26. *If (A, α, ε) is a unital commutative full symmetric Kreĭn C*-algebra with fundamental symmetry α and odd symmetry ε, the character $w \in \Omega(A, \alpha)$ is even if and only if $\gamma \circ w \in \Omega(A, \alpha)$ is odd. Hence in every equivalence class $[w] = \{w, \gamma \circ \omega\}$ there is one and only one even character and there is a bijection between $\Omega(A, \alpha, \varepsilon)$ and $\Omega_b(A, \alpha)$.*

Proof. Since $\varepsilon_{\mathbb{K}} \circ \gamma = -\gamma \circ \varepsilon_{\mathbb{K}}$, we see that w is even if and only if $\gamma \circ w$ is odd. More precisely, if w is even then $\varepsilon_{\mathbb{K}} \circ \gamma \circ w \circ \varepsilon = -\gamma \circ \varepsilon_{\mathbb{K}} \circ w \circ \varepsilon = -\gamma \circ w$, which implies that $\gamma \circ w$ is odd. For the other direction, assume that $\gamma \circ w$ is odd. Then

$$\begin{aligned} w &= \gamma \circ \gamma \circ w = \gamma \circ -\varepsilon_{\mathbb{K}} \circ \gamma \circ w \circ \varepsilon = -\gamma \circ \varepsilon_{\mathbb{K}} \circ \gamma \circ w \circ \varepsilon \\ &= \varepsilon_{\mathbb{K}} \circ \gamma \circ \gamma \circ w \circ \varepsilon = \varepsilon_{\mathbb{K}} \circ w \circ \varepsilon, \end{aligned}$$

and therefore w is even. To show that there exists an even character, let $w \in [w]$. Define $w^o := w|_{A_+}$ and $\tilde{w}(x) := w^o(x_+) + \varepsilon_{\mathbb{K}} \circ w^o \circ \varepsilon(x_-)$. By the properties of the odd symmetry, we see that $\tilde{w}$ is an even character compatible with α. Since $\tilde{w}|_{A_+} = w|_{A_+}$, by Lemma 3.21 c), we have $\ker \tilde{w} = \ker w$. Applying Corollary 3.24, we obtain $[\tilde{w}] = [w]$. $\square$

Definition 3.27. *Let A be a unital commutative full symmetric Kreĭn C*-algebra and let α be a fundamental symmetry of A and ε an odd symmetry of A. The **Gel'fand transform** of $x \in A$ is the map $\hat{x} : \Omega_b(A, \alpha) \to \mathbb{K}$ defined by:*

$$\hat{x}([w]) := w(x_+) + \varepsilon_{\mathbb{K}} \circ w \circ \varepsilon(x_-), \quad \forall x \in A.$$

By the previous theorem, it is clear that the Gel'fand transform $\hat{x}$ of x is just the function that to every even character $w \in \Omega(A, \alpha, \varepsilon)$ associates $w(x)$.

Although the following theorem is our goal, the proof is easy and straightforward.

Theorem 3.28 (Spectral theorem). *If (A, α, ε) is a unital commutative full symmetric Kreĭn C*-algebra with fundamental symmetry α and odd symmetry ε, then the Gelfand transform*

$$\varphi : A \to C(\Omega_b(A, \alpha), \mathbb{K}), \ a \mapsto \hat{a}$$

is an isometric $*$-isomorphism.

Proof. In view of Theorem 3.26, let $[w]$, with w even, be a point of $\Omega_b(A, \alpha)$. To prove that φ is a $*$-homomorphism of algebras, let $a, b \in A$ and $k \in \mathbb{C}$,

$$\begin{aligned}
&(\varphi(ab))([w]) = (\widehat{ab})([w]) = w(ab) = w(a)w(b) = \hat{a}([w])\hat{b}([w]) = (\varphi(a)\varphi(b))([w]),\\
&(\varphi(ka + b))([w]) = (\widehat{ka + b})([w]) = w(ka + b) = kw(a) + w(b) = k\hat{a}([w]) + \hat{b}([w])\\
&\qquad = (k\varphi(a) + \varphi(b))([w]),\\
&\varphi(a^*)[w] = \widehat{a^*}([w]) = w(a^*) = w(a)^* = \varphi(a)^*[w]
\end{aligned}$$

for all $[w] \in \Omega_b(A, \alpha)$. Clearly $\varphi(1_A) = 1_C$ so that φ is unital.

It is easy to verify that $\varphi \circ \alpha = \phi_C \circ \varphi$. In fact, for all $a \in A, [w] \in \Omega_b(A, \alpha)$,

$$(\varphi \circ \alpha)(a)[w] = \varphi(\alpha(a))[w] = w(\alpha(a)) = \gamma(w(a)) = \gamma(\varphi(a)[w])$$
$$= (\gamma \circ \varphi(a))[w] = \phi_C(\varphi(a))[w] = (\phi_C \circ \varphi)(a)[w].$$

We also have that $\varphi \circ \varepsilon = \varepsilon_C \circ \varphi$, in fact, for all $[w] \in \Omega_b(A, \alpha)$, and all $x \in A$,

$$\varepsilon_C \circ \varphi(x)[w] = \varepsilon_{\mathbb{K}}(w(x)) = \varepsilon_{\mathbb{K}} \circ w \circ \varepsilon^2(x) = w \circ \varepsilon(x) = \varphi \circ \varepsilon(x)[w]. \qquad (3.8)$$

Let A be a unital commutative Kreĭn C*-algebra with the fundamental symmetry α. Then $(A, \dagger_A)$ and $(C(\Omega_b(A, \alpha), \mathbb{K}), \dagger_C)$ become C*-algebras with the involutions $\dagger_A$ and $\dagger_C$ defined as in Theorem 3.3 respectively, that is,

$$a^{\dagger_A} = \alpha(a^{*_A}) \text{ for all } a \in A \text{ and } f^{\dagger_C} = \phi_C(f^{*_C}) \text{ for all } f \in C(\Omega_b(A, \alpha), \mathbb{K}).$$

To show φ is a $\dagger$-homomorphism, let $a \in A$ and $[w] \in \Omega_b(A, \alpha)$,

$$\varphi(a^{\dagger_A})[w] = \varphi(\alpha(a^{*_A}))[w] = \widehat{\alpha(a^{*_A})}[w] = w(\alpha(a^{*_A})).$$

Since $w \circ \alpha = \gamma \circ w$, we have,

$$\varphi(a^{\dagger_A})[w] = \gamma(w(a^{*_A})) = \gamma(w(a)^{*_{\mathbb{K}}}) = \gamma(\hat{a}([w])^{*_{\mathbb{K}}}) = \gamma(\hat{a}^{*_C}([w]))$$

Since $\phi_C(f) = \gamma \circ f$ for all $f \in C(\Omega_b(A), \mathbb{K})$,

$$\varphi(a^{\dagger_A})[w] = \phi_C(\hat{a}^{*_C})[w] = \hat{a}^{\dagger_C}[w] = \varphi(a)^{\dagger_C}[w].$$

By the spectral theorem for unital commutative C*-algebras, the restriction of the Gel'fand transform to the even part $\varphi_+ : A_+ \to C(\Omega(A, \alpha), \mathbb{K})_+$ is an isometric $\dagger$-isomorphism, for all $a \in A$ that coincides with the usual Gel'fand isomorphism for the commutative unital C*-algebra A_+.

Since ε and ε_C are linear surjective (because $\varepsilon^2 = i_A$) isometries, from the equation (3.8), we see that $\varphi_- = \varepsilon_- \circ \varphi_+ \circ \varepsilon_{C+}$ is isometric surjective too and hence $\varphi = \varphi_+ \oplus \varphi_-$ is a $\dagger$-epimorphism. Since φ is a $\dagger$-homomorphism, we have $\|\varphi(a)\| \leq \|a\|$ for all $a \in A$. Since φ_+ and φ_- are injective and the uniqueness property of a directsum, φ is also injective. Now φ is a $\dagger$-isomorphism, it implies that φ^{-1} exists and is also $\dagger$-isomorphism which implies that $\|a\| = \|\varphi^{-1}(\varphi(a))\| \leq \|\varphi(a)\|$ for all $a \in A$. $\square$

The previous theorem provide a complete characterization of those unital commutative Kreĭn C*-algebras that are full and symmetric. In this case, a posterori, with a bit more work, we might actually prove that the fundamental symmetry and the odd symmetry are indeed unique. Further analysis is required in order to provide a spectral theory of more general Kreĭn C*-algebras. Omitting the exchange symmetry requirement will lead us to algebras of sections of Banach bundles of Kreĭn C*-algebras and omitting the commutativity (or just the imprimitivity condition on the odd part) will lead us to a theory of Kreĭn spaceoids along very similar lines to those used to describe the spectrum of commutative full C*-categories in [1], we hope that the extra effort paid here to describe a "direct proof" of the spectral theory in the special case of commutative full symmetric Kreĭn C*-algebras will facilitate the analysis of those more general topics that we plan to address in the near future.

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VITA

Name Mr. Pichkitti Bannangkoon

Date of Birth 17 August 1984

Place of Birth Phatthalung, Thailand

Education B.Sc. (Mathematics) (First Class Honours), Prince of Songkla University, 2006

Scholarship The Development and Promotion of Science and Technology Talents Project (DPST)