CHAPTER IV

$\mathfrak{o}(5, \mathbf{C})$

Let V be a finite-dimensional irreducible $\mathfrak{o}(5, \mathbf{C})$-module, v^+ a maximal vector of V with highest weight λ. Suppose that $\lambda(h_1) = m_1$ and $\lambda(h_2) = m_2$. Then a Verma basis of V consists of all elements of the form

$$y_1^{a_4} y_2^{a_3} y_1^{a_2} y_2^{a_1} \cdot v^+$$

where $a_1, a_2, a_3, a_4 \in \mathbf{Z}_0^+$ and

$$0 \le a_1 \le m_2$$

$$0 \le a_2 \le m_1 + a_1$$

$$0 \le a_3 \le \min\{m_1 + a_2, 2a_2\}$$

$$0 \le a_4 \le \min\{m_1, \lfloor a_3/2 \rfloor\}$$

We want to be able to calculate the action of an arbitrary element of $\mathfrak{o}(5, \mathbf{C})$ on an arbitrary element of V; for this it is sufficient to know the action of the elements of a set of generators of $\mathfrak{o}(5, \mathbf{C})$ on the elements of a Verma basis of V. The purpose of this chapter to find formulas for the action of the Chevalley generators $\{x_1, x_2, y_1, y_2\}$ on the elements of the above Verma basis of V.

The following notations are used in this chapter. Let $\mathfrak{B}$ denote the above Verma basis. For each $i \in \{1, 2\}$, let

$$S_i = \operatorname{span}\{x_i, h_i, y_i\} \text{ and } B_i = \operatorname{span}\{x_i, h_i\}, \text{ where } h_i = [x_i, y_i].$$

Observe that for each $i \in \{1,2\}$, S_i is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$, while B_i is a two-dimensional nonabelian subalgebra of S_i.

Lemma 4.1.

i) $X = \operatorname{span}\{y_2^i \cdot v^+ \mid i \in \mathbb{Z}_0^+\}$ *is an irreducible* S_2*-module,*

$$\{y_2^i \cdot v^+ \mid i \in \mathbb{Z}_0^+ \text{ with } 0 \le i \le m_2\}$$

is a basis of X*, and* $y_2^i \cdot v^+ = 0$ *for all* $i \in \mathbb{Z}_0^+$ *with* $i > m_2$.

ii) *For each* $k \in \mathbb{Z}_0^+$ *with* $0 \le k \le m_2$,

$$Y_k = \operatorname{span}\{y_1^i y_2^k \cdot v^+ \mid i \in \mathbb{Z}_0^+\}$$

is an irreducible S_1*-module,*

$$\{y_1^i y_2^k \cdot v^+ \mid i \in \mathbb{Z}_0^+ \text{ with } 0 \le i \le m_1 + k\}$$

is a basis of Y_k*, and* $y_1^i y_2^k \cdot v^+ = 0$ *for all* $i \in \mathbb{Z}_0^+$ *with* $i > m_1 + k$.

Proof:

i) Since v^+ is a maximal vector for S_2, this follows from the theory of $\mathfrak{sl}(2,\mathbb{C})$-modules.

ii) Since $y_2^k \cdot v^+$ is a maximal vector of S_1, this follows from the theory of $\mathfrak{sl}(2,\mathbb{C})$-modules again. #

Lemma 4.2. *For each* $a_2 \in \mathbb{Z}_0^+$, *let*

$$X = \operatorname{span}\{y_1^{a_2} y_2^i \cdot v^+ \mid i \in \mathbb{Z}_0^+ \text{ with } \mu \le i \le m_2\},$$

and let

$$Y = \operatorname{span}\{y_2^j y_1^{a_2} y_2^i \cdot v^+ \mid i,j \in \mathbb{Z}_0^+ \text{ with } 0 \le j \le k,\ \mu \le i \le m_2\},$$

where $\mu = \max\{0, a_2 - m_1\}$, $k = 2a_2 - \mu$. *Also, for each* $j \in \mathbb{Z}_0^+$, *with* $0 \le j \le m_2$, *let*

$$C_j = \frac{m_2!}{j!}(m_2 - j)!.$$

i) X *is a* B_2*-module, and in fact* X *is the string module* $\mathcal{S}(k+n, k-n)$, *where* $n = m_2 - \mu$, *with standard basis*

$$\{C_{\mu+i} y_1^{a_2} y_1^{\mu+i} \cdot v^+ \mid i \in \mathbb{Z}_0^+ \text{ with } 0 \le i \le m_2 - \mu\}.$$

ii) Y *is a* B_2*-module and*

$$\{C_{\mu+i} y_2^j y_1^{a_2} y_2^{\mu+i} \cdot v^+ \mid i, j \in \mathbb{Z}_0^+ \text{ with } 0 \le j \le k,\ 0 \le i \le m_2 - \mu\}$$

is a basis of Y.

Proof: Fix $a_2 \in \mathbb{Z}_0^+$. i) We will show first that X is a B_2-module . This is obvious if $X = \{0\}$. Suppose $X \ne \{0\}$; it suffices to show that for any $i \in \mathbb{Z}_0^+$ with $\mu \le i \le m_2$, $x_2 \cdot (y_1^{a_2} y_2^i \cdot v^+)$, $h_2 \cdot (y_1^{a_2} y_2^i \cdot v^+) \in X$, and the fact that $y_1^{a_2} y_2^i \cdot v^+$ is a weight vector makes it clear that $h_2 \cdot (y_1^{a_2} y_2^i \cdot v^+) \in X$. On the other hand,

$$x_2 \cdot (y_1^{a_2} y_2^i \cdot v^+) = \begin{cases} 0 & \text{if } i = 0\ , \\ i(m_2 - i + 1) y_1^{a_2} y_2^{i-1} \cdot v^+ & \text{if } i > 0\ . \end{cases}$$

We consider $y_1^{a_2} y_2^{i-1} \cdot v^+$ in the case $i > 0$.

Case 1. $\mu < i \le m_2$. Then by Lemma 4.1 ii), $y_1^{a_2} y_2^{i-1} \cdot v^+ = 0$ if $a_2 \ge m_1 + i$ and it is an element of X if $a_2 < m_1 + i$.

Case 2. $i = \mu$. Then $a_2 = m_1 + i > m_1 + i - 1$. Thus by Lemma 4.1 ii), $y_1^{a_2} y_2^{i-1} \cdot v^+ = 0$.

Hence $x_2 \cdot (y_1^{a_2} y_2^{i} \cdot v^+)$, $h_2 \cdot (y_1^{a_2} y_2^{i} \cdot v^+) \in X$, and X is a B_2-module.

Next, we will show that

$$Z_1 = \{C_{\mu+i} y_1^{a_2} y_2^{\mu+i} \cdot v^+ \mid i \in \mathbf{Z}_0^+ \text{ with } 0 \leq i \leq m_2 - \mu\}$$

is a standard basis of X. This will also tell us that X is a string module. The elements $y_1^{a_2} y_2^{\mu+i} \cdot v^+$, $i \in \mathbf{Z}_0^+$ with $0 \leq i \leq m_2 - \mu$ are distinct elements of $\mathfrak{B}$, hence they form a basis of X. Thus Z_1 is a basis of X as well, because the coefficients $C_{\mu+i}$ are all nonzero.

Next, we will look at the action of x_2 and h_2 on the elements of Z_1. Fix $i \in \mathbf{Z}_0^+$ with $0 \leq i \leq m_2 - \mu$. Then

$$\begin{aligned} h_2 \cdot (C_{\mu+i} y_1^{a_2} y_2^{\mu+i} \cdot v^+) &= C_{\mu+i}(\lambda - (\mu+i)\alpha_2 - a_2\alpha_1)(h_2) y_1^{a_2} y_2^{\mu+i} \cdot v^+ \\ &= C_{\mu+i}(m_2 - 2\mu + 2a_2 - 2i) y_1^{a_2} y_2^{\mu+i} \cdot v^+ \\ &= C_{\mu+i}(k + (m_2 - \mu) - 2i) y_1^{a_2} y_2^{\mu+i} \cdot v^+ \end{aligned}$$

and

$$x_2 \cdot (C_{\mu+i} y_1^{a_2} y_2^{\mu+i} \cdot v^+) = \begin{cases} 0 & \text{if } i = 0, \\ C_{\mu+i-1} y_2^{a_2} y_1^{\mu+i-1} \cdot v^+ & \text{if } i > 0. \end{cases}$$

Therefore we have Z_1 is a standard basis of X. Observe that

$$h_2 \cdot (C_{\mu} y_1^{a_2} y_2^{\mu} \cdot v^+) = (m_2 - 2\mu + a_2) C_{\mu} y_1^{a_2} y_2^{\mu} \cdot v^+$$

while

$$h_2 \cdot (C_{m_2} y_1^{a_2} y_2^{m_2} \cdot v^+) = (2a_2 - m_2) C_{m_2} y_1^{a_2} y_2^{m_2} \cdot v^+$$

Thus $X = \mathcal{S}(m_2 - 2\mu + 2a_2, a_2 - m_2)$. Therefore $X = \mathcal{S}(k + n, k - n)$, where $n = m_2 - \mu$.

ii) First, we will show that

$$Z_2 = \{C_{\mu+i} y_2^j y_1^{a_2} y_2^{\mu+i} \cdot v^+ \mid i, j \in \mathbf{Z}_0^+ \text{ with } 0 \leq j \leq k,\ 0 \leq i \leq m_2 - \mu\}$$

is a basis of Y. It suffices to show that $k = \min\{m_1 + a_2, 2a_2\}$, for then the elements of Z_2 will be nonzero scalar multiples of elements of the Verma basis $\mathfrak{B}$. Indeed we have

$$\begin{aligned} k &= 2a_2 - \mu \\ &= 2a_2 - \max\{0, a_2 - m_1\} \\ &= 2a_2 + \min\{0, m_1 - a_2\} \\ &= \min\{2a_2, m_1 + a_2\} \end{aligned}$$

Thus Z_2 is a basis of Y.

Finally, we will show that Y is a B_2-module. This is obvious if $Y = \{0\}$. Suppose $Y \neq \{0\}$; it suffices to show that for any $i, j \in \mathbf{Z}_0^+$ with $0 \leq j \leq k$, $0 \leq i \leq m_2 - \mu$, we have $h_2 \cdot (C_{\mu+i} y_2^j y_1^{a_2} y_2^{\mu+i} \cdot v^+)$, $x_2 \cdot (C_{\mu+i} y_2^j y_1^{a_2} y_2^{\mu+i} \cdot v^+) \in Y$. As before, $h_2 \cdot (C_{\mu+i} y_2^j y_1^{a_2} y_2^{\mu+i} \cdot v^+) \in Y$, since $y_2^j y_1^{a_2} y_2^{\mu+i} \cdot v^+$ is a weight vector. On the other hand,

$$x_2 \cdot (C_{\mu+i} y_2^j y_1^{a_2} y_2^{\mu+i} \cdot v^+)$$

$$= \begin{cases} 0 & \text{if } i = 0 \text{ and } j = 0, \\ C_{\mu+i-1} y_1^{a_2} y_2^{\mu+i-1} \cdot v^+ & \text{if } i > 0 \text{ and } j = 0, \\ j(k+n-j+1) C_\mu y_2^{j-1} y_1^{a_2} y_2^{\mu} \cdot v^+ & \text{if } i = 0 \text{ and } j > 0, \\ j(k+n-2i-j+1) C_{\mu+i} y_2^{j-1} y_1^{a_2} y_2^{\mu+i} \cdot v^+ & \\ \quad + C_{\mu+i-1} y_2^j y_1^{a_2} y_2^{\mu+i-1} \cdot v^+ & \text{if } i > 0 \text{ and } j > 0. \end{cases}$$

Thus $x_2 \cdot (C_{\mu+i} y_2^j y_1^{a_2} y_2^{\mu+i} \cdot v^+) \in Y$, and Y is a B_2-module. #

Lemma 4.3. *Define Y as in Lemma 4.2. Then an action of y_2 can be defined on Y which makes Y into an S_2-module.*

Proof: By the proof of Lemma 4.2 ii) we have Y is a B_2-module such that for any i, $j \in \mathbf{Z}_0^+$ with $0 \le j \le k$, $0 \le i \le m_2 - \mu$,

1) $h_2 \cdot (C_{\mu+i} y_2^j y_1^{a_2} y_2^{\mu+i} \cdot v^+) = (k + m_2 - \mu - 2i - 2j) C_{\mu+i} y_2^j y_1^{a_2} y_2^{\mu+i} \cdot v^+$

and

2) $$x_2 \cdot (C_{\mu+i} y_2^j y_1^{a_2} y_2^{\mu+i} \cdot v^+) = j(k + m_2 - \mu - 2i - j + 1) C_{\mu+i} y_2^{j-1} y_1^{a_2} y_2^{\mu+i} \cdot v^+ + C_{\mu+i-1} y_2^j y_1^{a_2} y_2^{\mu+i-1} \cdot v^+$$

Then by Lemma 1.7, Y can be made into an S_2-module by keeping the same action for x_2 and h_2 and defining an action of y_2 by the following: for any i, $j \in \mathbf{Z}_0^+$ with $0 \le j \le k$, $0 \le i \le m_2 - \mu$,

3) If $0 \le j < k$, then

$$y_2 \cdot (C_{\mu+i} y_2^j y_1^{a_2} y_2^{\mu+i} \cdot v^+) = C_{\mu+i} y_2^{j+1} y_1^{a_2} y_2^{\mu+i} \cdot v^+$$

4) If $0 \le i < m_2 - \mu$, then

$$y_2 \cdot (C_{\mu+i} y_1^k y_2^{a_2} y_1^{\mu+i} \cdot v^+) = \sum_{p=1}^{m_2-\mu-i} (-1)^{p-1} \binom{m_2 - \mu - i}{p} \prod_{r=0}^{p-1} (k + 1 - r)(i + r + 1) \, C_{\mu+i+p} y_2^{k+1-p} y_1^{a_2} y_2^{\mu+i+p} \cdot v^+$$

5) $y_2 \cdot (C_{m_2} y_2^k y_1^{a_2} y_2^{m_2} \cdot v^+) = 0$

Thus we have Lemma 4.3. #

Note that from Lemmas 4.1, 4.3, Theorem 2.5 and Lemma 1.6, we have the following formulas for the actions of h_2, x_2 and y_2 on an element $y_2^{a_3} y_1^{a_2} y_2^{a_1} \cdot$

v^+ of the Verma basis $\mathfrak{B}$. Let $a_1, a_2, a_3 \in \mathbb{Z}_0^+$ with $0 \leq a_1 \leq m_2$, $0 \leq a_2 \leq m_1 + a_1$, $0 \leq a_3 \leq min\{m_1 + a_2, 2a_2\}$.

1) The formula

$$h_2 \cdot (y_2^{a_3} y_1^{a_2} y_2^{a_1} \cdot v^+) = (m_2 - 2a_1 + 2a_2 - 2a_3) y_2^{a_3} y_1^{a_2} y_2^{a_1} \cdot v^+$$

is clear.

2) By Lemmas 4.1, and 4.2,

$$x_2 \cdot (y_2^{a_3} y_1^{a_2} y_2^{a_1} \cdot v^+)$$

$$= \begin{cases} 0 & \text{if } a_1 = 0 = a_3 \,, \\ a_1(m_2 - a_1 + 1) y_1^{a_2} y_2^{a_1 - 1} \cdot v^+ & \text{if } a_1 > 0 \text{ and } a_3 = 0 \,, \\ a_3(m_2 - a_3 + 2a_2 + 1) y_2^{a_3 - 1} y_1^{a_2} \cdot v^+ & \text{if } a_1 = 0 \text{ and } a_3 > 0 \,, \\ a_1(m_2 - a_1 + 1) y_2^{a_3} y_1^{a_2} y_2^{a_1 - 1} \cdot v^+ & \\ \quad + a_3(m_2 - 2a_1 - a_3 + 2a_2 + 1) y_2^{a_3 - 1} y_1^{a_2} y_2^{a_1} \cdot v^+ & \text{if } a_1 > 0 \text{ and } a_3 > 0 \,. \end{cases}$$

Observe that if $a_3 > 0$, then $0 \leq a_3 - 1 \leq \min\{m_1 + a_2, 2a_2\}$, so $y_2^{a_3-1} y_1^{a_2} y_2^{a_1} \cdot v^+$ is an element of $\mathfrak{B}$ in this case. However, $y_2^{a_3} y_1^{a_2} y_2^{a_1-1} \cdot v^+$ may not be an element of $\mathfrak{B}$.

We consider $y_2^{a_3} y_1^{a_2} y_2^{a_1-1} \cdot v^+$ in the case $a_1 > 0$.

If $a_2 = m_1 + a_1$, then $y_2^{a_3} y_1^{a_2} y_2^{a_1-1} \cdot v^+ = 0$.

If $a_2 < m_1 + a_1$, then $y_2^{a_3} y_1^{a_2} y_2^{a_1-1} \cdot v^+ \in \mathfrak{B}$.

3) If $0 \leq a_3 < \min\{m_1 + a_2, 2a_2\}$, then

$$y_2 \cdot (y_2^{a_3} y_1^{a_2} y_2^{a_1} \cdot v^+) = y_2^{a_3+1} y_1^{a_2} y_2^{a_1} \cdot v^+$$

and this is an element of $\mathfrak{B}$.

4) If $0 \leq a_1 < m_2$ and $a_3 = \min\{m_1 + a_2, 2a_2\}$, then

$$y_2 \cdot (y_2^{a_3} y_1^{a_2} y_2^{a_1} \cdot v^+) = a_1! \sum_{s=a_3-m_2+a_1+1}^{a_3} (-1)^{a_3-s} \frac{\prod_{r=0}^{a_3-s}(a_3+1-r)(a_1-\mu+r+1)}{(a_3+1-s)!(a_1+a_3+1-s)!}$$

$$y_2^s y_1^{a_2} y_2^{a_1+a_3+1-s} \cdot v^+$$

where $\mu = \max\{0, a_2 - m_1\}$.

Observe that

a) $s \leq a_3$

b) $1 \leq a_3 + 1 - s$

c) $a_1 + 1 \leq a_1 + a_3 + 1 - s \leq m_2$

d) $0 < a_1 - \mu + r + 1$

e) $\prod_{r=0}^{a_3-s}(a_3+1-r) = 0$ iff $s \leq -1$

Let

$$s_0 = \begin{cases} 0 & \text{if } a_3 - m_2 + a_1 + 1 < 0\ , \\ a_3 - m_2 + a_1 + 1 & \text{if } a_3 - m_2 + a_1 + 1 \geq 0\ . \end{cases}$$

Then

$$y_2 \cdot (y_2^{a_3} y_1^{a_2} y_2^{a_1} \cdot v^+) = a_1! \sum_{s=s_o}^{a_3} (-1)^{a_3-s} \frac{\prod_{r=0}^{a_3-s}(a_3+1-r)(a_1-\mu+r+1)}{(a_3+1-s)!(a_1+a_3+1-s)!}$$

$$y_2^s y_1^{a_2} y_2^{a_1+a_3+1-s} \cdot v^+$$

We consider $y_2^s y_1^{a_2} y_2^{a_1+a_3+1-s} \cdot v^+$ for each $s \in \mathbb{Z}_0^+$ with $s_0 \leq s \leq a_3$. Since $a_1+1 \leq a_1+a_3+1-s \leq m_2$, $0 \leq a_2 \leq m_1+a_1+a_3+1-s$ and $0 \leq s_0 \leq s \leq a_3$, $y_2^s y_1^{a_2} y_2^{a_1+a_3+1-s} \cdot v^+ \in \mathfrak{B}$.

5) If $a_1 = m_2$ and $a_3 = \min\{m_1 + a_2, 2a_2\}$, then

$$y_2 \cdot (y_2^{a_3} y_1^{a_2} y_2^{a_1} \cdot v^+) = 0$$

Lemma 4.4. *Fix* $a_3 \in \mathbb{Z}_0^+$, *let*

$$\mathfrak{B}(a_3) = \{y_1^{b_4}y_2^{b_3}y_1^{b_2}y_2^{b_1} \cdot v^+ \mid b_1, b_2, b_3, b_4 \in \mathbb{Z}_0^+ \text{ with } 0 \le b_1 \le m_2,\ 0 \le b_2 \le m_1 + b_1,\ 0 \le b_3 \le \min\{m_1 + b_2, 2b_2, a_3 - 1\},\ 0 \le b_4 \le \min\{m_1, \lfloor b_3/2 \rfloor\}\},$$

and $Z(a_3) = \text{span}\mathfrak{B}(a_3)$. *Then* $\mathfrak{B}(a_3)$ *is a basis of* $Z(a_3)$ *and* $Z(a_3)$ *is an* S_1*-module.*

Proof: Since $\mathfrak{B}(a_3) \subseteq \mathfrak{B}$ and $\mathfrak{B}(a_3)$ spans $Z(a_3)$, it is a basis of $Z(a_3)$.

Next, we will show that $Z(a_3)$ is an S_1-module. If $a_3 = 0$, then $Z(a_3) = \{0\}$. Then it is an S_1-module. Suppose $a_3 > 0$. Note that

$$\begin{aligned} Z(a_3+1) &= \text{span}\{y_1^{b_4}y_2^{b_3}y_1^{b_2}y_2^{b_1} \cdot v^+ \mid b_1, b_2, b_3, b_4 \in \mathbb{Z}_0^+ \text{ with } \\ &\qquad 0 \le b_1 \le m_2,\ 0 \le b_2 \le m_1 + b_1, \\ &\qquad 0 \le b_3 \le \min\{m_1 + b_2, 2b_2, a_3\}, \\ &\qquad 0 \le b_4 \le \min\{m_1, \lfloor b_3/2 \rfloor\}\} \\ &= Z(a_3) + \text{span}\{y_1^{b_4}y_2^{a_3}y_1^{b_2}y_2^{b_1} \cdot v^+ \mid b_1, b_2, b_4 \in \mathbb{Z}_0^+ \text{ with } \\ &\qquad 0 \le b_1 \le m_2, \\ &\qquad \max\{a_3 - m_1, \lceil a_3/2 \rceil\} \le b_2 \le m_1 + b_1, \\ &\qquad 0 \le b_4 \le \min\{m_1, \lfloor a_3/2 \rfloor\}\} \\ &= Z(a_3) + \sum_{b_1=0}^{m_2} \text{span}\{y_1^{b_4}y_2^{a_3}y_1^{b_2}y_2^{b_1} \cdot v^+ \mid b_2, b_4 \in \mathbb{Z}_0^+ \text{ with } \\ &\qquad \max\{a_3 - m_1, \lceil a_3/2 \rceil\} \le b_2 \le m_1 + b_1, \\ &\qquad 0 \le b_4 \le \min\{m_1, \lfloor a_3/2 \rfloor\}\} \end{aligned}$$

We will finish the proof of this Lemma by induction on a_3.

Basis: $a_3 = 1$. Then

$$Z(a_3) = \operatorname{span}\{y_1^{b_4}y_2^{b_3}y_1^{b_2}y_2^{b_1} \cdot v^+ \mid b_1, b_2, b_3, b_4 \in \mathbf{Z}_0^+ \text{ with } 0 \le b_1 \le m_2,\ 0 \le b_2 \le m_1 + b_1,\ 0 \le b_3 \le \min\{m_1 + b_2, 2b_2, 0\},\ 0 \le b_4 \le \min\{m_1, \lfloor b_3/2 \rfloor\}\}$$
$$= \operatorname{span}\{y_1^{b_2}y_2^{b_1} \cdot v^+ \mid b_1, b_2 \in \mathbf{Z}_0^+ \text{ with } 0 \le b_1 \le m_2,\ 0 \le b_2 \le m_1 + b_1\}$$

which is a sum of S_1-modules by Lemma 4.1 part ii).

Induction step: Assume $Z(a_3)$ is an S_1-module, where $a_3 \in \mathbf{Z}^+$. Then $V/Z(a_3)$ is an S_1-module. We must show that $Z(a_3 + 1)$ is an S_1-module. For each $b_1 \in \mathbf{Z}_0^+$ with $0 \le b_1 \le m_2$, let

$$Y(a_3, b_1) = \operatorname{span}\{y_2^{a_3}y_1^{i}y_2^{b_1} \cdot v^+ + Z(a_3) \mid i \in \mathbf{Z}_0^+ \text{ with } 0 \le i \le m_1 + b_1\}$$

We will first show that $Y(a_3, b_1)$ is a B_1-module. Note that

$$h_1 \cdot \left(y_2^{a_3}y_1^{i}y_2^{b_1} \cdot v^+ + Z(a_3)\right) = (m_1 + b_1 - 2i + a_3)y_2^{a_3}y_1^{i}y_2^{b_1} \cdot v^+ + Z(a_3)$$

and

$$x_1 \cdot \left(y_2^{a_3}y_1^{i}y_2^{b_1} \cdot v^+ + Z(a_3)\right) = \begin{cases} i(1 - i + m_1 + b_1)y_2^{a_3}y_1^{i-1}y_2^{b_1} \cdot v^+ + Z(a_3) & \text{if } i > 0, \\ Z(a_3) & \text{if } i = 0. \end{cases}$$

Thus $h_1 \cdot \left(y_2^{a_3}y_1^{i}y_2^{b_1} \cdot v^+ + Z(a_3)\right), x_1 \cdot \left(y_2^{a_3}y_1^{i}y_2^{b_1} \cdot v^+ + Z(a_3)\right) \in Y(a_3, b_1)$, and $Y(a_3, b_1)$ is a B_1-module.

Now let $\mu = \max\{a_3 - m_1, \lceil a_3/2 \rceil\}$ and let

$$\mathfrak{B}_1 = \{y_2^{a_3}y_1^{i}y_2^{b_1} \cdot v^+ + Z(a_3) \mid i \in \mathbf{Z}_0^+ \text{ with } \mu \le i \le m_1 + b_1\}$$

We claim that $\mathfrak{B}_1$ is a basis of $Y(a_3, b_1)$. Clearly $\mathfrak{B}_1 \subseteq Y(a_3, b_1)$. First, we will show that $\mathfrak{B}_1$ spans $Y(a_3, b_1)$. Fix, $i \in \mathbf{Z}_0^+$ with $0 \leq i \leq m_1 + b_1$. We must prove $y_2^{a_3} y_1^i y_2^{b_1} \cdot v^+ + Z(a_3) \in \text{span}\mathfrak{B}_1$.

Case 1. $a_3 \leq \min\{m_1 + i, 2i\}$. Then

$$\max\{a_3 - m_1, \lceil a_3/2 \rceil\} \leq i \leq m_1 + b_1.$$

Thus

$$y_2^{a_3} y_1^i y_2^{b_1} \cdot v^+ + Z(a_3) \in \mathfrak{B}_1.$$

Case 2. $a_3 > \min\{m_1 + i, 2i\}$. Then

$$y_2^{a_3} y_1^i y_2^{b_1} \cdot v^+ = y_2^{a_3 - \min\{m_1+i,2i\}} (y_2^{\min\{m_1+i,2i\}} y_1^i y_2^{b_1} \cdot v^+)$$

Subcase 2.1. $b_1 = m_2$. Then by Lemma 4.3, $y_2^{a_3} y_1^i y_2^{b_1} \cdot v^+ = 0$.

Subcase 2.2. $b_1 \neq m_2$. Then by Lemma 4.3, $y_2^{a_3} y_1^i y_2^{b_1} \cdot v^+$ can be written as a linear combination of elements of the set

$$\{y_2^{b_3} y_1^i y_2^{a_1} \cdot v^+ \mid a_1, i, b_3 \in \mathbf{Z}_0^+ \text{ with } 0 \leq b_3 \leq \min\{m_1 + i, 2i, a_3 - 1\},$$
$$b_1 + 1 \leq a_1 \leq m_2\}$$

which is a subset of $Z(a_3)$.

Thus in Case 2, $y_2^{a_3} y_1^i y_2^{b_1} \cdot v^+ + Z(a_3) = Z(a_3)$.

Therefore $\mathfrak{B}_1$ spans $Y(a_3, b_1)$.

Secondly, we will show that $\mathfrak{B}_1$ is linearly independent. Suppose

$$\sum_{i=\mu}^{m_1+b_1} d_i y_2^{a_3} y_1^i y_2^{b_1} \cdot v^+ + Z(a_3) = Z(a_3),$$

where $d_i \in \mathbf{C}$ for all $i \in \mathbf{Z}_0^+$ with $\mu \leq i \leq m_1 + b_1$. Then

$$\sum_{i=\mu}^{m_1+b_1} d_i y_2^{a_3} y_1^i y_2^{b_1} \cdot v^+ \in Z(a_3).$$

Thus $\sum_{i=\mu}^{m_1+b_1} d_i y_2^{a_3} y_1^i y_2^{b_1} \cdot v^+$ can be written in a linear combination of elements in $\mathfrak{B}(a_3)$. Note that

$\{y_2^{a_3} y_1^i y_2^{b_1} \cdot v^+ \mid i \in \mathbf{Z}_0^+ \text{ with } \mu \leq i \leq m_1 + b_1\}$ and $\mathfrak{B}(a_3)$ are disjoint subsets of the Verma basis $\mathfrak{B}$. Hence $d_i = 0$ for all $i \in \mathbf{Z}_0^+$ with $\mu \leq i \leq m_1 + b_1$. Thus $\mathfrak{B}_1$ is linearly independent. Therefore $\mathfrak{B}_1$ is a basis of $Y(a_3, b_1)$.

Next, we will find a standard basis of $Y(a_3, b_1)$. Let

$$v_i = C_{\mu+i} y_2^{a_3} y_1^{\mu+i} y_2^{b_1} \cdot v^+ + Z(a_3),$$

where

$$C_{\mu+i} = \frac{(m_1 + b_1)!}{(\mu + i)!}(m_1 + b_1 - \mu - i)!$$

for all $i \in \mathbf{Z}_0^+$ with $0 \leq i \leq m_1 + b_1 - \mu$. Observe that $C_{\mu+i} \neq 0$ for all $i \in \mathbf{Z}_0^+$ with $0 \leq i \leq m_1 + b_1 - \mu$. Thus

$$\mathfrak{B}_2 = \{v_i \mid i \in \mathbf{Z}_0^+ \text{ with } 0 \leq i \leq m_1 + b_1 - \mu\}$$

is a basis of $Y(a_3, b_1)$.

We claim that $\mathfrak{B}_2$ is a standard basis of $Y(a_3, b_1)$. This will also tell us that $Y(a_3, b_1)$ is a string module. We start by checking $x_1 \cdot v_0$. We have

$$\begin{aligned} x_1 \cdot v_0 &= C_\mu x_1 (y_2^{a_3} y_1^\mu y_2^{b_1} \cdot v^+) + Z(a_3) \\ &= \begin{cases} Z(a_3) & \text{if } \mu = 0, \\ \mu(1 - \mu + m_1 + b_1) y_2^{a_3} y_1^{\mu-1} y_2^{b_1} \cdot v^+ + Z(a_3) & \text{if } \mu > 0. \end{cases} \end{aligned}$$

Consider $y_2^{a_3}y_1^{\mu-1}y_2^{b_1}\cdot v^+ + Z(a_3)$ in the case $\mu > 0$. Recall $\mu = \max\{a_3 - m_1, \lceil a_3/2\rceil\}$

Case 1. $\mu = a_3 - m_1$. Then

$$\min\{m_1 + (\mu - 1), 2(\mu - 1)\} = \min\{a_3 - 1, 2(a_3 - m_1) - 2\},$$

and $\frac{a_3}{2} \le \lceil a_3/2\rceil \le a_3 - m_1$, which implies $a_3 \le 2(a_3 - m_1)$.

If $a_3 < 2(a_3 - m_1)$, then $a_3 - 1 \le 2(a_3 - m_1) - 2$.

If $a_3 = 2(a_3 - m_1)$, then $a_3 - 1 > 2(a_3 - m_1) - 2 = a_3 - 2$.

Thus

$$\min\{m_1 + (\mu - 1), 2(\mu - 1)\} = \begin{cases} a_3 - 1 & \text{if } a_3 < 2(a_3 - m_1)\ , \\ a_3 - 2 & \text{if } a_3 = 2(a_3 - m_1)\ . \end{cases}$$

Hence

$$y_2^{a_3}y_1^{\mu-1}y_2^{b_1}\cdot v^+ = \begin{cases} y_2\cdot(y_2^{a_3}y_1^{\mu-1}y_2^{b_1}\cdot v^+) & \text{if } a_3 < 2(a_3 - m_1)\ , \\ y_2^2\cdot(y_2^{a_3}y_1^{\mu-1}y_2^{b_1}\cdot v^+) & \text{if } a_3 = 2(a_3 - m_1)\ . \end{cases}$$

Case 2. $\mu = \lceil a_3/2\rceil$. Then

$$\min\{m_1 + (\mu - 1), 2(\mu - 1)\} = \min\{m_1 + \lceil a_3/2\rceil - 1, 2\lceil a_3/2\rceil - 2\}$$

and $a_3 - m_1 \le \lceil a_3/2\rceil$, which implies $a_3 \le m_1 + \lceil a_3/2\rceil$.

Observe that

$$\lceil a_3/2\rceil = \begin{cases} \frac{a_3}{2} & \text{if } a_3 \text{ is even}\ , \\ \frac{a_3+1}{2} & \text{if } a_3 \text{ is odd}\ . \end{cases}$$

If $a_3 < m_1 + \lceil a_3/2 \rceil$, then $2\lceil a_3/2 \rceil < m_1 + \lceil a_3/2 \rceil$ which tell us $2\lceil a_3/2 \rceil - 2 < m_1 + \lceil a_3/2 \rceil - 1$. If $a_3 = m_1 + \lceil a_3/2 \rceil$, then $2\lceil a_3/2 \rceil - 1 \leq m_1 + \lceil a_3/2 \rceil$ which tell us $2\lceil a_3/2 \rceil - 2 \leq m_1 + \lceil a_3/2 \rceil - 1$. Thus

$$\min\{m_1 + (\mu - 1), 2(\mu - 1)\} = 2\lceil a_3/2 \rceil - 2$$
$$= \begin{cases} a_3 - 2 & \text{if } a_3 \text{ is even,} \\ a_3 - 1 & \text{if } a_3 \text{ is odd.} \end{cases}$$

Hence

$$y_2^{a_3} y_1^{\mu-1} y_2^{b_1} \cdot v^+ = \begin{cases} y_2^2 (y_2^{a_3-2} y_1^{\mu-1} y_2^{b_1} \cdot v^+) & \text{if } a_3 \text{ is even,} \\ y_2 \cdot (y_2^{a_3-1} y_1^{\mu-1} y_2^{b_1} \cdot v^+) & \text{if } a_3 \text{ is odd.} \end{cases}$$

Thus in both cases Lemma 4.3 implies $y_2^{a_3} y_1^{\mu-1} y_2^{b_1} \cdot v^+$ is zero if $b_1 = m_2$ and a linear combination of elements in the set

$$\{y_2^{b_3} y_1^{\mu-1} y_2^{a_1} \cdot v^+ \mid a_1, b_3 \in \mathbf{Z}_0^+ \text{ with } b_1 + 1 \leq a_1 \leq m_2,\ 0 \leq b_3 \leq a_1 - 1\}$$

if $b_1 \neq m_2$, and this last set is a subset of $\mathfrak{B}(a_3)$. Therefore

$$y_2^{a_3} y_1^{\mu-1} y_2^{b_1} \cdot v^+ + Z(a_3) = Z(a_3),$$

that is, $x_1 \cdot v_0 = Z(a_3)$.

Next, we will show that $x_1 \cdot v_i = v_{i-1}$ for all $i \in \mathbf{Z}_0^+$ with $0 < i \leq m_1 + b_1 - \mu$. Consider such an i.

$$\begin{aligned} x_1 \cdot v_i &= x_1 \cdot (C_{\mu+i} y_2^{a_3} y_1^{\mu+i} y_2^{b_1} \cdot v^+) + Z(a_3) \\ &= C_{\mu+i} y_2^{a_3} (x_1 y_1^{\mu+i} y_2^{b_1} \cdot v^+) + Z(a_3) \\ &= C_{\mu+i} (\mu + i)(1 - \mu - i + m_1 + b_1) y_2^{a_3} y_1^{\mu+i-1} y_2^{b_1} \cdot v^+ + Z(a_3) \\ &= C_{\mu+i-1} y_2^{a_3} y_1^{\mu+i-1} y_2^{b_1} \cdot v^+ + Z(a_3) \\ &= v_{i-1} \end{aligned}$$

Finally, we calculate

$$h_1 \cdot v_i = C_{\mu+i} h_1 \cdot (y_2^{a_3} y_1^{\mu+i} y_2^{b_1} \cdot v^+) + Z(a_3)$$

$$= (m_1 + b_1 - 2(\mu+i) + a_3) C_{\mu+i} y_2^{a_3} y_1^{\mu+i} y_2^{b_1} \cdot v^+ + Z(a_3)$$

$$= (m_1 + b_1 - 2(\mu+i) + a_3) v_i$$

Therefore $\mathfrak{B}_2$ is a standard basis of $Y(a_3, b_1)$.

Observe that

$$h_1 \cdot v_0 = (m_1 + b_1 + -2\mu + a_3) v_0$$

$$h_1 \cdot v_{m_1+b_1-\mu} = (m_1 + b_1 + -2(m_1 + b_1) + a_3) v_{m_1+b_1-\mu}$$

Then $Y(a_3, b_1)$ is a string module $\mathcal{S}(m_1 + b_1 + a_3 - 2\mu, a_3 - m_1 - b_1)$. Thus $Y(a_3, b_1) = \mathcal{S}(k+n, k-n)$, where $k = a_3 - \mu$, $n = m_1 + b_1 - \mu$.

Next, let

$$T(a_3, b_1) = \operatorname{span}\{y_1^j y_2^{a_3} y_1^i y_2^{b_1} \cdot v^+ + Z(a_3) \mid i, j \in \mathbf{Z}_0^+ \text{ with } 0 \le j \le k,\ \mu \le i \le m_1 + b_1\}$$

We will show that $T(a_3, b_1)$ is an S_1-module. We start by claiming that

$$\mathfrak{B}_3 = \{y_1^j \cdot v_i \mid i, j \in \mathbf{Z}_0^+ \text{ with } 0 \le j \le k,\ 0 \le i \le m_1 + a_1 - \mu\}$$

is a basis of $T(a_3, b_1)$. Since $C_{\mu+i} \neq 0$ for all $i \in \mathbf{Z}_0^+$ with $0 \le i \le m_1 + a_1 - \mu$, it suffices to show

$$\{y_1^j y_2^{a_3} y_1^{\mu+i} y_2^{b_1} \cdot v^+ + Z(a_3) \mid i, j \in \mathbf{Z}_0^+ \text{ with } 0 \le j \le k,\ 0 \le i \le m_1 + b_1 - \mu\}$$

is linearly independent. Observe that

$$\begin{aligned} k &= a_3 - \mu \\ &= a_3 - \max\{a_3 - m_1, \lceil a_3/2 \rceil\} \\ &= a_3 + \min\{m_1 - a_3, -\lceil a_3/2 \rceil\} \\ &= \min\{m_1, a_3 - \lceil a_3/2 \rceil\} \\ &= \min\{m_1, \lfloor a_3/2 \rfloor\} \end{aligned}$$

Thus

$$\{y_1^j y_2^{a_3} y_1^{\mu+i} y_2^{b_1} \cdot v^+ + Z(a_3) \mid i,j \in \mathbb{Z}_0^+ \text{ with } 0 \le j \le k,\ 0 \le i \le m_1 + a_1 - \mu\}$$

is a subset of $\mathfrak{B}$, the Verma basis, and for any $i, j \in \mathbb{Z}_0^+$ with $0 \le j \le k$, $0 \le i \le m_1 + a_1 - \mu$, $y_1^j y_2^{a_3} y_1^{\mu+i} y_2^{b_1} \cdot v^+ \notin Z(a_3)$. Therefore

$$\{y_1^j y_2^{a_3} y_1^{\mu+i} y_2^{b_1} \cdot v^+ + Z(a_3) \mid i,j \in \mathbb{Z}_0^+ \text{ with } 0 \le j \le k,\ 0 \le i \le m_1 + a_1 - \mu\}$$

is linearly independent, as needed, and the claim is proved.

By Lemma 1.6, $T(a_3, b_1)$ is an S_1-submodule of $V/Z(a_3)$. Observe that

$$Z(a_3+1)/Z(a_3) = \sum_{b_1=0}^{m_2} T(a_3, b_1)$$

will then be an S_1-submodule of $V/Z(a_3)$ as well.

We claim that $Z(a_3+1)$ must be an S_1-submodule of V. Let $z \in Z(a_3+1)$. Then $z + Z(a_3) \in Z(a_3+1)/Z(a_3)$ and

$$h_1 \cdot (z + Z(a_3)) = h_1 \cdot z + Z(a_3) \in Z(a_3+1)/Z(a_3)$$

Thus

$$h_1 \cdot (z + Z(a_3)) = z' + Z(a_3)$$

for some $z' \in Z(a_3+1)$, so $h_1 \cdot z = z + z''$ for some $z'' \in Z(a_3) \subseteq Z(a_3+1)$. Thus $h_1 \cdot z \in Z(a_3+1)$. By the same argument we have $x_1 \cdot z$, $y_1 \cdot z \in Z(a_3+1)$ as well. Hence $Z(a_3+1)$ is an S_1-submodule of V. Thus the result holds in the case a_3+1.

Therefore we have Lemma 4.4. #

The following procedures I, II, III, IV and V are important tools. They will be used for finding the action of the Chevalley generators of $\mathfrak{o}(5, \mathbf{C})$ on an arbitrary element of the Verma basis $\mathfrak{B}$.

I. Fix $a_3, a_1 \in \mathbf{Z}_0^+$. Let

$$\mu = \max\{a_3 - m_1, \lceil a_3/2 \rceil\}.$$

Basis elements with the given exponents a_3, and a_1 are:

$$\{y_1^j y_2^{a_3} y_1^i y_2^{a_1} \cdot v^+ \mid j \in \mathbf{Z}_0^+ \text{ with } 0 \le j \le \min\{m_1, \lfloor a_3/2 \rfloor\},\ \mu \le i \le m_1 + a_1\}$$

We consider $y_2^{a_3} y_1^{\mu-1} y_2^{a_1} \cdot v^+$. If $\mu = 0$ or $y_2^{a_3} y_1^{\mu-1} y_2^{a_1} \cdot v^+ = 0$, then we will consider the action of y_1 on this basis element in II. If $\mu \ne 0$, then $\mu = a_3 - m_1$, or $\mu = \lceil a_3/2 \rceil$ and by the same argument as in the proof of Lemma 4.4, we have that $\min\{m_1 + (\mu-1), 2(\mu-1)\}$ is $a_3 - 1$ or $a_3 - 2$.

Case : $a_1 = m_2$. Then by Lemma 4.3, $y_2^{a_3} y_1^{\mu-1} y_2^{a_1} \cdot v^+ = 0$. Thus we will consider the action of y_1 on this basis element in II.

Case : $a_1 \ne m_2$.

Subcase : $\min\{m_1 + (\mu-1), 2(\mu-1)\}$ is $a_3 - 1$. Then we will consider the action of y_1 on this basis element in III.

Subcase : $\min\{m_1 + (\mu-1), 2(\mu-1)\}$ is $a_3 - 2$. Then we will consider the action of y_1 on this basis element in IV.

II. Fix $a_3, a_1 \in \mathbf{Z}_0^+$. Let

$$\mu = \max\{a_3 - m_1, \lceil a_3/2 \rceil\}.$$

Basis elements with the given exponents a_3 and a_1 are:

$$\{y_1^j y_2^{a_3} y_1^i y_2^{a_1} \cdot v^+ \mid i, j \in \mathbf{Z}_0^+ \text{ with } 0 \leq j \leq \min\{m_1, \lfloor a_3/2 \rfloor\},\ \mu \leq i \leq m_1 + a_1\}$$

Suppose $y_2^{a_3} y_1^{\mu-1} y_2^{a_1} \cdot v^+ = 0$.

We will find the formulas of the action of x_1 and y_1 on the basis elements with the given exponents a_3, a_1. Let

$$X = \operatorname{span}\{y_2^{a_3} y_1^i y_2^{a_1} \cdot v^+ \mid i \in \mathbf{Z}_0^+ \text{ with } \mu \leq i \leq m_1 + a_1\}$$

$$= \operatorname{span}\{y_2^{a_3} y_1^{\mu+i} y_2^{a_1} \cdot v^+ \mid i \in \mathbf{Z}_0^+ \text{ with } 0 \leq i \leq m_1 + a_1 - \mu\}$$

and let $c_i = \dfrac{(m_1 + a_1)!(m_1 + a_1 - i)!}{i!}$ for all $i \in \mathbf{Z}_0^+$ with $0 \leq i \leq m_1 + a_1$.

It is clear that $\{y_2^{a_3} y_1^{\mu+i} y_2^{a_1} \cdot v^+ \mid i \in \mathbf{Z}_0^+ \text{ with } 0 \leq i \leq m_1 + a_1 - \mu\}$ is a basis of X. We claim X is a B_1-module and

$$D_1 = \{c_{\mu+i} y_2^{a_3} y_1^{\mu+i} y_2^{a_1} \cdot v^+ \mid i \in \mathbf{Z}_0^+ \text{ with } 0 \leq i \leq m_1 + a_1 - \mu\}$$

is a standard basis of X. This will also tell us that X is a string module. It is clear that D_1 is a basis of X. For $i \in \mathbf{Z}_0^+$ with $0 \leq i \leq m_1 + a_1 - \mu$, we have

1) $x_1 \cdot (c_{\mu+i} y_2^{a_3} y_1^{\mu+i} y_2^{a_1} \cdot v^+) = c_{\mu+i}\{[x_1, y_2^{a_3}] + y_2^{a_3} x_1\} y_1^{\mu+i} y_2^{a_1} \cdot v^+$

$$= c_{\mu+i} y_2^{a_3}\{[x_1, y_1^{\mu+i}] + y_1^{\mu+i} x_1\} y_2^{a_1} \cdot v^+$$

$$= c_{\mu+i}(\mu + i)(1 - \mu - i + m_1 + a_1) y_2^{a_3} y_1^{\mu+i-1} y_2^{a_1} \cdot v^+$$

$$= c_{\mu+i-1} y_2^{a_3} y_1^{\mu+i-1} y_2^{a_1} \cdot v^+$$

(observe that if $i = 0$, then $x_1 \cdot (c_\mu y_2^{a_3} y_1^\mu y_2^{a_1} \cdot v^+) = 0$) and

2) $h_1 \cdot (c_{\mu+i} y_2^{a_3} y_1^{\mu+i} y_2^{a_1} \cdot v^+) = (m_1 + a_1 - 2(\mu + i) + a_3) c_{\mu+i} y_2^{a_3} y_1^{\mu+i} y_2^{a_1} \cdot v^+$

Thus X is a B_1-module and D_1 is a standard basis of X. Observe that

$$h_1 \cdot (c_\mu y_2^{a_3} y_1^{\mu} y_2^{a_1} \cdot v^+) = (m_1 + a_1 - 2\mu + a_3) c_\mu y_2^{a_3} y_1^{\mu} y_2^{a_1} \cdot v^+$$

while

$$h_1 \cdot (c_{m_1+a_1} y_2^{a_3} y_1^{m_1+a_1} y_2^{a_1} \cdot v^+) = (a_3 - m_1 - a_1) c_{m_1+a_1} y_2^{a_3} y_1^{m_1+a_1} y_2^{a_1} \cdot v^+$$

Then $X = \mathcal{S}(m_1 + a_1 - 2\mu + a_3, a_3 - m_1 - a_1)$, which is the same as $\mathcal{S}(k+n, k-n)$ if $k = a_3 - \mu$ and $n = m_1 + a_1 - \mu$.

Let

$$Y = \text{span}\{y_1^j y_2^{a_3} y_1^i y_2^{a_1} \cdot v^+ \mid i, j \in \mathbf{Z}_0^+ \text{ with } 0 \le j \le k,\ \mu \le i \le m_1 + a_1\}$$

We claim that

$$D_2 = \{c_{\mu+i} y_1^j y_2^{a_3} y_1^{\mu+i} y_2^{a_1} \cdot v^+ \mid i, j \in \mathbf{Z}_0^+ \text{ with } 0 \le j \le k,\ 0 \le i \le m_1 + a_1 - \mu\}$$

is a basis of Y. It suffices to show $k = \min\{m_1, \lfloor a_3/2 \rfloor\}$, for then the elements of D_2 will be nonzero scalar multiples of elements of the Verma basis, $\mathfrak{B}$. Indeed we have

$$\begin{aligned} k &= a_3 - \mu \\ &= a_3 - \max\{a_3 - m_1, \lceil a_3/2 \rceil\} \\ &= a_3 + \min\{m_1 - a_3, -\lceil a_3/2 \rceil\} \\ &= \min\{m_1, \lfloor a_3/2 \rfloor\} \end{aligned}$$

Thus D_2 is a basis of Y.

By Lemma 1.6, Y is an S_1-module . We will use Lemma 1.7 and Theorem 2.5 to find the action of y_1 on the elements of D_2. At the same time, we will write down the actions of h_1, x_1 and y_1 on the corresponding elements of $\mathfrak{B}$. First, we will show that the actions of h_1 and x_1 on D_2 satisfy Lemma 1.7. For $i,j \in \mathbf{Z}_0^+$ with $0 \le j \le k$, $0 \le i \le m_1 + a_1 - \mu$.

1) $h_1 \cdot (c_{\mu+i} y_1^j y_2^{a_3} y_1^{\mu+i} y_2^{a_1} \cdot v^+) = (k+n-2i-2j) c_{\mu+i} y_1^j y_2^{a_3} y_1^{\mu+i} y_2^{a_1} \cdot v^+$

That is,

$$h_1 \cdot (y_1^j y_2^{a_3} y_1^{\mu+i} y_2^{a_1} \cdot v^+) = (k+n-2i-2j) y_1^j y_2^{a_3} y_1^{\mu+i} y_2^{a_1} \cdot v^+$$

2)
$$\begin{aligned} x_1 \cdot (c_{\mu+i} y_1^j y_2^{a_3} y_1^{\mu+i} y_2^{a_1} \cdot v^+) &= \{[x_1, y_1^j] + y_1^j x_1\} c_{\mu+i} y_2^{a_3} y_1^{\mu+i} y_2^{a_1} \cdot v^+ \\ &= [x_1, y_1^j] c_{\mu+i} y_2^{a_3} y_1^{\mu+i} y_2^{a_1} \cdot v^+ \\ &\quad + c_{\mu+i} y_1^j y_2^{a_3} ([x_1, y_1^{\mu+i}] + y_1^{\mu+i} x_1) y_2^{a_1} \cdot v^+ \end{aligned}$$

That is,

$$x_1 \cdot (y_1^j y_2^{a_3} y_1^{\mu+i} y_2^{a_1} \cdot v^+)$$
$$= \begin{cases} j(1-j+m_1+a_1+a_3-2\mu-2i) y_1^{j-1} y_2^{a_3} y_1^{\mu+i} y_2^{a_1} \cdot v^+ & \\ +(\mu+i)(m_1+a_1-\mu-i+1) y_1^j y_2^{a_3} y_1^{\mu+i-1} y_2^{a_1} \cdot v^+ & \text{if } j>0,\ i>0, \\ (\mu+i)(m_1+a_1-\mu-i+1) y_2^{a_3} y_1^{\mu+i-1} y_2^{a_1} \cdot v^+ & \text{if } j=0,\ i>0, \\ j(1-j+m_1+a_1+a_3-2\mu) y_1^{j-1} y_2^{a_3} y_1^{\mu} y_2^{a_1} \cdot v^+ & \text{if } j>0,\ i=0, \\ 0 & \text{if } j=0,\ i=0. \end{cases}$$

First, we consider $y_1^{j-1} y_2^{a_3} y_1^{\mu+i} y_2^{a_1} \cdot v^+$ in the case $j > 0, i \ge 0$. Then $0 \le j-1 \le k$, $\mu \le \mu + i \le m_1 + a_1$. Thus $y_1^{j-1} y_2^{a_3} y_1^{\mu+i} y_2^{a_1} \cdot v^+ \in Y$.

Next, we consider $y_1^j y_2^{a_3} y_1^{\mu+i-1} y_2^{a_1} \cdot v^+$ in the case $j \ge 0, i > 0$. Then $\mu \le \mu+i-1 \le m_1+a_1$. Thus $y_1^j y_2^{a_3} y_1^{\mu+i-1} y_2^{a_1} \cdot v^+ \in Y$. Hence the actions of h_1 and x_1 on the elements of D_2 satisfy Lemma 1.7. Therefore Y can be made into an S_1-module by keeping the same action for x_1 and h_1 and defining an action of y_1 by the following: for any $i,j \in \mathbf{Z}_0^+$ with $0 \le j \le k$, $0 \le i \le m_1 + a_1 - \mu$,

3) If $0 \le j < k$, then

$$y_1 \cdot (y_1^j y_2^{a_3} y_1^{\mu+i} y_2^{a_1} \cdot v^+) = y_1^{j+1} y_2^{a_3} y_1^{\mu+i} y_2^{a_1} \cdot v^+$$

4) If $0 \le i < m_1 + a_1 - \mu$, then

$$y_1 \cdot (c_{\mu+i} y_1^k y_2^{a_3} y_1^{\mu+i} y_2^{a_1} \cdot v^+) = \sum_{p=1}^{m_1+a_1-\mu-i} \left\{ (-1)^{p-1} \binom{m_1 + a_1 - \mu - i}{p} \prod_{r=0}^{p-1} (k+1-r)(i+r+1) \right\} c_{\mu+i+p} y_1^{k+1-p} y_2^{a_3} y_1^{\mu+i+p} y_2^{a_1} \cdot v^+$$

That is,

$$y_1 \cdot (y_1^k y_2^{a_3} y_1^{\mu+i} y_2^{a_1} \cdot v^+) = (\mu+i)! \sum_{s=s_0}^{k} (-1)^{k-s} \frac{\prod_{r=0}^{k-s} (k+1-r)(i+r+1)}{(k+1-s)!(\mu+i+k+1-s)!} y_1^s y_2^{a_3} y_1^{\mu+i+k+1-s} y_2^{a_1} \cdot v^+$$

where

$$s_0 = \begin{cases} 0 & \text{if } k+1-(m_1+a_1-i) < 0 \ , \\ k+1-(m_1+a_1-i) & \text{if } k+1-(m_1+a_1-i) \ge 0 \ . \end{cases}$$

Observe that

a) $(k+1-s) > 0$

b) $\mu + 1 \le \mu + i + k + 1 - s \le m_1 + a_1$

c) $\prod_{r=0}^{k-s} (k+1-r) = 0$ iff $s < 0$

d) $0 \le s_0 \le k$

Thus $y_1^s y_2^{a_3} y_1^{\mu+i+k+1-s} y_2^{a_1} \cdot v^+$ is an element of Y, moreover it is an element of $\mathfrak{B}$ for all $s \in \mathbf{Z}_0^+$ with $s_0 \le s \le k$.

5) $y_1 \cdot (c_{m_1+a_1} y_1^k y_2^{a_3} y_1^{m_1+a_1} y_2^{a_1} \cdot v^+) = 0$. That is,

$$y_1 \cdot (y_1^k y_2^{a_3} y_1^{m_1+a_1} y_2^{a_1} \cdot v^+) = 0.$$

III. Fix $a_3, a_1 \in \mathbf{Z}_0^+$. Let $\mu = \max\{a_3 - m_1, \lceil a_3/2 \rceil\}$. Basis elements with the given exponents a_3, a_1 are:

$\{y_1^j y_2^{a_3} y_1^i y_2^{a_1} \cdot v^+ \mid i, j \in \mathbf{Z}_0^+ \text{ with } 0 \le j \le \min\{m_1, \lfloor a_3/2 \rfloor\},\ \mu \le i \le m_1 + a_1\}$.

Suppose $\min\{m_1 + \mu - 1, 2(\mu - 1)\} = a_3 - 1$ and $a_1 \neq m_2$.

We will find the formulas of the action of y_1 and x_1 on the basis elements with the given exponents a_3, a_1. Let $\mu_0 = \max\{0, \mu - 1 - m_1\}$. Then by Lemma 4.3,

$$\begin{aligned} y_2^{a_3} y_1^{\mu-1} y_2^{a_1} \cdot v^+ &= y_2 \cdot (y_2^{a_3-1} y_1^{\mu-1} y_2^{a_1} \cdot v^+) \\ &= a_1! \sum_{s=s_0}^{a_3-1} \left\{ (-1)^{a_3-1-s} \frac{\prod_{r=0}^{a_3-1-s} (a_3 - r)(a_1 - \mu_0 + r + 1)}{(a_3 - s)!(a_1 + a_3 - s)!} \right\} \\ &\qquad y_2^s y_1^{\mu-1} y_2^{a_1+a_3-s} \cdot v^+ \end{aligned}$$

where

$$s_0 = \begin{cases} 0 & \text{if } a_1 + a_3 - m_2 < 0\,, \\ a_1 + a_3 - m_2 & \text{if } a_1 + a_3 - m_2 \ge 0\,. \end{cases}$$

For each $s \in \mathbf{Z}_0^+$ with $s_0 \le s \le a_3 - 1$, let

$$q_s = a_1!(-1)^{a_3-1-s} \frac{\prod_{r=0}^{a_3-1-s} (a_3 - r)(a_1 - \mu_0 + r + 1)}{(a_3 - s)!(a_1 + a_3 - s)!}$$

Observe that $q_s \neq 0$. Then

$$y_2^{a_3} y_1^{\mu-1} y_2^{a_1} \cdot v^+ = \sum_{s=s_0}^{a_3-1} q_s y_2^s y_1^{\mu-1} y_2^{a_1+a_3-s} \cdot v^+ \qquad (1)$$

For each $i, s \in \mathbf{Z}_0^+$ with $0 \le i \le m_1 + a_1 - \mu$, $s_0 \le s \le a_3$, let

$$\rho_i = \frac{(m_1 + a_1)!}{(\mu + i)!}$$

$$d_s^{(i)} = \frac{(m_1 + a_1 - \mu - i + a_3 - s)!}{(a_3 - s)!}, \text{ and}$$

$$r_s = \mu(1 - \mu + m_1 + a_1 + a_3 - s)$$

Observe that $\rho_i, d_s^{(i)}$ and r_s are nonzero. For each $i \in \mathrm{Z}_0^+$ with $0 \le i \le m_1 + a_1 - \mu$, let

$$v_i = \rho_i \left(\frac{d_{a_3}^{(i)}}{d_{a_3}^{(0)} r_{a_3}} y_2^{a_3} y_1^{\mu+i} y_2^{a_1} \cdot v^+ - \sum_{s=s_0}^{a_3-1} \frac{q_s d_s^{(i)}}{d_s^{(0)} r_s} y_2^s y_1^{\mu+i} y_2^{a_1+a_3-s} \cdot v^+ \right).$$

Observe that

a) $s_0 \le s \le a_3 - 1$

b) $\mu \le \mu + i \le m_1 + a_1$

c) $a_1 + 1 \le a_1 + a_3 - s \le m_2$

Then the elements $y_2^s y_1^{\mu+i} y_2^{a_1+a_3-s} \cdot v^+$ with $i,\ s \in \mathrm{Z}_0^+$, $0 \le i \le m_1 + a_1 - \mu$, $s_0 \le s \le a_3$ are all district elements of $\mathfrak{B}$. Hence

$$D_3 = \{v_i \mid i \in \mathbf{Z}_0^+ \text{ with } 0 \le i \le m_1 + a_1 - \mu\}$$

is linearly independent.

Let $X = \mathrm{span} D_3$. We claim that X is a B_1-module and D_3 is a standard basis of X. This will also tell us that X is a string module. For each $i \in \mathrm{Z}_0^+$

with $0 < i \leq m_1 + a_1 - \mu$, we have

$$
\begin{aligned}
&x_1 \cdot v_i \\
&= \rho_i \left\{ \frac{d_{a_3}^{(i)}}{d_{a_3}^{(0)} r_{a_3}} x_1 \cdot (y_2^{a_3} y_1^{\mu+i} y_2^{a_1} \cdot v^+) \right. \\
&\qquad \left. - \sum_{s=s_0}^{a_3-1} \frac{q_s d_s^{(i)}}{d_s^{(0)} r_s} x_1 \cdot (y_2^s y_1^{\mu+i} y_2^{a_1+a_3-s} \cdot v^+) \right\} \\
&= \rho_i \left\{ \frac{d_{a_3}^{(i)} (\mu+i)(m_1+a_1-\mu-i+1)}{d_{a_3}^{(0)} r_{a_3}} y_2^{a_3} y_1^{\mu+i-1} y_2^{a_1} \cdot v^+ \right. \\
&\qquad \left. - \sum_{s=s_0}^{a_3-1} \frac{q_s d_s^{(i)} (\mu+i)(m_1+a_1+a_3-s-\mu-i+1)}{d_s^{(0)} r_s} y_2^s y_1^{\mu+i-1} y_2^{a_1+a_3-s} \cdot v^+ \right\} \\
&= \rho_{i-1} \left\{ \frac{d_{a_3}^{(i-1)}}{d_{a_3}^{(0)} r_{a_3}} y_2^{a_3} y_1^{\mu+i-1} y_2^{a_1} \cdot v^+ \right. \\
&\qquad \left. - \sum_{s=s_0}^{a_3-1} \frac{q_s d_s^{(i-1)}}{d_s^{(0)} r_s} y_2^s y_1^{\mu+i-1} y_2^{a_1+a_3-s} \cdot v^+ \right\} \\
&= v_{i-1}
\end{aligned}
$$

$$
\begin{aligned}
x_1 \cdot v_0 &= \rho_0 \left\{ \frac{1}{r_{a_3}} x_1 \cdot (y_2^{a_3} y_1^{\mu} y_2^{a_1} \cdot v^+) - \sum_{s=s_0}^{a_3-1} \frac{q_s}{r_s} x_1 \cdot (y_2^s y_1^{\mu} y_2^{a_1+a_3-s} \cdot v^+) \right\} \\
&= \rho_0 \left\{ \frac{\mu(m_1+a_1-\mu+1)}{r_{a_3}} y_2^{a_3} y_1^{\mu-1} y_2^{a_1} \cdot v^+ \right. \\
&\qquad \left. - \sum_{s=s_0}^{a_3-1} \frac{q_s(\mu)(m_1+a_1+a_3-s-\mu+1)}{r_s} y_2^s y_1^{\mu-1} y_2^{a_1+a_3-s} \cdot v^+ \right\} \\
&= \rho_0 \left\{ y_2^{a_3} y_1^{\mu-1} y_2^{a_1} \cdot v^+ - \sum_{s=s_0}^{a_3-1} q_s y_2^s y_1^{\mu-1} y_2^{a_1+a_3-s} \cdot v^+ \right\} \\
&= 0
\end{aligned}
$$

Also, for each $i \in \mathbf{Z}_0^+$ with $0 \leq i \leq m_1 + a_1 - \mu$,

$$h_1 \cdot v_i = (m_1 + a_1 + a_3 - 2\mu - 2i)v_i.$$

Thus X is a B_1-module with standard basis D_3.

Observe that

$$h_1 \cdot v_0 = (m_1 + a_1 + a_3 - 2\mu)v_0$$

while,

$$h_1 \cdot v_{m_1+a_1-\mu} = (a_3 - a_1 - m_1)v_{m_1+a_1-\mu}$$

Thus X is the string module $\mathcal{S}(m_1 + a_1 + a_3 - 2\mu, a_3 - m_1 - a_1)$. Then $X = \mathcal{S}(k+n, k-n)$, where $k = a_3 - \mu$, $n = m_1 + a_1 - \mu$.

Let

$$D_4 = \{y_1^j \cdot v_i \mid i, j \in \mathbf{Z}_0^+ \text{ with } 0 \leq j \leq k,\ 0 \leq i \leq m_1 + a_1 - \mu\},$$

and let $Y = \text{span} D_4$. We claim that D_4 is a basis of Y. It suffices to show linear independence.

First, observe that

$$\begin{aligned} k =& a_3 - \mu \\ =& a_3 - \max\{a_3 - m_1, \lceil a_3/2 \rceil\} \\ =& \min\{m_1, a_3 - \lceil a_3/2 \rceil\} \\ =& \min\{m_1, \lfloor a_3/2 \rfloor\} \end{aligned}$$

For each $i, j \in \mathbf{Z}_0^+$ with $0 \leq j \leq k$, $0 \leq i \leq m_1 + a_1 - \mu$,

$$y_1^j \cdot v_i = \rho_i \left\{ \frac{d_{a_3}^{(i)}}{d_{a_3}^{(0)} r_{a_3}} y_1^j y_2^{a_3} y_1^{\mu+i} y_2^{a_1} \cdot v^+ \right.$$

$$\left. - \sum_{s=s_0}^{a_3-1} \frac{q_s d_s^{(i)}}{d_s^{(0)} r_s} y_1^j y_2^s y_1^{\mu+i} y_2^{a_1+a_3-s} \cdot v^+ \right\}$$

Observe that,

$a)\ \dfrac{d_{a_3}^{(i)}}{d_{a_3}^{(0)} r_{a_3}} \neq 0$

$b)\ y_1^j y_2^{a_3} y_1^{\mu+i} y_2^{a_1} \cdot v^+ \in \mathfrak{B}$

$c)$ By Lemma 4.4, $\displaystyle\sum_{s=s_0}^{a_3-1} \frac{q_s d_s^{(i)}}{d_s^{(0)} r_s} y_1^j y_2^s y_1^{\mu+i} y_2^{a_1+a_3-s} \cdot v^+ \in Z(a_3)$

Then by Lemma 2.10, D_4 is a basis of Y. For each $i, j \in \mathbb{Z}_0^+$ with $0 \leq j \leq k$, $0 \leq i \leq m_1 + a_1 - \mu$,

1) $h_1 \cdot (y_1^j \cdot v_i) = \big(m_1 + a_1 + a_3 - 2(\mu + i + j)\big) y_1^j \cdot v_i$

$$= (k + n - 2i - 2j) y_1^j \cdot v_i$$

2) $x_1 \cdot (y_1^j \cdot v_i) = \big\{[x_1, y_1^j] + y_1^j x_1\big\} \cdot v_i$

$$= j(1 - j + m_1 + a_1 + a_3 - 2\mu - 2i) y_1^j \cdot v_i + y_1^j \cdot v_{i-1}$$

$$= j(k + n - 2i - j + 1) y_1^{j-1} \cdot v_i + y_1^j \cdot v_{i-1}$$

Thus Y is a B_1-module and $h_1 \cdot (y_1^j \cdot v_i)$ and $x_1 \cdot (y_1^j \cdot v_i)$ satisfy 1) and 2) of Lemma 1.7. Therefore Y can be made into an S_1-module by keeping the same action for x_1 and h_1 and defining an action of y_1 by the following: for any $i, j \in \mathbb{Z}_0^+$ with $0 \leq j \leq k$, $0 \leq i \leq m_1 + a_1 - \mu$,

3) If $0 \leq j < k$, then

$$y_1 \cdot (y_1^j \cdot v_i) = y_1^{j+1} \cdot v_i$$

4) If $0 \leq i < m_1 + a_1 - \mu$, then

$$y_1 \cdot (y_1^k \cdot v_i) = \sum_{p=1}^{m_1+a_1-\mu-i} \left\{ (-1)^{p-1} \binom{m_1 + a_1 - \mu - i}{p} \prod_{r=0}^{p-1} (k + 1 - r)(i + r + 1) \right\}$$

$$y_1^{k+1-p} \cdot v_{i+p}$$

Let $t = k+1-p$. Then

$$y_1 \cdot (y_1^k \cdot v_i) = \sum_{t=t_0}^{k} \left\{ (-1)^{k-t} \binom{m_1 + a_1 - \mu - i}{k+1-t} \prod_{r=0}^{k-t} (k+1-r)(i+r+1) \right\} y_1^t \cdot v_{i+k+1-t}$$

where

$$t_0 = \begin{cases} 0 & \text{if } k+1-m_1-a_1+\mu+i < 0\,, \\ k+1-m_1-a_1+\mu+i & \text{if } k+1-m_1-a_1+\mu+i \geq 0\,. \end{cases}$$

For each $i,\ t \in \mathbf{Z}_0^+$ with $t_0 \leq t \leq k$, $0 \leq i \leq m_1 + a_1 - \mu$, let

$$\varepsilon_t^{(i)} = (-1)^{k-t} \binom{m_1 + a_1 - \mu - i}{k+1-t} \prod_{r=0}^{k-t} (k+1-r)(i+r+1)$$

Then

$$\begin{aligned} y_1 \cdot (y_1^k \cdot v_i) &= \sum_{t=t_0}^{k} \varepsilon_t^{(i)} y_1^t \cdot v_{i+k+1-t} \\ &= \sum_{t=t_0}^{k} \varepsilon_t^{(i)} \rho_i \left(\frac{d_{a_3}^{(i+k+1-t)}}{d_{a_3}^{(0)} r_{a_3}} y_1^t y_2^{a_3} y_1^{\mu+i+k+1-t} y_2^{a_1} \cdot v^+ \right. \\ &\quad \left. - \sum_{s=s_0}^{a_3-1} \frac{q_s d_s^{(i+k+1-t)}}{d_s^{(0)} r_s} y_1^t y_2^s y_1^{\mu+i+k+1-t} y_2^{a_1+a_3-s} \cdot v^+ \right) \end{aligned}$$

while on the other hand,

$$\begin{aligned} y_1 \cdot (y_1^k \cdot v_i) = \rho_i &\left\{ \frac{d_{a_3}^{(i)}}{d_{a_3}^{(0)} r_{a_3}} y_1 \cdot (y_1^k y_2^{a_3} y_1^{\mu+i} y_2^{a_1} \cdot v^+) \right. \\ &\quad \left. - \sum_{s=s_0}^{a_3-1} \frac{q_s d_s^{(i)}}{d_s^{(0)} r_s} y_1 \cdot (y_1^k y_2^s y_1^{\mu+i} y_2^{a_1+a_3-s} \cdot v^+) \right\} \end{aligned}$$

Thus

$$y_1\cdot(y_1^k y_2^{a_3} y_1^{\mu+i} y_2^{a_1}\cdot v^+) = \frac{d_{a_3}^{(0)} r_{a_3}}{d_{a_3}^{(i)}}\left\{\sum_{s=s_0}^{a_3-1}\frac{q_s d_s^{(i)}}{d_s^{(0)} r_s} y_1\cdot(y_1^k y_2^s y_1^{\mu+i} y_2^{a_1+a_3-s}\cdot v^+)\right.$$

$$+\sum_{t=t_0}^{k}\varepsilon_t^{(i)}\left(\frac{d_{a_3}^{(i+k+1-t)}}{d_{a_3}^{(0)} r_{a_3}} y_1^t y_2^{a_3} y_1^{\mu+i+k+1-t} y_2^{a_1}\cdot v^+\right.$$

$$\left.\left.-\sum_{s=s_0}^{a_3-1}\frac{q_s d_s^{(i+k+1-t)}}{d_s^{(0)} r_s} y_1^t y_2^s y_1^{\mu+i+k+1-t} y_2^{a_1+a_3-s}\cdot v^+\right)\right\} \qquad (*)$$

The usefulness of equation $(*)$ lies in fact that if we know how to express $y_1\cdot(y_1^k y_2^s y_1^{\mu+i} y_2^{a_1+a_3-s}\cdot v^+)$ and $y_1^t y_2^s y_1^{\mu+i+k+1-t} y_2^{a_1+a_3-s}\cdot v^+$ in term of $\mathfrak{B}$, then we know how to express $y_1\cdot(y_1^k y_2^{a_3} y_1^{\mu+i} y_2^{a_1}\cdot v^+)$ in terms of $\mathfrak{B}$.

5) $y_1\cdot(y_1^k\cdot v_{m_1+a_1-\mu}) = 0$. This tells us

$$0 = \rho_{m_1+a_1-\mu}\left\{\frac{1}{d_{a_3}^{(0)} r_{a_3}} y_1\cdot(y_1^k y_2^{a_3} y_1^{m_1+a_1} y_2^{a_1}\cdot v^+)\right.$$

$$\left.-\sum_{s=s_0}^{a_3-1}\frac{q_s}{d_s^{(0)} r_s} y_1\cdot(y_1^k y_2^s y_1^{m_1+a_1} y_2^{a_1+a_3-s}\cdot v^+)\right\}$$

Thus

$$y_1\cdot(y_1^k y_2^{a_3} y_1^{m_1+a_1} y_2^{a_1}\cdot v^+) = d_{a_3}^{(0)} r_{a_3}\left\{\sum_{s=s_0}^{a_3-1}\frac{q_s}{d_s^{(0)} r_s} y_1\cdot(y_1^k y_2^s y_1^{m_1+a_1} y_2^{a_1+a_3-s}\cdot v^+)\right\} \qquad (**)$$

The usefulness of equation $(**)$ lies in the fact that if we know how to express $y_1\cdot(y_1^k y_2^s y_1^{m_1+a_1} y_2^{a_1+a_3-s}\cdot v^+)$ in terms of $\mathfrak{B}$, then we know how to express $y_1\cdot(y_1^k y_2^{a_3} y_1^{m_1+a_1} y_2^{a_1}\cdot v^+)$ in terms of $\mathfrak{B}$.

We can find formulas for $x_1 \cdot (y_1^j y_2^{a_3} y_1^{\mu+i} y_2^{a_1} \cdot v^+)$ for all $i, j \in \mathbf{Z}_0^+$ with $0 \le j \le k$, $0 \le i \le m_1 + a_1 - \mu$ with the help of formula 2). We obtain

$$\begin{aligned} x_1 \cdot (y_1^j \cdot v_i) \;&= j(k+n-2i-j+1)\rho_i \left\{ \frac{d_{a_3}^{(i)}}{d_{a_3}^{(0)} r_{a_3}} y_1^{j-1} y_2^{a_3} y_1^{\mu+i} y_2^{a_1} \cdot v^+ \right. \\ &\qquad \left. - \sum_{s=s_0}^{a_3-1} \frac{q_s d_s^{(i)}}{d_s^{(0)} r_s} y_1^{j-1} y_2^s y_1^{\mu+i} y_2^{a_1+a_3-s} \cdot v^+ \right\} \\ &\qquad + \rho_{i-1} \left\{ \frac{d_{a_3}^{(i-1)}}{d_{a_3}^{(0)} r_{a_3}} y_1^j y_2^{a_3} y_1^{\mu+i-1} y_2^{a_1} \cdot v^+ \right. \\ &\qquad \left. - \sum_{s=s_0}^{a_3-1} \frac{q_s d_s^{(i-1)}}{d_s^{(0)} r_s} y_1^j y_2^s y_1^{\mu+i-1} y_2^{a_1+a_3-s} \cdot v^+ \right\} \end{aligned}$$

while direct substitution of the expression for $x_1 \cdot (y_1^j \cdot v_i)$ yields

$$\begin{aligned} x_1 \cdot (y_1^j \cdot v_i) \;&= \rho_i \left\{ \frac{d_{a_3}^{(i)}}{d_{a_3}^{(0)} r_{a_3}} x_1 \cdot (y_1^j y_2^{a_3} y_1^{\mu+i} y_2^{a_1} \cdot v^+) \right. \\ &\qquad \left. - \sum_{s=s_0}^{a_3-1} \frac{q_s d_s^{(i)}}{d_s^{(0)} r_s} x_1 \cdot (y_1^j y_2^s y_1^{\mu+i} y_2^{a_1+a_3-s} \cdot v^+) \right\} \end{aligned}$$

Thus

$$\begin{aligned} x_1 \cdot (y_1^j y_2^{a_3} y_1^{\mu+i} y_2^{a_1} \cdot v^+) \;&= \frac{d_{a_3}^{(0)} r_{a_3}}{\rho_i d_{a_3}^{(i)}} \left\{ \sum_{s=s_0}^{a_3-1} \frac{q_s}{d_s^{(0)} r_s} (\rho_i d_s^{(i)} (x_1 y_1^j \right. \\ &\qquad - j(k+n-2i-j+1) y_1^{j-1}) y_2^s y_1^{\mu+i} \\ &\qquad - \rho_{i-1} d_s^{(i-1)} y_1^j y_2^s y_1^{\mu+i-1}) y_2^{a_1+a_3-s} \cdot v^+ \\ &\qquad + \frac{1}{d_{a_3}^{(0)} r_{a_3}} (j(k+n-2i-j+1)\rho_i d_{a_3}^{(i)} y_1^{j-1} y_2^{a_3} y_1^{\mu+i} \\ &\qquad \left. + \rho_{i-1} d_{a_3}^{(i-1)} y_1^j y_2^{a_3} y_1^{\mu+i-1}) y_2^{a_1} \cdot v^+ \right\} \end{aligned} \qquad (***)$$

This shows us that if we know how to express $y_1^{j-1}y_2^s y_1^{\mu+i}y_2^{a_1+a_3-s}\cdot v^+$, $x_1\cdot(y_1^j y_2^s y_1^{\mu+i}y_2^{a_1+a_3-s}\cdot v^+)$, $y_1^j y_2^{a_3}y_1^{\mu+i-1}y_2^{a_1}\cdot v^+$ and $y_1^j y_2^s y_1^{\mu+i-1}y_2^{a_1+a_3-s}\cdot v^+$ in terms of $\mathfrak{B}$, then we will know how to express $x_1\cdot(y_1^j y_2^{a_3}y_1^{\mu+i}y_2^{a_1}\cdot v^+)$ in term of $\mathfrak{B}$.

IV. Fix $a_3, a_1 \in \mathbf{Z}_0^+$. Let $\mu = \max\{a_3 - m_1, \lceil a_3/2\rceil\}$. Basis elements with the exponents a_3, a_1 are:

$$\{y_1^j y_2^{a_3} y_1^i y_2^{a_1}\cdot v^+ \mid i,j\in \mathbf{Z}_0^+ \text{ with } 0\le j\le \min\{m_1, \lfloor a_3/2\rfloor\},\ \mu\le i\le m_1+a_1\}$$

Suppose

$$\min\{m_1+\mu-1, 2(\mu-1)\} = a_3-2 \text{ and } a_1\ne m_2.$$

We will find the formulas of the action of y_1 and x_1 on the basis elements with the exponents a_3, a_1. Let $\mu_0 = \max\{0,\ \mu-1-m_1\}$. By Lemma 4.3,

$$\begin{aligned} y_2^{a_3}y_1^{\mu-1}y_2^{a_1}\cdot v^+ &= y_2^2\cdot(y_2^{a_3-2}y_1^{\mu-1}y_2^{a_1}\cdot v^+)\\ &= y_2\cdot\big(y_2\cdot(y_2^{a_3-2}y_1^{\mu-1}y_2^{a_1}\cdot v^+)\big)\\ &= a_1!\sum_{s=s_0}^{a_3-2}(-1)^{a_3-2-s}\frac{\prod_{r=0}^{a_3-2-s}(a_3-1-r)(a_1-\mu_0+r+1)}{(a_3-1-s)!(a_1+a_3-1-s)!}\\ &\qquad\qquad y_2\cdot y_2^s y_1^{\mu-1}y_2^{a_1+a_3-1-s}\cdot v^+\end{aligned}$$

where

$$s_0=\begin{cases}0 & \text{if } a_1+a_3-m_2-1<0\,,\\ a_1+a_3-m_2-1 & \text{if } a_1+a_3-m_2-1\ge 0\,.\end{cases}$$

For each $s\in\mathbf{Z}_0^+$ with $s_0\le s\le a_3-2$, let

$$q_s = a_1!(-1)^{a_3-2-s}\frac{\prod_{r=0}^{a_3-2-s}(a_3-1-r)(a_1-\mu_0+r+1)}{(a_3-1-s)!(a_1+a_3-1-s)!}$$

Observe that $q_s \neq 0$. Then

$$
\begin{aligned}
y_2^{a_3} y_1^{\mu-1} y_2^{a_1} \cdot v^+ &= \sum_{s=s_0}^{a_3-2} q_s y_2 \cdot (y_2^s y_1^{\mu-1} y_2^{a_1+a_3-1-s} \cdot v^+) \\
&= q_{a_3-2} y_2 \cdot (y_2^{a_3-2} y_1^{\mu-1} y_2^{a_1+1} \cdot v^+) \\
&\quad + \sum_{s=s_0}^{a_3-3} q_s (y_2^{s+1} y_1^{\mu-1} y_2^{a_1+a_3-1-s} \cdot v^+)
\end{aligned}
$$

Observe that for each $s \in \mathbb{Z}_0^+$ with $s_0 \leq s \leq a_3 - 3$, $y_2^{s+1} y_1^{\mu-1} y_2^{a_1+a_3-1-s} \cdot v^+$ is an element of the Verma basis $\mathfrak{B}$. By Lemma 4.3 again,

$$
y_2 \cdot (y_2^{a_3-2} y_1^{\mu-1} y_2^{a_1+1} \cdot v^+) =
$$

$$
(a_1+1)! \sum_{s=t_0}^{a_3-2} (-1)^{a_3-2-s} \frac{\prod_{r=0}^{a_3-2-s} (a_3 - 1 - r)(a_1 - \mu_0 + r + 2)}{(a_3 - 1 - s)!(a_1 + a_3 - 1 - s)!}
$$

$$
y_2^s y_1^{\mu-1} y_2^{a_1+a_3-s} \cdot v^+
$$

where

$$
t_0 = \begin{cases} 0 & \text{if } a_1 + a_3 - m_2 < 0\,, \\ a_1 + a_3 - m_2 & \text{if } a_1 + a_3 - m_2 \geq 0\,. \end{cases}
$$

For each $s \in \mathbb{Z}_0^+$ with $t_0 \leq s \leq a_3 - 2$, let

$$
q'_s = q_{a_3-2}(a_1+1)! \sum_{s=t_0}^{a_3-2} (-1)^{a_3-2-s} \frac{\prod_{r=0}^{a_3-2-s} (a_3 - 1 - r)(a_1 - \mu_0 + r + 2)}{(a_3 - 1 - s)!(a_1 + a_3 - s)!}
$$

Observe that $q'_s \neq 0$ and $y_2^s y_1^{\mu-1} y_2^{a_1+a_3-s} \cdot v^+$ are elements of the Verma basis,

$\mathfrak{B}$. Hence

$$y_2^{a_3}y_1^{\mu-1}y_2^{a_1}\cdot v^+ = \sum_{s=t_0}^{a_3-2} q'_s y_2^s y_1^{\mu-1} y_2^{a_1+a_3-s}\cdot v^+$$

$$+\sum_{s=s_0}^{a_3-3} q_s y_2^{s+1} y_1^{\mu-1} y_2^{a_1+a_3-1-s}\cdot v^+$$

$$=\sum_{s=t_0}^{a_3-2} q'_s y_2^s y_1^{\mu-1} y_2^{a_1+a_3-s}\cdot v^+$$

$$+\sum_{s=s_0+1}^{a_3-2} q_{s-1} y_2^s y_1^{\mu-1} y_2^{a_1+a_3-s}\cdot v^+$$

We need to compare values of t_0 and s_0+1.

Case 1. $a_3+a_1-m_2-1<0$. Then $t_0=0$ and $s_0=0$.

Case 2. $a_3+a_1-m_2-1\geq 0$. Then $t_0=a_3+a_1-m_2=s_0+1$.

Thus

$$y_2^{a_3}y_1^{\mu-1}y_2^{a_1}\cdot v^+ = \begin{cases} \sum_{s=1}^{a_3-2}(q_s+q'_s)y_2^s y_1^{\mu-1} y_2^{a_1+a_3-s}\cdot v^+ & \\ +q'_0 y_1^{\mu-1} y_2^{a_1+a_3}\cdot v^+ & \text{if } a_3+a_1-m_2-1<0, \\ \sum_{s=t_0}^{a_3-2}(q_s+q'_s)y_2^s y_1^{\mu-1} y_2^{a_1+a_3-s}\cdot v^+ & \text{if } a_3+a_1-m_2-1\geq 0. \end{cases}$$

In Case 1, let $l_0=q'_0$ and $l_s=q'_s+q_s$ for all $s\in \mathbf{Z}^+$ with $s\leq a_3-2$.

In Case 2, let $l_s=q'_s+q_s$ for all $s\in \mathbf{Z}_0^+$ with $t_0\leq s\leq a_3-2$. Then

$$y_2^{a_3}y_1^{\mu-1}y_2^{a_1}\cdot v^+ = \sum_{s=t_0}^{a_3-2} l_s y_2^s y_1^{\mu-1} y_2^{a_1+a_3-s}\cdot v^+$$

for both cases.

For all $i,s\in \mathbf{Z}_0^+$ with $0\leq i\leq m_1+a_1-\mu$, and $t_0\leq s\leq a_3$, let

$$v_i = \rho_i\left\{\frac{d_{a_3}^{(i)}}{d_{a_3}^{(0)}r_{a_3}}y_2^{a_3}y_1^{\mu+i}y_2^{a_1}\cdot v^+ - \sum_{s=t_0}^{a_3-2}\frac{l_s d_s^{(i)}}{d_s^{(0)}r_s}y_2^s y_1^{\mu+i}y_2^{a_1+a_3-s}\cdot v^+\right\}$$

where

$$\rho_i = \frac{(m_1+a_1)!}{(\mu+i)!}$$

$$d_s^{(i)} = \frac{(m_1+a_1-\mu-i+a_3-s)!}{(a_3-s)!}$$

$$r_s = \mu(1-\mu+m_1+a_1+a_3-s)$$

for all $s \in \mathbf{Z}_0^+$ with $t_0 \le s \le a_3$. By the same argument as in II, we have that

$$Y = \operatorname{span}\{y_1^j \cdot v_i \mid i,j \in \mathbf{Z}_0^+ \text{ with } 0 \le j \le k,\ 0 \le i \le m_1+a_1-\mu\}$$

is an S_1-module, where $k = \min\{m_1, \lfloor a_3/2\}$. Then we have that for each $i, j \in \mathbf{Z}_0^+$ with $0 \le j \le k$, $0 \le i \le m_1+a_1-\mu$,

$$\begin{aligned} 1)\ y_1 \cdot (y_1^k y_2^{a_3} y_1^{\mu+i} y_2^{a_1} \cdot v^+) = \frac{d_{a_3}^{(0)} r_{a_3}}{d_{a_3}^{(i)}} \Bigg\{ & \sum_{s=t_0}^{a_3-2} \frac{l_s d_s^{(i)}}{d_s^{(0)} r_s} y_1 \cdot (y_1^k y_2^s y_1^{\mu+i} y_2^{a_1+a_3-s} \cdot v^+) \\ & + \sum_{t=j_0}^{k} \varepsilon_t^{(i)} \Bigg(\frac{d_{a_3}^{(i+k+1-t)}}{d_{a_3}^{(0)} r_{a_3}} y_1^t y_2^{a_3} y_1^{\mu+i+k+1-t} y_2^{a_1} \cdot v^+ \\ & - \sum_{s=t_0}^{a_3-2} \frac{l_s d_s^{(i+k+1-t)}}{d_s^{(0)} r_s} y_1^t y_2^s y_1^{\mu+i+k+1-t} y_2^{a_1+a_3-s} \cdot v^+ \Bigg) \Bigg\} \end{aligned}$$

where

$$j_0 = \begin{cases} 0 & \text{if } \ k+1-m_1-a_1-\mu+i < 0, \\ k+1-m_1-a_1-\mu+i & \text{if } \ k+1-m_1-a_1-\mu+i \ge 0. \end{cases}$$

and

$$\varepsilon_t^{(i)} = (-1)^{k-t} \binom{m_1+a_1-\mu-i}{k+1-t} \prod_{r=0}^{k-t} (k+1-r)(i+r+1)$$

for all $i, t \in \mathbf{Z}_0^+$ with $0 \le i \le m_1+a_1-\mu$, $j_0 \le t \le k$.

2) $y_1 \cdot (y_1^k y_2^{a_3} y_1^{m_1+a_1} y_2^{a_1} \cdot v^+) = d_{a_3}^{(0)} r_{a_3} \left\{ \sum_{s=t_0}^{a_3-2} \frac{l_s}{d_s^{(0)} r_s} y_1 \cdot (y_1^k y_2^s y_1^{m_1+a_1} y_2^{a_1+a_3-s} \cdot v^+) \right\}$

3)

$$
\begin{aligned}
x_1 \cdot (y_1^j y_2^{a_3} y_1^{\mu+i} y_2^{a_1} \cdot v^+) = \frac{d_{a_3}^{(0)} r_{a_3}}{\rho_i d_{a_3}^{(i)}} \Bigg\{ & \sum_{s=t_0}^{a_3-2} \frac{l_s}{d_s^{(0)} r_s} (\rho_i d_s^{(i)} (j(k+n-2i-j+1)) y_1^{j-1} \\
& + x_1 y_1^j) y_2^s y_1^{\mu+i} - \rho_{i-1} d_s^{(i-1)} y_1^i y_2^s y_1 \mu + i - 1) y_2^{a_1+a_3-s} \cdot v^+) \\
& + \frac{1}{d_{a_3}^{(0)} r_{a_3}} (\rho_i d_{a_3}^{(i)} (j(k+n-2i-j+1) y_1^{j-1} y_2^{a_3} y_1^{\mu+i} \\
& + \rho_{i-1} d_{a_3}^{(i-1)} y_1^j y_2^{a_3} y_1^{\mu+i-1}) y_2^{a_1} \cdot v^+) \Bigg\}
\end{aligned}
$$

where $n = m_1 + a_1 - \mu$.

Fix $l,\ b_3,\ b_1 \in \mathbb{Z}_0^+$ with $b_3 + b_1 = l$. Define

$$X(b_3, b_1) \quad = \quad \operatorname{span}\{y_2^{b_3} y_1^i y_2^{b_1} \cdot v^+ \in \mathfrak{B} \mid i \in \mathbb{Z}_0^+\}$$

Observe that

$$\{y_2^{b_3} y_1^i y_2^{b_1} \cdot v^+ \mid i \in \mathbb{Z}_0^+ \text{ with } \mu \le i \le m_1 + b_1\}$$

is a basis of $X(b_3, b_1)$, where $\mu = \max\{b_3 - m_1, \lceil b_3/2 \rceil\}$.

Lemma 4.5. *Let* $l \in \mathbb{Z}_0^+$. *Suppose that there exist* $a_3,\ a_1 \in \mathbb{Z}_0^+$ *such that* $l = a_3 + a_1$ *and* $\{i \in \mathbb{Z}_0^+ \mid y_2^{a_3} y_1^i y_2^{a_1} \cdot v^+ \in \mathfrak{B}\} \neq \emptyset$. *Let* b_3 *be the smallest nonnegative integer such that* $\{i \in \mathbb{Z}_0^+ \mid y_2^{b_3} y_1^i y_2^{b_1} \cdot v^+ \in \mathfrak{B}\} \neq \emptyset$, *where* $b_1 = l - b_3$. *Then*

i) $x_1 \cdot (y_2^{b_3} y_1^{\mu} y_2^{b_1} \cdot v^+) = 0$, *where* $\mu = \max\{b_3 - m_1, \lceil b_3/2 \rceil\}$.

ii) $X(b_3, b_1)$ *is a* B_1*-module.*

Proof: i) We will prove this by contradiction. Suppose $x_1 \cdot (y_2^{b_3} y_1^{\mu} y_2^{b_1} \cdot v^+) \neq 0$. Then $\mu > 0$ and

$$x_1 \cdot (y_2^{b_3} y_1^{\mu} y_2^{b_1} \cdot v^+) = \mu(1 - \mu + m_1 + b_1) y_2^{b_3} y_1^{\mu - 1} y_2^{b_1} \cdot v^+$$

is nonzero. Hence $y_2^{b_3} y_1^{\mu-1} y_2^{b_1} \cdot v^+ \neq 0$. Then by I, we have that

$$\min\left\{m_1 + (\mu - 1), 2(\mu - 1)\right\} = b_3 - 1 \text{ or } b_3 - 2$$

and $b_1 \neq m_2$. By Lemma 4.3, $y_2^{b_3} y_1^{\mu-1} y_2^{b_1} \cdot v^+$ can be written as a linear combination of elements in the subset

$$\{y_2^{s} y_1^{\mu-1} y_2^{b_1 + b_3 - s} \cdot v^+ \mid s \in \mathbf{Z}_0^+ \text{ with } s_0 \leq s \leq b_3 - 1\}$$

of $\mathfrak{B}$, where

$$s_0 = \begin{cases} 0 & \text{if} \quad b_1 + b_3 - m_2 < 0, \\ b_1 + b_3 - m_2 & \text{if} \quad b_1 + b_3 - m_2 \leq 0. \end{cases}$$

Thus there exists $s \in \mathbf{Z}_0^+$ with $s < b_3$ such that $y_2^{s} y_1^{\mu-1} y_2^{b_1 + b_3 - s} \cdot v^+ \in \mathfrak{B}$. Since $s + (b_1 + b_3 - s) = l$, this contradicts the choice of b_3. Therefore $x_1 \cdot (y_2^{b_3} y_1^{\mu} y_2^{b_1} \cdot v^+) = 0$.

ii) It is clear that $X(b_3, b_1)$ is a B_1-module. #

Application :

V. Next, we will explain how to find the action of x_1 and y_1 on an arbitrary element of the Verma basis, $\mathfrak{B}$. We must go through the following procedure. Fix $a_3, a_1 \in \mathbf{Z}_0^+$ with $\{i \in \mathbf{Z}_0^+ \mid y_2^{a_3} y_1^{i} y_2^{a_1} \cdot v^+ \in \mathfrak{B}\} \neq \emptyset$. The basis elements with the given exponents a_3, a_1 are:

$$\{y_1^j y_2^{a_3} y_1^i y_2^{a_1} \cdot v^+ \mid i, j \in \mathbf{Z}_0^+ \text{ with}$$

$$0 \leq j \leq \min\{m_1, \lfloor a_3/2 \rfloor\},$$

$$\max\{a_3 - m_1, \lceil a_3/2 \rceil\} \leq i \leq m_1 + a_1\}$$

Let $l = a_3 + a_1$, b_3 be the smallest integer such that $\{i \in \mathbf{Z}_0^+ \mid y_2^{b_3} y_1^i y_2^{b_1} \cdot v^+ \in \mathfrak{B}\} \neq \emptyset$, where $b_3 + b_1 = l$.

First, look at $X(b_3, b_1)$. The basis elements with the given exponents b_3, b_1 are:

$$\{y_1^j y_2^{b_3} y_1^i y_2^{b_1} \cdot v^+ \mid i, j \in \mathbf{Z}_0^+ \text{ with}$$

$$0 \leq j \leq \min\{m_1, \lfloor b_3/2 \rfloor\},$$

$$\max\{b_3 - m_1, \lceil b_3/2 \rceil\} \leq i \leq m_1 + a_1\}$$

We can determine the actions of y_1 and x_1 on the basis elements with the given exponents b_3, b_1. by using II.

Next, look at $X(b_3 + 1, b_1 - 1)$. The basis elements with the given exponents $b_3 + 1, b_1 - 1$ are:

$$\{y_1^j y_2^{b_3+1} y_1^i y_2^{b_1-1} \cdot v^+ \mid i, j \in \mathbf{Z}_0^+ \text{ with}$$

$$0 \leq j \leq \min\{m_1, \lfloor (b_3+1)/2 \rfloor\},$$

$$\max\{b_3 + 1 - m_1, \lceil (b_3+1)/2 \rceil\} \leq i \leq m_1 + b_1 - 1\}$$

Let $k = \min\{m_1, \lfloor (b_3+1)/2 \rfloor\}$ and $\mu = \max\{b_3 + 1 - m_1, \lceil (b_3+1)/2 \rceil\}$.

Case : $x_1 \cdot (y_2^{b_3+1} y_1^{\mu} y_2^{b_1-1} \cdot v^+) = 0$. Then we can find the action of y_1 and x_1 on the basis elements with the given exponents $b_3 + 1, b_1 - 1$ by using II again.

Case : $x_1 \cdot (y_2^{b_3+1} y_1^{\mu} y_2^{b_1-1} \cdot v^+) \neq 0$.

Subcase : $\min\{m_1+(\mu-1), 2(\mu-1)\} = b_3$. Then we can find the action of x_1, y_1 on the basis elements with the given exponents b_3+1, b_1-1 by using III and our knowledge of the b_3, b_1 case.

Subcase : $\min\{m_1+(\mu-1), 2(\mu-1)\} = b_3 - 1$. Then we can find the action of x_1, y_1 on the basis elements with the given exponents b_3+1, b_1-1 by using IV and our knowledge of the b_3, b_1 case.

Continue this process with the pairs (b_3+i, b_1-i) , as i goes from 2 to $a_3 - b_3$.

Finally, we will describe the action of x_2 and y_2 on an arbitrary element of the Verma basis $\mathfrak{B}$. Fix $a_1, a_2, a_3, a_4 \in \mathbf{Z}_0^+$ with $0 \leq a_1 \leq m_2$, $0 \leq a_2 \leq m_1 + a_1$, $0 \leq a_3 \leq \min\{m_1+a_2, 2a_2\}$, $0 \leq a_4 \leq \min\{m_1, \lfloor a_3/2 \rfloor\}$.

We consider the action of x_2 first,

$$
\begin{aligned}
x_2 \cdot (y_1^{a_4} y_2^{a_3} y_1^{a_2} y_2^{a_1} \cdot v^+) &= \{[x_2, y_1^{a_4}] + y_1^{a_4} x_2\} y_2^{a_3} y_1^{a_2} y_2^{a_1} \cdot v^+ \\
&= y_1^{a_4} \{[x_2, y_2^{a_3}] + y_2^{a_3} x_2\} y_1^{a_2} y_2^{a_1} \cdot v^+ \\
&= y_1^{a_4} [x_2, y_2^{a_3}] y_2^{a_3} y_1^{a_2} y_2^{a_1} \cdot v^+ \\
&\qquad + y_1^{a_4} y_2^{a_3} ([x_2, y_1^{a_2}] + y_1^{a_2} x_2) y_2^{a_1} \cdot v^+ \\
&= y_1^{a_4} [x_2, y_2^{a_3}] y_1^{a_2} y_2^{a_1} \cdot v^+ \\
&\qquad + y_1^{a_4} y_2^{a_3} y_1^{a_2} ([x_2, y_2^{a_1}] + y_2^{a_1} x_2) \cdot v^+ \\
&= y_1^{a_4} [x_2, y_2^{a_3}] y_1^{a_2} y_2^{a_1} \cdot v^+ \\
&\qquad + y_1^{a_4} y_2^{a_3} y_1^{a_2} [x_2, y_2^{a_1}] \cdot v^+ \\
&= a_3(1 - a_3 + m_2 - 2a_1 + 2a_2) y_1^{a_4} y_2^{a_3-1} y_1^{a_2} y_2^{a_1} \cdot v^+ \\
&\qquad + a_1(1 - a_1 + m_2) y_1^{a_4} y_2^{a_3} y_1^{a_2} y_2^{a_1-1} \cdot v^+
\end{aligned}
$$

We must be able to express $y_1^{a_4} y_2^{a_3-1} y_1^{a_2} y_2^{a_1} \cdot v^+$ as a linear combination of elements of $\mathfrak{B}$ whenever $a_3 > 0$ and likewise for $y_1^{a_4}y_2^{a_3}y_1^{a_2}y_2^{a_1-1} \cdot v^+$ whenever $a_1 > 0$.

First, consider $y_1^{a_4}y_2^{a_3-1}y_1^{a_2}y_2^{a_1} \cdot v^+$.

Case : $a_4 \le \min\{m_1, \lfloor(a_3-1)/2\rfloor\}$. Then $y_1^{a_4}y_2^{a_3-1}y_1^{a_2}y_2^{a_1} \cdot v^+ \in \mathfrak{B}$.

Case : $a_4 > \min\{m_1, \lfloor(a_3-1)/2\rfloor\}$. Let $k = \min\{m_1, \lfloor(a_3-1)/2\rfloor\}$ Then

$$y_1^{a_4}y_2^{a_3-1}y_1^{a_2}y_2^{a_1} \cdot v^+ = y_1^{a_4-k}(y_1^k y_2^{a_3-1}y_1^{a_2}y_2^{a_1} \cdot v^+)$$

and we can express $y_1^{a_4}y_2^{a_3-1}y_1^{a_2}y_2^{a_1} \cdot v^+$ in terms of the elements of $\mathfrak{B}$ by using V.

Next, we consider $y_1^{a_4}y_2^{a_3}y_1^{a_2}y_2^{a_1-1} \cdot v^+$.

Case : $a_2 = m_1 + a_1$. Then $y_1^{a_4}y_2^{a_3}y_1^{a_2}y_2^{a_1-1} \cdot v^+ = 0$.

Case : $a_2 < m_1 + a_1$. Then $y_1^{a_4}y_2^{a_3}y_1^{a_2}y_2^{a_1-1} \cdot v^+ \in \mathfrak{B}$.

Finally, we look at $y_2 \cdot (y_1^{a_4}y_2^{a_3}y_1^{a_2}y_2^{a_1} \cdot v^+)$.

Case : $a_4 = 0$. Then $y_2 \cdot (y_1^{a_4}y_2^{a_3}y_1^{a_2}y_2^{a_1} \cdot v^+) = y_2 \cdot (y_2^{a_3}y_1^{a_2}y_2^{a_1} \cdot v^+)$ which can be expressed as a linear combination of elements of $\mathfrak{B}$ using Lemma 4.3.

Case : $a_4 \neq 0$. Then $a_3 \ge 2$. Since $\mathrm{ad}_{y_2}^3(y_1) \in L_{-(\alpha_1+3\alpha_2)} = \{0\}$ and

$\mathrm{ad}^2_{y_1}(y_2) \in L_{-(\alpha_2+2\alpha_1)} = \{0\}$ and $a_3 \geq 2$, by Lemma 2.9, we have that

$$
\begin{aligned}
y_2 \cdot (y_1^{a_4} y_2^{a_3} y_1^{a_2} y_2^{a_1} \cdot v^+) &= (y_2 y_1^{a_4} y_2^{a_3}) y_1^{a_2} y_2^{a_1} \cdot v^+ \\
&= \Big\{ \frac{a_3 + 1 - 2a_4}{a_3 + 1} y_1^{a_4} y_2^{a_3+1} \\
&\qquad - \frac{a_4(a_3 - 1)}{a_3 + 1} y_1^{a_4 - 1} y_2^{a_3+1} y_1 \\
&\qquad + a_4 y_1^{a_4-1} y_2^{a_3} y_1 y_2 \Big\} y_1^{a_2} y_2^{a_1} \cdot v^+ \\
&= \frac{a_3 + 1 - 2a_4}{a_3 + 1} y_1^{a_4} y_2^{a_3+1} y_1^{a_2} y_2^{a_1} \cdot v^+ \\
&\qquad - \frac{a_4(a_3 - 1)}{a_3 + 1} y_1^{a_4 - 1} y_2^{a_3+1} y_1^{a_2+1} y_2^{a_1} \cdot v^+ \\
&\qquad + a_4 y_1^{a_4-1} y_2^{a_3} y_1 y_2 y_1^{a_2} y_2^{a_1} \cdot v^+
\end{aligned}
$$

We start our discussion of the terms in this sum with $y_1^{a_4} y_2^{a_3+1} y_1^{a_2} y_2^{a_1} \cdot v^+$

Case : $a_3 < \min\{m_1 + a_2, 2a_2\}$. Then $y_1^{a_4} y_2^{a_3+1} y_1^{a_2} y_2^{a_1} \cdot v^+ \in \mathfrak{B}$.

Case : $a_3 = \min\{m_1 + a_2, 2a_2\}$. Then

$$y_1^{a_4} y_2^{a_3+1} y_1^{a_2} y_2^{a_1} \cdot v^+ = y_1^{a_4}\big(y_2 \cdot (y_2^{a_3} y_1^{a_2} y_2^{a_1} \cdot v^+)\big),$$

which can be expressed in terms of the elements of $\mathfrak{B}$ by using Lemma 4.3 and V.

Next comes $y_1^{a_4-1} y_2^{a_3+1} y_1^{a_2+1} y_2^{a_1} \cdot v^+$.

Case : $a_2 = m_1 + a_1$. Then by Lemma 4.1 ii), $y_1^{a_4-1} y_2^{a_3+1} y_1^{a_2+1} y_2^{a_1} \cdot v^+ = 0$.

Case : $a_2 < m_1 + a_1$.

Subcase : $a_3 < \min\{m_1 + a_2 + 1, 2(a_2 + 1)\}$. Then

$$y_1^{a_4-1} y_2^{a_3+1} y_1^{a_2+1} y_2^{a_1} \cdot v^+ \in \mathfrak{B}.$$

Subcase : $a_3 \geq \min\{m_1+a_2+1, 2(a_2+1)\}$. Let

$$k = \min\{m_1+a_2+1, 2(a_2+1)\}.$$

Then

$$y_1^{a_4-1}y_2^{a_3+1}y_1^{a_2+1}y_2^{a_1}\cdot v^+ = y_1^{a_4-1}\left(y_2^{a_3+1-k}(y_2^k y_1^{a_2+1}y_2^{a_1}\cdot v^+)\right),$$

which can be expressed in terms of the elements of $\mathfrak{B}$ by using Lemma 4.3 and V.

Last comes $y_1^{a_4-1}y_2^{a_3}y_1y_2y_1^{a_2}y_2^{a_1}\cdot v^+$.

First, we look at $y_2y_1^{a_2}y_2^{a_1}\cdot v^+$. Since $\min\{m_1+a_2, 2a_2\} \geq 2$, $y_2y_1^{a_2}y_2^{a_1}\cdot v^+ \in \mathfrak{B}$.

Next, we look at $y_1y_2y_1^{a_2}y_2^{a_1}\cdot v^+$. Since $\min\{m_1, \lfloor 1/2 \rfloor\} = 0$, $y_1y_2y_1^{a_2}y_2^{a_1}\cdot v^+ \notin \mathfrak{B}$.

Observe that

$$\max\{1-m_1, \lceil 1/2 \rceil\} = 1$$

and

$$\min\left\{m_1 + (\max\{1-m_1, \lceil 1/2 \rceil\} - 1), 2\big(\max\{1-m_1, \lceil 1/2 \rceil\} - 1\big)\right\} = 0.$$

Then

$$x_1\cdot(y_2y_1y_2^{a_1}\cdot v^+) = \begin{cases} 0 & \text{if } a_1 = m_2, \\ (m_1+a_1)y_2^{a_1+1}\cdot v^+ \in \mathfrak{B} & \text{if } a_1 < m_2. \end{cases}$$

Case : $a_1 = m_2$. Then by using II,

$$y_1\cdot(y_2y_1^{a_2}y_2^{a_1}\cdot v^+) = \begin{cases} 0 & \text{if } a_2 = m_1+a_1, \\ \dfrac{a_2}{a_2+1}y_2y_1^{a_2+1}y_2^{a_1}\cdot v^+ \in \mathfrak{B} & \text{if } a_2 < m_1+a_1. \end{cases}$$

Case : $a_1 < m_2$. Then by using III,

$$y_1 \cdot (y_2 y_1^{a_2} y_2^{a_1} \cdot v^+) = \begin{cases} \dfrac{d_1^{(0)} r_1 q_0}{d_0^{(0)} r_0} y_1^{a_2+1} y_2^{a_1+1} \cdot v^+ \in \mathfrak{B} & \text{if } \; a_2 = m_1 + a_1, \\ \dfrac{d_1^{(0)} r_1 q_0}{d_1^{(a_2-1)}} \left\{ \dfrac{q_0 d_0^{(a_2-1)}}{d_0^{(0)} r_0} y_1^{a_2+1} y_2^{a_1+1} \cdot v^+ \right. & \\ + \varepsilon_0^{(0)} \left(\dfrac{d_1^{(a_2)}}{d_1^{(0)} r_1} y_2 y_1^{a_2+1} y_2^{a_1} \cdot v^+ \right. & \\ \left. \left. - \frac{q_0 d_0^{(a_2)}}{d_0^{(0)} r_0} y_1^{a_2+1} y_2^{a_1+1} \cdot v^+ \right) \right\} & \text{if } \; a_2 < m_1 + a_1. \end{cases}$$

Observe that $y_1^{a_2+1} y_2^{a_1+1} \cdot v^+$ and $y_2 y_1^{a_2+1} y_2^{a_1} \cdot v^+$ are elements of $\mathfrak{B}$. Then $y_2^{a_3} y_1 y_2 y_1^{a_2} y_2^{a_1} \cdot v^+$ can be expressed in terms of the elements of $\mathfrak{B}$ by using Lemma 4.3 and $y_1^{a_4-1} y_2^{a_3} y_1 y_2 y_1^{a_2} y_2^{a_1} \cdot v^+$ can be expressed in terms of the elements of $\mathfrak{B}$ by using V.