CHAPTER III

$\mathfrak{sl}(3, \mathbf{C})$

Let V be a finite-dimensional irreducible $\mathfrak{sl}(3, \mathbf{C})$-module, v^+ a maximal vector of V with highest weight λ. Suppose that $\lambda(h_1) = m_1$ and $\lambda(h_2) = m_2$. Then a Verma basis of V consists of all elements of the form

$$y_1^{a_3} y_2^{a_2} y_1^{a_1} \cdot v^+$$

where $a_1, a_2, a_3 \in \mathbf{Z}_0^+$ and

$$0 \leq a_1 \leq m_1$$

$$0 \leq a_2 \leq m_2 + a_1$$

$$0 \leq a_3 \leq \min\{a_2, m_2\}$$

We want to be able to calculate the action of an arbitrary element of $\mathfrak{sl}(3, \mathbf{C})$ on an arbitrary element of V; for this it is sufficient to know the action of the elements of a set of generators of $\mathfrak{sl}(3, \mathbf{C})$ on the elements of a Verma basis of V. The purpose of this chapter is to find formulas for the action of the Chevalley generators $\{x_1, x_2, y_1, y_2\}$ on the elements of the above Verma basis.

The following notations are used in this chapter. Let $\mathfrak{B}$ denote the above Verma basis. For each $i \in \{1, 2\}$, let

$$S_i = \text{span}\{x_i, h_i, y_i\} \quad \text{and} \quad B_i = \text{span}\{x_i, h_i\},$$

where $h_i = [x_i, y_i]$. Observe that for each $i \in \{1, 2\}$, S_i is isomorphic to $\mathfrak{sl}(2, \mathbf{C})$, while B_i is a two-dimensional nonabelian subalgebra of S_i. For each $a_2 \in \mathbf{Z}_0^+$, let $\mu = \max\{0, a_2 - m_2\}$, $k = a_2 - \mu$, and $n = m_1 - \mu$.

Lemma 3.1.

i) $X = \text{span}\{y_1^i \cdot v^+ \mid i \in \mathbf{Z}_0^+\}$ *is an irreducible* S_1*-module,*

$$\{y_1^i \cdot v^+ \mid i \in \mathbf{Z}_0^+ \text{ with } 0 \leq i \leq m_1\}$$

is a basis of X*, and* $y_1^i \cdot v^+ = 0$ *for all* $i \in \mathbf{Z}_0^+$ *with* $i > m_1$.

ii) *For each* $t \in \mathbf{Z}_0^+$ *with* $0 \leq t \leq m_1$,

$$Y_t = \text{span}\{y_2^i y_1^t \cdot v^+ \mid i \in \mathbf{Z}_0^+\}$$

is an irreducible S_2*-module,*

$$\{y_2^i y_1^t \cdot v^+ \mid i \in \mathbf{Z}_0^+ \text{ with } 0 \leq i \leq m_2 + t\}$$

is a basis of Y_t*, and* $y_2^i y_1^t \cdot v^+ = 0$ *for all* $i \in \mathbf{Z}_0^+$ *with* $i > m_2 + t$.

Proof: i) Since v^+ is a maximal vector for S_1, this follows from the theory of $\mathfrak{sl}(2, \mathbf{C})$-modules.

ii) Since $y_1^t \cdot v^+$ is a maximal vector of S_2, this follows from the theory of $\mathfrak{sl}(2, \mathbf{C})$-modules again. #

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Lemma 3.2. *For each* $a_2 \in \mathbf{Z}_0^+$*, let*

$$X = \text{span}\{y_2^{a_2} y_1^i \cdot v^+ \mid i \in \mathbf{Z}_0^+ \text{ with } \mu \leq i \leq m_1\}, \quad \text{and let}$$

$$Y = \text{span}\{y_1^j y_2^{a_2} y_1^i \cdot v^+ \mid i, j \in \mathbf{Z}_0^+ \text{ with } 0 \leq j \leq k,\ \mu \leq i \leq m_1\}.$$

Also, for each $j \in \mathbf{Z}_0^+$ *with* $0 \leq j \leq m_1$*, let*

$$C_j = \frac{m_1!}{j!}(m_1 - j)!.$$

i) *X is a B_1-module, and in fact X is the string module $\mathcal{S}(k+n,\ k-n)$, with standard basis*

$$\{C_{\mu+i}y_2^{a_2}y_1^{\mu+i}\cdot v^+ \mid i \in \mathbf{Z}_0^+ \text{ with } 0 \le i \le m_1-\mu\}.$$

ii) *Y is a B_1-module and*

$$\{C_{\mu+i}y_1^j y_2^{a_2}y_1^{\mu+i}\cdot v^+ \mid i,j \in \mathbf{Z}_0^+ \text{ with } 0 \le j \le k,\ 0 \le i \le m_1-\mu\}$$

is a basis of Y.

Proof: Fix $a_2 \in \mathbf{Z}_0^+$.

i) We will show first that X is a B_1-module. This is obvious if $X = \{0\}$. Suppose $X \neq \{0\}$; it suffices to show that for any $i \in \mathbf{Z}_0^+$ with $\mu \le i \le m_1$, $x_1\cdot(y_2^{a_2}y_1^i\cdot v^+)$, $h_1\cdot(y_2^{a_2}y_1^i\cdot v^+) \in X$. The fact that $y_2^{a_2}y_1^i\cdot v^+$ is a weight vector makes it clear that $h_1\cdot(y_2^{a_2}y_1^i\cdot v^+) \in X$. On the other hand,

$$x_1\cdot(y_2^{a_2}y_1^i\cdot v^+) = \begin{cases} 0 & \text{if } i=0\,, \\ i(m_1-i+1)y_2^{a_2}y_1^{i-1}\cdot v^+ & \text{if } i>0\,. \end{cases}$$

We consider $y_2^{a_2}y_1^{i-1}\cdot v^+$ in the case $i>0$.

Case 1. $\mu < i \le m_1$. Then by Lemma 3.1(ii), $y_2^{a_2}y_1^{i-1}\cdot v^+ = 0$ if $a_2 \ge m_2+i$ and it is an element of X if $a_2 < m_2+i$.

Case 2. $i=\mu$. Then $a_2 = m_2+i > m_2+i-1$. Thus by Lemma 3.1 ii), $y_2^{a_2}y_1^{i-1}\cdot v^+ = 0$.

Hence $x_1\cdot(y_2^{a_2}y_1^i\cdot v^+) \in X$, and X is a B_1-module.

Next, we will show that

$$Z_1 = \{C_{\mu+i}y_2^{a_2}y_1^{\mu+i}\cdot v^+ \mid i \in \mathbf{Z}_0^+ \text{ with } 0 \le i \le m_1-\mu\}$$

is a standard basis of X. This will also tell us that X is a string module. The elements $y_2^{a_2}y_1^{\mu+i}\cdot v^+$, $i\in\mathbf{Z}_0^+$ with $0\le i\le m_1-\mu$, are distinct elements of $\mathfrak{B}$, so they form a basis of X. Thus Z_1 is a basis of X as well, because the coefficients $C_{\mu+i}$ are all nonzero.

Next, we will look at the action of x_1 and h_1 on the elements of Z_1. Fix $i\in\mathbf{Z}_0^+$ with $0\le i\le m_1-\mu$. Then

$$\begin{aligned} h_1\cdot(C_{\mu+i}y_2^{a_2}y_1^{\mu+i}\cdot v^+) &= C_{\mu+i}(\lambda-(\mu+i)\alpha_1-a_2\alpha_2)(h_1)y_2^{a_2}y_1^{\mu+i}\cdot v^+\\ &= C_{\mu+i}(m_1-2\mu+a_2-2i)y_2^{a_2}y_1^{\mu+i}\cdot v^+\\ &= C_{\mu+i}(k+n-2i)y_2^{a_2}y_1^{\mu+i}\cdot v^+\end{aligned}$$

and

$$x_1\cdot(C_{\mu+i}y_2^{a_2}y_1^{\mu+i}\cdot v^+)=\begin{cases}0 & \text{if } i=0\,,\\ C_{\mu+i-1}y_2^{a_2}y_1^{\mu+i-1}\cdot v^+ & \text{if } i>0\,.\end{cases}$$

Therefore we have Z_1 is a standard basis of X. Observe that

$$h_1\cdot(C_\mu y_2^{a_2}y_1^{\mu}\cdot v^+)=(m_1-2\mu+a_2)C_\mu y_2^{a_2}y_1^{\mu}\cdot v^+$$

while

$$h_1\cdot(C_{m_1}y_2^{a_2}y_1^{m_1}\cdot v^+)=(a_2-m_1)C_{m_1}y_2^{a_2}y_1^{m_1}\cdot v^+$$

Thus $X=\mathcal{S}(m_1-2\mu+a_2,a_2-m_1)=\mathcal{S}(k+n,k-n)$.

ii) First, we will show that

$$Z_2=\{C_{\mu+i}y_1^{j}y_2^{a_2}y_1^{\mu+i}\cdot v^+\mid i,j\in\mathbf{Z}_0^+ \text{ with } 0\le j\le k,\ 0\le i\le m_1-\mu\}$$

is a basis of Y. It suffices to show that $k=\min\{a_2,m_2\}$, for then the elements of Z_2 will be nonzero scalar multiples of elements of the Verma basis $\mathfrak{B}$. Indeed we have

$$\begin{aligned}k &= a_2 - \mu \\ &= a_2 - \max\{0, a_2 - m_2\} \\ &= a_2 + \min\{0, m_2 - a_2\} \\ &= \min\{a_2, m_2\}\end{aligned}$$

Thus Z_2 is a basis of Y.

Finally, we will show that Y is a B_1-module. This is obvious if $Y = \{0\}$. Suppose $Y \neq \{0\}$; it suffices to show that for any $i, j \in \mathbf{Z}_0^+$ with $0 \leq j \leq k$, $0 \leq i \leq m_1 - \mu$, $x_1 \cdot (C_{\mu+i} y_1^j y_2^{a_2} y_1^{\mu+i} \cdot v^+)$, $h_1 \cdot (C_{\mu+i} y_1^j y_2^{a_2} y_1^{\mu+i} \cdot v^+) \in Y$. As before, $h_1 \cdot (C_{\mu+i} y_1^j y_2^{a_2} y_1^{\mu+i} \cdot v^+) \in Y$, since $y_1^j y_2^{a_2} y_1^{\mu+i} \cdot v^+$ is a weight vector. Also,

$$\begin{aligned}x_1 \cdot (C_{\mu+i} y_1^j y_2^{a_2} y_1^{\mu+i} \cdot v^+) &= C_{\mu+i}\{[x_1, y_1^j] + y_1^j x_1\} y_2^{a_2} y_1^{\mu+i} \cdot v^+ \\ &= C_{\mu+i}\{([x_1, y_1^j] y_2^{a_2} + y_1^j([x_1, y_2^{a_2}] + y_2^{a_2} x_1)) y_1^{\mu+i}\} \cdot v^+ \\ &= C_{\mu+i}\{[x_1, y_1^j] y_2^{a_2} y_1^{\mu+i} + y_1^j y_2^{a_2}([x_1, y_1^{\mu+i}] + y_1^{\mu+i} x_1)\} \cdot v^+ \\ &= j(k + n - 2i - j + 1) C_{\mu+i} y_1^{j-1} y_2^{a_2} y_1^{\mu+i} \cdot v^+ \\ &\quad + C_{\mu+i-1} y_1^j y_2^{a_2} y_1^{\mu+i-1} \cdot v^+\end{aligned}$$

Thus $x_1 \cdot (C_{\mu+i} y_1^j y_2^{a_2} y_1^{\mu+i} \cdot v^+) \in Y$ and Y is a B_1-module. #

Lemma 3.3. *Define Y as in Lemma 3.2. Then an action of y_1 can be defined on Y which makes Y in to an S_1-module.*

Proof: By the proof of Lemma 3.2 ii), we have Y is a B_1-module such that for any i, $j \in \mathbf{Z}_0^+$ with $0 \leq j \leq k$, $0 \leq i \leq m_1 - \mu$,

1) $h_1 \cdot (C_{\mu+i} y_1^j y_2^{a_2} y_1^{\mu+i} \cdot v^+) = (k + n - 2i - 2j) C_{\mu+i} y_1^j y_2^{a_2} y_1^{\mu+i} \cdot v^+$

2) $x_1 \cdot (C_{\mu+i}y_1^j y_2^{a_2} y_1^{\mu+i} \cdot v^+) = j(k+n-2i-j+1)C_{\mu+i}y_1^{j-1}y_2^{a_2}y_1^{\mu+i} \cdot v^+$

$$+ C_{\mu+i-1}y_1^j y_2^{a_2} y_1^{\mu+i-1} \cdot v^+$$

Then by Lemma 1.7, Y can be made into an S_1-module by keeping the same action for x_1 and h_1 and defining an action of y_1 by the following: for any $i, j \in \mathbf{Z}_0^+$ with $0 \le j \le k$, $0 \le i \le m_1 - \mu$,

3) If $0 \le j < k$, then

$$y_1 \cdot (C_{\mu+i}y_1^j y_2^{a_2} y_1^{\mu+i} \cdot v^+) = C_{\mu+i}y_1^{j+1}y_2^{a_2}y_1^{\mu+i} \cdot v^+$$

4) If $0 \le i < m_1 - \mu$, then

$$y_1 \cdot (C_{\mu+i}y_1^k y_2^{a_2} y_1^{\mu+i} \cdot v^+) = \sum_{p=1}^{n-i} \left\{ (-1)^{p-1} \binom{m_1 - \mu - i}{p} \prod_{r=0}^{p-1} (k+1-r)(i+r+1) \right\} C_{\mu+i+p}y_1^{k+1-p}y_2^{a_2}y_1^{\mu+i+p} \cdot v^+$$

5) $y_1 \cdot (C_{m_1} y_1^k y_2^{a_2} y_1^{m_1} \cdot v^+) = 0.$

Thus we have Lemma 3.3. #

Application :

Note that from Lemmas 3.1, 3.2, 3.3, Theorem 2.5, and Lemma 1.6 we have that for any $a_2 \in \mathbf{Z}_0^+$, and for any $i, j \in \mathbf{Z}_0^+$ with $0 \le j \le k$, $0 \le i \le m_1 - \mu$,

1) $h_1 \cdot (y_1^j y_2^{a_2} y_1^{\mu+i} \cdot v^+) = (m_1 + a_2 - 2\mu - 2j - 2i) y_1^j y_2^{a_2} y_1^{\mu+i} \cdot v^+$

2) $x_1 \cdot (y_1^j y_2^{a_2} y_1^{\mu+i} \cdot v^+) = (\mu+i)(m_1 - \mu - i + 1) y_1^j y_2^{a_2} y_1^{\mu+i-1} \cdot v^+$

$$+ j(m_1 - 2\mu - 2 - j + a_2 + 1) y_1^{j-1} y_2^{a_2} y_1^{\mu+i} \cdot v^+$$

3) If $0 \le j < k$, then

$$y_1 \cdot (y_1^j y_2^{a_2} y_1^{\mu+i} \cdot v^+) = y_1^{j+1} y_2^{a_2} y_1^{\mu+i} \cdot v^+$$

4) If $0 \leq i \leq m_1 - \mu$, then

$$y_1 \cdot (y_1^k y_2^{a_2} y_1^{\mu+i} \cdot v^+) = (\mu+i)! \sum_{s=k+1-m_1+\mu+i}^{k} (-1)^{k-s} \frac{\prod_{r=0}^{k-s}(k+1-r)(i+r+1)}{(k+1-s)!(\mu+i+k+1-s)!} y_1^s y_2^{a_2} y_1^{\mu+i+k+1-s} \cdot v^+$$

5) $y_1 \cdot (y_1^k y_2^{a_2} y_1^{m_1} \cdot v^+) = 0$

Then we have the following formulas for the actions of h_1, x_1, y_1, h_2, x_2 and y_2 on an element $y_1^{a_3} y_2^{a_2} y_1^{a_1} \cdot v^+$ of the Verma basis. Let $a_1, a_2, a_3 \in \mathbb{Z}_0^+$ with $0 \leq a_1 \leq m_1$, $0 \leq a_2 \leq m_2 + a_1$, and $0 \leq a_3 \leq \min\{a_2, m_2\}$.

1) The formula

$$h_1 \cdot (y_1^{a_3} y_2^{a_2} y_1^{a_1} \cdot v^+) = (m_1 + 2a_1 + a_2 - 2a_3) y_1^{a_3} y_2^{a_2} y_1^{a_1} \cdot v^+$$

is clear.

2) By Lemmas 3.1 and 3.2,

$$x_1 \cdot (y_1^{a_3} y_2^{a_2} y_1^{a_1} \cdot v^+)$$

$$= \begin{cases} 0 & \text{if } a_1 = 0 = a_3\,, \\ a_1(m_1 - a_1 + 1) y_2^{a_2} y_1^{a_1 - 1} \cdot v^+ & \text{if } a_1 > 0 \text{ and } a_3 = 0\,, \\ a_3(m_1 - 2a_1 - a_3 + a_2 + 1) y_1^{a_3 - 1} y_2^{a_2} \cdot v^+ & \text{if } a_1 = 0 \text{ and } a_3 > 0\,, \\ a_1(m_1 - a_1 + 1) y_1^{a_3} y_2^{a_2} y_1^{a_1 - 1} \cdot v^+ & \\ \quad + a_3(m_1 - 2a_1 - a_3 + a_2 + 1) y_1^{a_3 - 1} y_2^{a_2} y_1^{a_1} \cdot v^+ & \text{if } a_1 > 0 \text{ and } a_3 > 0\,. \end{cases}$$

Observe that if $a_3 > 0$, then $0 \leq a_3 - 1 \leq \min\{a_2, m_2\}$, so $y_1^{a_3-1} y_2^{a_2} y_1^{a_1} \cdot v^+$ is an element of $\mathfrak{B}$ in this case. However, $y_1^{a_3} y_2^{a_2} y_1^{a_1 - 1} \cdot v^+$ may not be an element of $\mathfrak{B}$.

We consider $y_1^{a_3} y_2^{a_2} y_1^{a_1 - 1} \cdot v^+$ in the case $a_1 > 0$.

If $a_2 = m_2 + a_1$, then $y_1^{a_3} y_2^{a_2} y_1^{a_1 - 1} \cdot v^+ = 0$.

If $a_2 < m_2 + a_1$, then $y_1^{a_3} y_2^{a_2} y_1^{a_1 - 1} \cdot v^+ \in \mathfrak{B}$.

3) If $0 \leq a_3 < \min\{m_2, a_2\}$, then

$$y_1 \cdot (y_1^{a_3} y_2^{a_2} y_1^{a_1} \cdot v^+) = y_1^{a_3+1} y_2^{a_2} y_1^{a_1} \cdot v^+$$

and this is an element of $\mathfrak{B}$.

4) If $0 \leq a_1 < m_1$ and $a_3 = \min\{m_2, a_2\}$, then

$$y_1 \cdot (y_1^{a_3} y_2^{a_2} y_1^{a_1} \cdot v^+) = a_1! \sum_{s=a_3-m_1+a_1+1}^{a_3} (-1)^{a_3-s} \frac{\prod_{r=0}^{a_3-s}(a_3+1-r)(a_1-\mu+r+1)}{(a_3+1-s)!(a_1+a_3+1-s)!}$$
$$y_1^s y_2^{a_2} y_1^{a_1+a_3+1-s} \cdot v^+$$

Note that

a) $s \leq a_3$

b) $1 \leq a_3 + 1 - s \leq m_1 - a_1$

c) $a_1 + 1 \leq a_1 + a_3 + 1 - s \leq m_1$

d) $0 < a_1 - \mu + r + 1$

e) $\prod_{r=0}^{a_3-s}(a_3 + 1 - r) = 0$ iff $s \leq -1$.

Let

$$s_0 = \begin{cases} 0 & \text{if } a_3 - m_1 + a_1 + 1 < 0 \text{ ,} \\ a_3 - m_1 + a_1 + 1 & \text{if } a_3 - m_1 + a_1 + 1 \geq 0 \text{ .} \end{cases}$$

Then

$$y_1 \cdot (y_1^{a_3} y_2^{a_2} y_1^{a_1} \cdot v^+) = a_1! \sum_{s=s_0}^{a_3} (-1)^{a_3-s} \frac{\prod_{r=0}^{a_3-s}(a_3+1-r)(a_1-\mu+r+1)}{(a_3+1-s)!(a_1+a_3+1-s)!}$$
$$y_1^s y_2^{a_2} y_1^{a_1+a_3+1-s} \cdot v^+$$

We consider $y_1^s y_2^{a_2} y_1^{a_1+a_3+1-s} \cdot v^+$ for each $s \in \mathbb{Z}_0^+$ with $s_0 \leq s \leq a_3$. Since $a_1+1 \leq a_1+a_3+1-s \leq m_1$, $0 \leq a_2 \leq m_2+a_1+a_3+1-s$, and $0 \leq s_0 \leq s \leq a_3$, $y_1^s y_2^{a_2} y_1^{a_1+a_3+1-s} \cdot v^+ \in \mathfrak{B}$.

5) If $a_1 = m_1$ and $a_3 = \min\{m_2, a_2\}$, then

$$y_1 \cdot (y_1^{a_3} y_2^{a_2} y_1^{a_1} \cdot v^+) = 0$$

6) Clearly,

$$h_2 \cdot (y_1^{a_3} y_2^{a_2} y_1^{a_1} \cdot v^+) = (m_2 + a_1 - 2a_2 + a_3) y_1^{a_3} y_2^{a_2} y_1^{a_1} \cdot v^+$$

7) A simple calculation shows

$$x_2 \cdot (y_1^{a_3} y_2^{a_2} y_1^{a_1} \cdot v^+) = \begin{cases} 0 & \text{if } a_2 = 0 \text{ ,} \\ a_2(m_2 + a_1 - a_2 + 1) y_1^{a_3} y_2^{a_2-1} y_1^{a_1} \cdot v^+ & \text{if } a_2 > 0 \text{ .} \end{cases}$$

We must consider $y_1^{a_3} y_2^{a_2-1} y_1^{a_1} \cdot v^+$ in the case $a_2 > 0$, since it may not be an element of $\mathfrak{B}$.

Case 1. $a_3 = 0$. Then $y_1^{a_3} y_2^{a_2-1} y_1^{a_1} \cdot v^+ = y_2^{a_2-1} y_1^{a_1} \cdot v^+ \in \mathfrak{B}$.

Case 2. $a_3 \neq 0$.

Subcase 2.1. $a_3 \leq \min\{a_2 - 1, m_2\}$. Then $y_1^{a_3} y_2^{a_2-1} y_1^{a_1} \cdot v^+ \in \mathfrak{B}$.

Subcase 2.2. $a_3 > \min\{a_2 - 1, m_2\}$. Then

$$y_1^{a_3} y_2^{a_2-1} y_1^{a_1} \cdot v^+ = y_1 \cdot (y_1^{a_3-1} y_2^{a_2-1} y_1^{a_1} \cdot v^+)$$

and $a_3 - 1 = \min\{a_2 - 1, m_2\}$, so we can express $y_1^{a_3} y_2^{a_2-1} y_1^{a_1} \cdot v^+$ as a linear combination of elements of $\mathfrak{B}$ by formula 4).

8) By Lemma 2.6,

$$y_2 \cdot (y_1^{a_3} y_2^{a_2} y_1^{a_1} \cdot v^+) = \frac{a_2 + 1 - a_3}{a_2 + 1} y_1^{a_3} y_2^{a_2+1} y_1^{a_1} \cdot v^+ + \frac{a_3}{a_2 + 1} y_1^{a_3-1} y_2^{a_2+1} y_1^{a_1+1} \cdot v^+$$

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Again, we must consider whether this expression shows us how to express $y_2 \cdot (y_1^{a_3} y_2^{a_2} y_1^{a_1} \cdot v^+)$ as a linear combination of elements of $\mathfrak{B}$. First, we consider $y_1^{a_3} y_2^{a_2+1} y_1^{a_1} \cdot v^+$.

Case 1. $a_2 = m_2 + a_1$. Then $a_2 + 1 > m_2 + a_1$. Thus $y_1^{a_3} y_2^{a_2+1} y_1^{a_1} \cdot v^+ = 0$.

Case 2. $a_2 < m_2 + a_1$. Then $y_1^{a_3} y_2^{a_2+1} y_1^{a_1} \cdot v^+ \in \mathfrak{B}$.

Finally, we consider $y_1^{a_3-1} y_2^{a_2+1} y_1^{a_1+1} \cdot v^+$.

If $a_3 = 0$ or $a_1 = m_1$, then it is zero.

If $a_3 > 0$ and $a_1 < m_1$ then $0 \leq a_1 + 1 \leq m_1$, $0 \leq a_2 + 1 \leq m_2 + (a_1 + 1)$, and $0 \leq a_3 - 1 \leq \min\{m_2, a_2 + 1\}$, so $y_1^{a_3-1} y_2^{a_2+1} y_1^{a_1+1} \cdot v^+ \in \mathfrak{B}$.

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