CHAPTER II

BACKGROUND

This chapter presents the background material needed for the rest of the thesis. The main theorems and lemmas in this chapter are Theorem 2.5, and Lemmas 2.6 and 2.9. We will use them in an important way in Chapters III and IV.

Lemma 2.1. *Let V be a finite-dimensional L-module, where L is a finite-dimensional semisimple Lie algebra and suppose $V = \oplus\sum_{i=1}^{n} I_i$, where $I_1, \ldots, I_n$ are irreducible L-submodules of V. Then for each weight μ of V, the weight space $V_\mu = \sum_{i=1}^{n}(I_i)_\mu$. In particular, $V_\mu = \oplus\sum_{i=1}^{n}(I_i)_\mu$.*

Proof: Let μ be a weight of V. First, we will show that $V_\mu \subseteq \sum_{i=1}^{n}(I_i)_\mu$. Let $z \in V_\mu$. Then $z = \sum_{i=1}^{n} z_i$, where $z_i \in I_i$ for all $i \in \bar{n}$. Fix $t \in H$, where H is a maximal toral subalgebra of L. Then

$$\begin{aligned}\sum_{i=1}^{n} \mu(t)z_i &= \mu(t)z \\ &= t \cdot z \\ &= \sum_{i=1}^{n} t \cdot z_i\end{aligned}$$

Thus $\sum_{i=1}^{n}(t \cdot z_i - \mu(t)z_i) = 0$. But for each $i \in \bar{n}$, $t \cdot z_i - \mu(t)z_i \in I_i$ and $V = \oplus\sum_{i=1}^{n} I_i$, hence $t \cdot z_i = \mu(t)z_i$. Therefore $z_i \in (I_i)_\mu$ for all $i \in \bar{n}$. Thus $z \in \sum_{i=1}^{n}(I_i)_\mu$. Hence $V_\mu \subseteq \sum_{i=1}^{n}(I_i)_\mu$.

We observe that for each $i \in \bar{n}$, $(I_i)_\mu \subseteq V_\mu$ and V_μ is a vector space, hence $\sum_{i=1}^{n}(I_i)_\mu \subseteq V_\mu$. Therefore $V_\mu = \oplus\sum_{i=1}^{n}(I_i)_\mu$. #

Lemma 2.2. *Let V be an $\mathfrak{sl}(2, \mathbf{C})$-module with a maximal vector v^+ of weight m, and let $k \in \mathbf{Z}_0^+$ with $k \le m$. Then for any $t \in \mathbf{Z}_0^+$ with $0 \le t \le k$,*

$$x^t \cdot (y^k \cdot v^+) = \frac{k!(m+t-k)!}{(k-t)!(m-k)!} y^{k-t} \cdot v^+ \qquad (1)$$

Proof: We will prove (1) by induction on t. Clearly it is true for $t = 0$. Suppose (1) is true for $t \in \mathbf{Z}_0^+$ with $0 \le t < k$. Then

$$\begin{aligned}
x^{t+1} \cdot (y^k \cdot v^+) &= x \cdot (x^t \cdot (y^k \cdot v^+)) \\
&= \frac{k!(m+t-k)!}{(k-t)!(m-k)!} x \cdot (y^{k-t} \cdot v^+) \\
&= \frac{k!(m+t-k)!}{(k-t)!(m-k)!} \left\{ [x, y^{k-t}] + y^{k-t}x \right\} \cdot v^+ \\
&= \frac{k!(m+t-k)!}{(k-t)!(m-k)!} (k-t) \left\{ (1-(k-t)) y^{(k-t)-1} + y^{(k-t)-1} h \right\} \cdot v^+ \\
&= \frac{k!(m+(t+1)-k)!}{(k-(t+1))!(m-k)!} y^{k-(t+1)} \cdot v^+
\end{aligned}$$

Thus we have Lemma 2.2. #

Our next goal is Theorem 2.5, which basically says that in an $\mathfrak{sl}(2, \mathbf{C})$-module, the action of y is completely determined by the actions of x and h. Lemmas 2.3 and 2.4 are used in the proof of Theorem 2.5.

Lemma 2.3. *Let V be a finite-dimensional vector space over* $\mathbf{C}$ *and suppose $\varphi_1, \varphi_2 : \mathfrak{sl}(2, \mathbf{C}) \to \mathfrak{gl}(V)$ are two representations such that $\varphi_1|_B = \varphi_2|_B$, where*

$B = \text{span}\{x, h\}$. *Let*

$$H = \varphi_1(h),\ X = \varphi_1(x),\ Y_1 = \varphi_1(y) \text{ and } Y_2 = \varphi_2(y).$$

Using the module structure on V corresponding to φ_1, write V as a direct sum of irreducible submodules, $V = \oplus\sum_{i=1}^{n} I_i$, and for each $i \in \bar{n}$, let v_i^+ be a maximal vector of I_i, with highest weight m_i. Then for any $i \in \bar{n}$ and any $t \in \mathbf{Z}_0^+$ with $1 \leq t \leq m_i$,

$$X^t\big(Y_2\big(Y_1^{m_i}(v_i^+)\big)\big) = \frac{m_i!t!}{(m_i - t)!}\big\{ - Y_1^{m_i - t + 1}(v_i^+) + Y_2\big(Y_1^{m_i - t}(v_i^+)\big)\big\} \qquad (2)$$

Proof: Note that $H = \varphi_2(h)$ and $X = \varphi_2(x)$ as well. Fix $i \in \bar{n}$. We will prove equation (2) by induction on t.

Basis: $t = 1$. We have

$$\begin{aligned} X\big(Y_2\big(Y_1^{m_i}(v_i^+)\big)\big) &= \big\{[X, Y_2] + Y_2 X\big\} Y_1^{m_i}(v_i^+) \\ &= \{HY_1^{m_i} + m_i Y_2 Y_1^{m_i - 1}\}(v_i^+) \\ &= \frac{m_i!}{(m_i - 1)!}\{-Y_1^{m_i} + Y_2 Y_1^{m_i - 1}\}(v_i^+) \end{aligned}$$

Thus, it is true in the case $t = 1$.

Induction: Suppose (2) is true for $t \in \mathbf{Z}^+$ with $1 \leq t < m_i$. Then

$$\begin{aligned} X^{t+1}\big(Y_2\big(Y_1^{m_i}(v_i^+)\big)\big) &= X\big(X^t\big(Y_2\big(Y_1^{m_i}(v_i^+)\big)\big)\big) \\ &= \frac{m_i!t!}{(m_i - t)!} X\big(\{-Y_1^{m_i - t + 1} + Y_2 Y_1^{m_i - t}\}(v_i^+)\big) \\ &= \frac{m_i!t!}{(m_i - t)!}(m_i - t)(t + 1)\{-Y_1^{m_i - t} + Y_2 Y_1^{m_i - (t+1)}\}(v_i^+) \\ &= \frac{m_i!(t + 1)!}{(m_i - (t + 1))!}\{-Y_1^{m_i - t} + Y_2 Y_1^{m_i - (t+1)}\}(v_i^+) \end{aligned}$$

Thus, it is true in the case $t + 1$. Therefore, we have Lemma 2.3. #

Lemma 2.4. *Let V be a finite-dimensional $\mathfrak{sl}(2, \mathbf{C})$-module, $m \in \mathbf{Z}^+$ and suppose w is a nonzero element of the weight space V_{-m}. Then $x^m \cdot w \neq 0$.*

Proof: By Weyl's Theorem, we may write $V = \oplus\sum_{i=1}^{n} I_i$, where for each $i \in \bar{n}$, I_i is an irreducible $\mathfrak{sl}(2, \mathbf{C})$-submodule of V with highest weight m_i. Since w is an element of the weight space V_{-m}, Lemma 2.1 implies $w = \sum_{i=1}^{n} w_i$, where $w_i \in (I_i)_{-m}$ for all $i \in \bar{n}$. Since $w \neq 0$, there exists $i_0 \in \bar{n}$ such that $w_{i_0} \neq 0$. Thus $-m$ is a weight of I_{i_0}, which implies $-m = m_{i_0} - 2k$, where $k = \frac{1}{2}(m_{i_0} + m)$ is in $\mathbf{Z}_0^+$ and $k \leq m_{i_0}$. Let v^+ be a maximal vector of I_{i_0}. Then $y^k \cdot v^+$ is a nonzero element of $(I_{i_0})_{-m}$ and $\dim(I_{i_0})_{-m} = 1$, hence $w_{i_0} = cy^k \cdot v^+$ for some nonzero $c \in \mathbf{C}$.

To show $x^m \cdot w \neq 0$, it suffices to show $x^m \cdot w_{i_0} \neq 0$. Because $c \neq 0$, for this it suffices to show $x^m \cdot (y^k \cdot v^+) \neq 0$. Observe that $m \in \mathbf{Z}^+$ with $1 \leq m \leq k$, so by Lemma 2.2 we have that

$$x^m \cdot (y^k \cdot v^+) = \frac{k!(m_{i_0} + m - k)!}{(k-m)!(m_{i_0} - k)!} y^{k-m} \cdot v^+$$

Because the scalars in this expression are nonzero and $k - m \in \mathbf{Z}_0^+$ with $k - m \leq k \leq m_{i_0}$, we have that $x^m \cdot (y^k \cdot v^+) \neq 0$. Therefore $x^m \cdot w \neq 0$. #

Theorem 2.5. *Let V be a finite-dimensional vector space over $\mathbf{C}$ and suppose $\varphi_1, \varphi_2 : \mathfrak{sl}(2, \mathbf{C}) \to \mathfrak{gl}(V)$ are two representations such that $\varphi_1|_B = \varphi_2|_B$, where $B = \operatorname{span}\{x, h\}$. Then $\varphi_1 = \varphi_2$.*

Proof: Let

$$H = \varphi_1(h), \; X = \varphi_1(x), \; Y_1 = \varphi_1(y) \text{ and } Y_2 = \varphi_2(y).$$

Note that $H = \varphi_2(h)$ and $X = \varphi_2(x)$ as well. Let $\cdot_1$ be the module structure on V corresponding to φ_1. Using this module structure, write V as a direct sum of irreducible submodules, $V = \oplus\sum_{i=1}^{n} I_i$. To show $\varphi_1 = \varphi_2$, it suffices to show $Y_1|_{I_i} = Y_2|_{I_i}$ for all $i \in \bar{n}$.

Fix $i \in \bar{n}$ and let v^+ be a maximal vector of I_i. Let m be the highest weight of I_i. Then $\{v^+, y \cdot_1 v^+, \ldots, y^m \cdot_1 v^+\}$ is a basis of I_i. To show $Y_1|_{I_i} = Y_2|_{I_i}$, it suffices to show $Y_1(y^t \cdot_1 v^+) = Y_2(y^t \cdot_1 v^+)$ for all $t \in \{0, 1, \ldots, m\}$. Observing that $y^t \cdot_1 v^+ = Y_1^t(v^+)$, we must show that $Y_1(Y_1^t(v^+)) = Y_2(Y_1^t(v^+))$ for all $t \in \{0, 1, \ldots, m\}$. Suppose $\exists t \in \{0, 1, \ldots, m\}$ such that $Y_1(Y_1^t(v^+)) \neq Y_2(Y_1^t(v^+))$, and let t_0 be the largest such t.

First, we claim that $t_0 \neq m$. Note that $Y_1(Y_1^m(v^+)) = y^{m+1} \cdot_1 v^+ = 0$. Thus if we can show $Y_2(Y_1^m(v^+)) = 0$, then $t_0 \neq m$. Suppose $Y_2(Y_1^m(v^+)) \neq 0$. We see that $Y_2(Y_1^m(v^+)) \in V_{-(m+2)}$. Then by Lemma 2.4, $X^{m+2}(Y_2(Y_1^m(v^+))) \neq 0$. But by Lemma 2.3,

$$\begin{aligned} X^{m+2}(Y_2(Y_1^m(v^+))) &= X^2(X^m(Y_2(Y_1^m(v^+)))) \\ &= X^2\{(m!)^2(-Y_1 + Y_2)(v^+)\} \\ &= (m!)^2 X\{(-XY_1 + XY_2)(v^+)\} \\ &= (m!)^2 X\{(-H + H)(v^+)\} \\ &= 0 \end{aligned}$$

which is a contradiction. Thus $Y_2(Y_1^m(v^+)) = 0$. Therefore we have the claim $t_0 \neq m$.

Next, we will show that $Y_1(Y_1^{t_0}(v^+)) = Y_2(Y_1^{t_0}(v^+))$. Since $t_0 \neq m$, $t_0 < m$ and $t_0 \in \{0, 1, \ldots, m\}$, by our choice of t_0, $Y_1(Y_1^{t_0+1}(v^+)) = Y_2(Y_1^{t_0+1}(v^+))$.

Therefore

$$X\big(Y_1(Y_1^{t_0+1}(v^+))\big) = X\big(Y_2(Y_1^{t_0+1}(v^+))\big) \tag{3}$$

By Lemma 2.2,

$$\begin{aligned} X\big(Y_1(Y_1^{t_0+1}(v^+))\big) &= X(Y_1^{t_0+2}(v^+)) \\ &= (t_0+2)(m-(t_0+1))Y_1^{t_0+1}(v^+) \end{aligned}$$

$$\begin{aligned} X\big(Y_2(Y_1^{t_0+1}(v^+))\big) &= \big([X,Y_2]+Y_2X\big)Y_1^{t_0+1}(v^+) \\ &= (H+Y_2X)Y_1^{t_0+1}(v^+) \\ &= HY_1^{t_0+1} + Y_2\big\{[X,Y_1^{t_0+1}] + Y_1^{t_0+1}X\big\}(v^+) \\ &= HY_1^{t_0+1}(v^+) + (t_0+1)(-t_0)Y_2Y_1^{t_0}(v^+) + Y_2Y_1^{t_0}H(v^+) \\ &= (m-2(t_0+1))Y_1^{t_0+1}(v^+) + (t_0+1)(m-t_0)Y_2(Y_1^{t_0}(v^+)) \end{aligned}$$

From (3), we have

$$(t_0+1)(m-t_0)Y_2(Y_1^{t_0}(v^+)) = (m-t_0)(t_0+1)Y_1^{t_0+1}(v^+)$$

But $(t_0+1) \neq 0$ and $(m-t_0) \neq 0$, hence $Y_1(Y_1^{t_0}(v^+)) = Y_2(Y_1^{t_0}(v^+))$, which contradicts the choice of t_0. Thus $Y_1(Y_1^t(v^+)) = Y_2(Y_1^t(v^+))$ for all $t \in \{0,1,\dots,m\}$. Hence $Y_1|_{I_i} = Y_2|_{I_i}$. Therefore $Y_1 = Y_2$, i.e, $\varphi_1 = \varphi_2$. #

Lemma 2.6 will be useful in Chapter III, where it will tell us the formula for the action of the Chevalley generator y_2 of $\mathfrak{sl}(3,\mathbf{C})$ on the elements of a Verma basis of a finite-dimensional irreducible $\mathfrak{sl}(3,\mathbf{C})$-module.

Lemma 2.6. *Let A be an associative algebra over a field of characteristic zero. Let a, $b \in A$ and suppose that $\mathrm{ad}_a^2(b) = 0$ and $\mathrm{ad}_b^2(a) = 0$. Then for any k, $m \in \mathbf{Z}^+$,*

i) $ba^k b^m = \dfrac{m+1-k}{m+1} a^k b^{m+1} + \dfrac{k}{m+1} a^{k-1} b^{m+1} a$

ii) $b^m a^k b = \dfrac{m+1-k}{m+1} b^{m+1} a^k + \dfrac{k}{m+1} a b^{m+1} a^{k-1}$

Proof: i) We will prove i) by induction on k.

Basis: $k = 1$. We must prove

$$bab^m = \frac{m}{m+1} ab^{m+1} + \frac{1}{m+1} b^{m+1} a$$

for all $m \in \mathbf{Z}^+$. We will prove this basis step by induction on m.

Basis: $m = 1$. Then

$$\begin{aligned} bab &= \mathrm{ad}_b(a)b + ab^2 \\ &= b\,\mathrm{ad}_b(a) + ab^2 \\ &= b^2 a - bab + ab^2 \end{aligned}$$

Thus

$$bab = \frac{1}{2} ab^2 + \frac{1}{2} b^2 a.$$

Hence the result holds in the case $m = 1$.

Induction: Suppose that

$$bab^m = \frac{m}{m+1} ab^{m+1} + \frac{1}{m+1} b^{m+1} a,$$

where $m \in \mathbf{Z}^+$. Then

$$\begin{aligned} bab^{m+1} &= (bab)b^m \\ &= \frac{1}{2} ab^{m+2} + \frac{1}{2} b^2 ab^m \end{aligned} \tag{4}$$

Look at

$$b^2ab^m = b(bab^m)$$

$$= \frac{m}{m+1}bab^{m+1} + \frac{1}{m+1}b^{m+2}a$$

From equation (4) we have

$$\frac{m+2}{m+1}bab^{m+1} = ab^{m+2} + \frac{1}{m+1}b^{m+2}a$$

That is,

$$bab^{m+1} = \frac{m+1}{m+2}ab^{m+2} + \frac{1}{m+2}b^{m+2}a$$

Thus the result holds in the case $m+1$.

Induction: Suppose that

$$ba^kb^m = \frac{m+1-k}{m+1}a^kb^{m+1} + \frac{k}{m+1}a^{k-1}b^{m+1}a$$

for all $m \in \mathbf{Z}^+$, where $k \in \mathbf{Z}^+$. Then

$$ba^{k+1}b^m = ad_b(a)a^kb^m + aba^kb^m$$

$$= a^kad_b(a)b^m + a(ba^kb^m)$$

$$= \{a^k(bab^m) - a^{k+1}b^{m+1}\} + \left\{\frac{m+1-k}{m+1}a^{k+1}b^{m+1} + \frac{k}{m+1}a^kb^{m+1}\right\}$$

$$= a^k\left\{\frac{m}{m+1}ab^{m+1} + \frac{1}{m+1}b^{m+1}a\right\} - \frac{k}{m+1}a^{k+1}b^{m+1}$$

$$+ \frac{k}{m+1}a^kb^{m+1}a$$

$$= \frac{m-k}{m+1}a^{k+1}b^{m+1} + \frac{k+1}{m+1}a^kb^{m+1}a$$

Thus the result holds in the case $k+1$. Therefore we have Lemma 2.6 i).

ii) The proof is the same as that of part i). #

Our next goal is Lemma 2.9, which will be useful in Chapter IV, where it will tell us the formula for the action of the Chevalley generator y_2 of $\mathfrak{o}(5, \mathbf{C})$ on the elements of a Verma basis of a finite-dimensional irreducible $\mathfrak{o}(5, \mathbf{C})$-module. Lemmas 2.7 and 2.8 are used in the proof of Lemma 2.9.

Lemma 2.7. *Let F be a field, A an F-algebra, and $\delta \in \mathrm{Der}A$. If $a \in A$ is such that $a\delta(a) = \delta(a)a$, then for any $k \in \mathbf{Z}^+$ with $k \geq 2$,*

$$\delta(a^k) = ka^{k-1}\delta(a) \tag{5}$$

Proof: We will prove (5) by induction on k.

Basis: $k = 2$. A quick calculation shows

$$\begin{aligned} \delta(a^2) &= a\delta(a) + \delta(a)a \\ &= a\delta(a) + a\delta(a) \\ &= 2a\delta(a) \end{aligned}$$

Induction: Suppose that (5) is true for some $k \in \mathbf{Z}^+$ with $k \geq 2$. Then

$$\begin{aligned} \delta(a^{k+1}) &= \delta(a^k a) \\ &= a^k\delta(a) + \delta(a^k)a \\ &= a^k\delta(a) + (ka^{k-1}\delta(a))a \\ &= (k+1)a^k\delta(a) \end{aligned}$$

Thus the result holds in the case $k + 1$. Hence we have Lemma 2.7. #

Lemma 2.8. *Let A be an associative algebra over a field of characteristic zero. Let a, $b \in A$ and suppose that* $\mathrm{ad}_b^3(a) = 0$. *Then for any $m \in \mathbf{Z}^+$ with $m \geq 2$,*

i) $bab^m = \frac{m-1}{m+1}(ab^{m+1} - b^{m+1}a) + b^m ab$

ii) $b^{m-2}ab^2 = \frac{2}{m(m-1)}ab^m + \frac{2(m-2)}{m-1}b^{m-1}ab - \frac{m-2}{m}b^m a$

Proof: We will prove i) and ii) together by induction on m.

Basis: $m = 2$. Then

$$\begin{aligned} bab^2 &= (\mathrm{ad}_b(a) + ab)b^2 \\ &= (-ad_b^2(a) + b\mathrm{ad}_b(a))b + ab^3 \\ &= \mathrm{ad}_b^3(a) - b\mathrm{ad}_b^2(a) + b\mathrm{ad}_b(a)b + ab^3 \\ &= -b^2\mathrm{ad}_b(a) + 2b\mathrm{ad}_b(a)b + ab^3 \\ &= ab^3 - b^3a + 3b^2ab - 2bab^2 \end{aligned}$$

Thus $3bab^2 = ab^3 - b^3a + b^2ab$. That is, $bab^2 = \frac{1}{3}(ab^3 - b^3a) + b^2ab$. Hence i) is done for the case $m = 2$; ii) is clearly true in this case.

Induction: Suppose i) and ii) are true for $m \in \mathbb{Z}^+$ with $m \geq 2$. Then

$$\begin{aligned}
bab^{m+1} &= (bab^m)b \\
&= \left\{\frac{m-1}{m+1}(ab^{m+1} - b^{m+1}a) + b^m ab\right\} b \\
&= \frac{m-1}{m+1}(ab^{m+2} - b^{m+1}ab) + b^m ab^2 \\
&= \frac{m-1}{m+1}(ab^{m+2} - b^{m+1}ab) + b^2(b^{m-2}ab^2) \\
&= \frac{m-1}{m+1}(ab^{m+2} - b^{m+1}ab) + \frac{2}{m(m-1)}b^2ab^m + \frac{2(m-2)}{m-1}b^{m+1}ab \\
&\quad - \frac{m-2}{m}b^{m+2}a
\end{aligned}$$

Look at

$$\begin{aligned}
b^2ab^m &= b(bab^m) \\
&= \frac{m-1}{m+1}(bab^{m+1} - b^{m+2}a) + b^{m+1}ab
\end{aligned}$$

Then

$$\frac{(m+2)(m-1)}{m(m+1)}bab^{m+1} = \frac{m-1}{m+1}(ab^{m+2} - b^{m+2}a) + \frac{(m-1)(m+2)}{m(m+1)}b^{m+1}ab$$

Thus

$$bab^{m+1} = \frac{m}{m+2}(ab^{m+2} - b^{m+2}a) + b^{m+1}ab.$$

Hence i) is done for the case $m+1$.

For ii) we calculate

$$
\begin{aligned}
b^{(m+1)-2}ab^2 &= b(b^{m-2}ab^2) \\
&= \frac{2}{m(m-1)}bab^m + \frac{2(m-2)}{m-1}b^m ab - \frac{m-2}{m}b^{m+1}a \\
&= \frac{2}{m(m-1)}\left(\frac{m-1}{m+1}(ab^{m+1} - b^{m+1}a) + b^m ab\right) + \frac{2(m-2)}{m-1}b^m ab \\
&\quad - \frac{m-2}{m}b^{m+1}a \\
&= \frac{2}{(m+1)m}ab^{m+1} + \frac{2(m-1)}{m}b^m ab - \frac{m-1}{m+1}b^{m+1}a
\end{aligned}
$$

Thus ii) is done for the case $m+1$. Therefore we have Lemma 2.8. #

Lemma 2.9. *Let A be an associative algebra over a field of characteristic zero. Let $a, b \in A$ and suppose that $\mathrm{ad}_a^2(b) = 0$ and $\mathrm{ad}_b^3(a) = 0$. Then for any k, $m \in \mathbf{Z}^+$ with $m \geq 2$,*

$$
ba^k b^m = \frac{m+1-2k}{m+1}a^k b^{m+1} - \frac{k(m-1)}{m+1}a^{k-1}b^{m+1}a + ka^{k-1}b^m ab
$$

Proof: Since $\mathrm{ad}_a^2(b) = 0$ and $\mathrm{ad}_b^2(b) = 0$, $a\mathrm{ad}_a(b) = \mathrm{ad}_a(b)a$ and $b\mathrm{ad}_b(b) = \mathrm{ad}_b(b)b$. Then by Lemma 2.7, $\mathrm{ad}_b(a^k) = ka^{k-1}\mathrm{ad}_b(a)$ and $\mathrm{ad}_b(b^m) = mb^{m-1}\mathrm{ad}_b(b) = 0$. Then

$$
\begin{aligned}
ba^k b^m - a^k b^{m+1} &= \mathrm{ad}_b(a^k b^m) \\
&= a^k \mathrm{ad}_b(b^m) + \mathrm{ad}_b(a^k)b^m \\
&= ka^{k-1}\mathrm{ad}_b(a)b^m \\
&= ka^{k-1}bab^m - ka^k b^{m+1}
\end{aligned}
$$

Thus by Lemma 2.8 i),

$$\begin{aligned} ba^kb^m &= (1-k)a^kb^{m+1} + ka^{k-1}(bab^m) \\ &= (1-k)a^kb^{m+1} + ka^{k-1}\left\{\frac{m-1}{m+1}(ab^{m+1} - b^{m+1}a) + b^mab\right\} \\ &= \frac{m+1-2k}{m+1}a^kb^{m+1} - \frac{k(m-1)}{m+1}a^{k-1}b^ma + ka^{k-1}b^mab \end{aligned}$$

Therefore we have Lemma 2.9. #

Lemma 2.10 will be used in Chapter IV.

Lemma 2.10. *Let V be a vector space over a field F with a basis $w_1, w_2, \ldots, w_n$. Let $I \subseteq \bar{n}$. For each $i \in I$, let $z_i \in \operatorname{span}\{w_j \mid j \in \bar{n} - I\}$. Then $\{w_i + z_i \mid i \in I\}$ is linearly independent.*

Proof: Suppose $\sum_{i \in I} b_i(w_i + z_i) = 0$, where $b_i \in F$ for all $i \in I$. Then

$$\sum_{i\in I} b_iw_i + \sum_{i\in I} b_iz_i = 0$$

Because for any $i \in I$, $z_i \in \operatorname{span}\{w_j \mid j \in \bar{n} - I\}$, there exist scalars d_i, $i \in \bar{n} - I$, such that

$$\sum_{i\in I} b_iz_i = \sum_{i\in \bar{n}-I} d_iw_i$$

Hence

$$\sum_{i\in I} b_iw_i + \sum_{i\in \bar{n}-I} d_iw_i = 0$$

Since $w_1, w_2, \ldots, w_n$ are linearly independent, $b_i = 0$ for all $i \in I$ and $d_i = 0$ for all $i \in \bar{n} - I$. Therefore $\{w_i + z_i \mid i \in I\}$ is linearly independent. #