CHAPTER I

PRELIMINARIES

In this chapter we shall give some notations, definitions and results proved by other authers which will be used in this thesis. Our general notations are:

$\mathbf{Z}$ = the set of all integers,

$\mathbf{Z}^+$ = the set of all positive integers,

$\mathbf{Z}_0^+ = \mathbf{Z}^+ \cup \{0\}$,

$\mathbf{C}$ = the set of all complex numbers,

$End_F(V)$ = the set of all linear transformations from V to V, where F is a field and V is a vector space over F,

$\lceil i \rceil$ = the smallest integer which is greater than or equal to i, where i is a real number,

$\lfloor i \rfloor$ = the largest integer which is less than or equal to i, where i is a real number,

$\bar{n} = \{1, 2, \ldots, n\}$, where $n \in \mathbf{Z}^+$.

For a good account of the basics of Lie algebras and their representations, the reader is advised to consult Humphreys' book [3]. We will follow the notation established therein very closely. However, there are a few notational conventions, definitions, and results which play such an important role in this thesis that they deserve special mention. First, some notation.

1) Given a vector space V (over some field F), $\mathfrak{gl}(V)$ will denote the Lie algebra consisting of the set $End_F(V)$ together with the bracket $[a,b] = ab - ba$ for all $a, b \in End_F(V)$.

2) For $n \in \mathbf{Z}^+$, $\mathfrak{sl}(n, \mathbf{C})$ will denote the Lie algebra consisting of all $n \times n$ matrices over $\mathbf{C}$ which have trace zero, with bracket $[a,b] = ab - ba$ for all $a, b \in \mathfrak{sl}(n, \mathbf{C})$.

3) The standard basis for $\mathfrak{sl}(2, \mathbf{C})$ consists of the three matrices

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

4) Let s be the 5×5 matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_2 \\ 0 & I_2 & 0 \end{pmatrix}$, where I_2 is the 2×2 identity matrix; $\mathfrak{o}(5, \mathbf{C})$ will denote the Lie algebra consisting of all 5×5 matrices a over $\mathbf{C}$ which satisfy $sa = -a^t s$, with bracket $[a,b] = ab - ba$ for all $a, b \in \mathfrak{o}(5, \mathbf{C})$.

5) Suppose L is a finite-dimensional semisimple Lie algebra over $\mathbf{C}$ and H is a maximal toral subalgebra of L. For any L-module V and any linear functional $\mu \in H^*$, we will use V_μ to denote the *weight space*

$$V_\mu = \{v \in V \mid h \cdot v = \mu(h)v \text{ for all } h \in H\}$$

The elements of V_μ are called *weight vectors* of *weight* μ, and μ is said to be a *weight* of V whenever $V_\mu \neq \{0\}$.

Next, a couple of definitions.

6) Let L be a finite-dimensional semisimple Lie algebra over $\mathbf{C}$, H a maximal toral subalgebra of L, Φ the set of roots of L with respect to H, and $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ a basis of simple roots. For each $i \in \bar{l}$, let x_i be a

nonzero element of the root space L_{α_i}. Then we may choose $y_i \in L_{-\alpha_i}$ such that $S_i = \text{span}\{x_i, h_i, y_i\} \cong \mathfrak{sl}(2, \mathbf{C})$, where $h_i = [x_i, y_i] \in H$. The set $\{x_1, \ldots, x_l, y_1, \ldots, y_l\}$ is called a set of *Chevalley generators* of L.

7) Let V be a finite-dimensional irreducible L-module and let λ be the highest weight of V. Then any nonzero element v^+ of V_λ is called a *maximal vector* of V. Recall that if $\{x_1, \ldots, x_l, y_1, \ldots, y_l\}$ is a set of Chevalley generators of L and v^+ is a maximal vector of V, then $x_i \cdot v^+ = 0$ for all $i \in \bar{l}$.

Finally, some heavily used results.

Theorem 1.1. *([3], Theorem 6.3)(Weyl's theorem) Let L be a finite-dimensional semisimple Lie algebra over an algebraically closed field of characteristic zero. Then every finite-dimensional L-module is completely reducible.*

Lemma 1.2. *([3], Lemma 7.2) Let V be a finite-dimensional irreducible $\mathfrak{sl}(2, \mathbf{C})$-module with highest weight λ, let $v^+ \in V_\lambda$ be a maximal vector of V, and set*

$$
\begin{aligned}
v_{-1} &= 0, \\
v_0 &= v^+, \\
v_{i+1} &= \frac{1}{i+1} y \cdot v_i
\end{aligned}
$$

for all $i \in \mathbf{Z}_0^+$. Then for each $i \in \mathbf{Z}_0^+$,

i) $h \cdot v_i = (\lambda - 2i) v_i$

ii) $y \cdot v_i = (i+1) v_{i+1}$

iii) $x \cdot v_i = (\lambda - i + 1) v_{i-1}$

Theorem 1.3. *([3], Theorem 7.2) Let V be a finite-dimensional irreducible $\mathfrak{sl}(2, \mathbf{C})$-module.*

i) *If $m = \dim V - 1$, then V is the direct sum of weight spaces*

$$V = \oplus \sum_{i=0}^{m} V_{m-2i}$$

and $\dim V_k = 1$ for all weights $k \in \{m, m-2, \ldots, -m\}$.

ii) *If we ignore nonzero scalar multiples then V has a unique maximal vector, whose weight is m.*

iii) *Given a maximal vector v^+ of V, if $v_0, v_1, \ldots, v_m$ are defined as in Lemma 1.2, then they form a basis of V, with the actions of x, y and h given by formulas i), ii) and iii) of Lemma 1.2.*

Lemma 1.4. *([3], Lemma 21.2) Let L be a semisimple Lie algebra with Chevalley generators $x_1, \ldots, x_l$ and $y_1, \ldots, y_l$, H a maximal toral subalgebra of L. For each $k \in \mathbf{Z}_0^+$ and $i, j \in \bar{l}$, the following identities hold in the universal enveloping algebra of L.*

i) $[x_j, y_i^k] = 0$ *for* $i \neq j$

ii) $[x_i, y_i^k] = k(1-k)y_i^{k-1} + k y_i^{k-1} h_i$

iii) $[h, y_i^k] = -k\alpha_i(h) y_i^k$ *for all* $h \in H$.

Let S denote $\mathfrak{sl}(2, \mathbf{C})$ for the moment, and let B denote the subalgebra $\mathrm{span}\{x, h\}$. The finite-dimensional indecomposable B-modules on which h acts semisimply will play a central role in Chapters III and IV. A complete exposition of this topic may be found in [2], Chapters 5 and 6. For our purposes, however, it suffices to record the following definitions and results.

Proposition 1.5. *([2], Proposition 5.3) Let I be an indecomposable B-module. Then there is a scalar λ and a basis $\{v_0, v_1, \ldots, v_k\}$ of I such that $x \cdot v_i = v_{i-1}$ and $h \cdot v_i = (\lambda - 2i)v_i$ (where $v_{-1} = 0$).*

As in [2], page 79, if λ and ν are in $\mathbf{C}$ with $\lambda - \nu = 2k$ for some $k \in \mathbf{Z}_0^+$, then we can construct a vector space X with basis $\{v_0, v_1, \ldots, v_k\}$ and define an action of B on X by $x \cdot v_i = v_{i-1}(v_{-1} = 0)$ and $h \cdot v_i = (\lambda - 2i)v_i$. It can be checked that this makes X into an indecomposable B-module with $h \cdot v_0 = \lambda v_0$ and $h \cdot v_k = \nu v_k$. This B-module will be denoted by $\mathcal{S}(\lambda, \nu)$ and will be refered to as a *string module.* Any basis $v_0, \ldots, v_k$ such that $x \cdot v_i = v_{i-1}$ and $h \cdot v_i = (\lambda - 2i)v_i$ will be called a *standard basis* of $\mathcal{S}(\lambda, \nu)$.

Lemma 1.6. *([2], Lemma 5.7) Let V be a finite-dimensional $\mathfrak{sl}(2, \mathbf{C})$-module, let $m \in \mathbf{Z}$ and $n \in \mathbf{Z}_0^+$ and let W be a B-submodule of V isomorphic to $\mathcal{S}(m+n, m-n)$. Let $w_0, \ldots, w_n$ be a standard basis for W. Then $m \geq 0$ and the set*

$$\{y^j \cdot w_i \mid i, j \in \mathbf{Z}_0^+ \text{ with } 0 \leq j \leq m,\ 0 \leq i \leq n\}$$

spans the $\mathfrak{sl}(2, \mathbf{C})$-submodule of V generated by W.

Lemma 1.7. *([2], Lemma 6.3) Let $m, n \in \mathbf{Z}_0^+$ and let V be a B-module with basis $\{v_{i,j} \mid i, j \in \mathbf{Z}_0^+ \text{ with } 0 \leq j \leq m,\ 0 \leq i \leq n\}$ satisfying*

1) $h \cdot v_{j,i} = (m + n - 2j - 2i)v_{j,i}$

2) $x \cdot v_{j,i} = v_{j,i-1} + j(m + n - 2i - j + 1)v_{j-1,i}$, where we make the convention that $v_{j,i} = 0$ if at least one of j or i is negative.

Then V can be made into an S-module by keeping the same action for x and h and defining an action of y by

3) $y \cdot v_{j,i} = v_{j+1,i}$ for $0 \le j < m$, $0 \le i \le n$.

4) $$y \cdot v_{m,i} = \sum_{p=1}^{n-i} \left\{ (-1)^{p-1} \binom{n-i}{p} \prod_{r=0}^{p-1} (m+1-r)(i+r+1) \right\} v_{m+1-p,i+p}$$

for $0 \le i < n$

5) $y \cdot v_{m,n} = 0$.

The last topic which needs to be discussed in this chapter is that of Verma bases. These are bases for the finite-dimensional irreducible modules of many of the finite-dimensional simple Lie algebras which are constructed in a special way. Again, the reader is advised to consult other sources (such as [4] and [2]) for the full story. Here we only need to know Verma bases for modules over the algebras $\mathfrak{sl}(3, \mathbf{C})$ and $\mathfrak{o}(5, \mathbf{C})$.

Lemma 1.8. *([2], Proposition 9.2) Let L be the Lie algebra $\mathfrak{sl}(3, \mathbf{C})$ with Chevalley generators x_1, x_2, y_1 and y_2. Let V be a finite-dimensional irreducible L-module with highest weight λ, and let v^+ be a maximal vector for V. Set $m_1 = \lambda(h_1)$, $m_2 = \lambda(h_2)$, where $h_i = [x_i, y_i]$ for $i \in \{1, 2\}$. The set of all elements of V of the form*

$$y_1^{a_3} y_2^{a_2} y_1^{a_1} \cdot v^+$$

where $a_1, a_2, a_3 \in \mathbf{Z}_0^+$ and

$$0 \le a_1 \le m_1$$

$$0 \le a_2 \le m_2 + a_1$$

$$0 \le a_3 \le \min\{a_2, m_2\}$$

forms a basis of V.

Lemma 1.9. *([2], Proposition 9.4) Let L be the Lie algebra $\mathfrak{o}(5,\mathbb{C})$ with Chevalley generators x_1, x_2, y_1 and y_2. Let V be a finite-dimensional irreducible L-module with highest weight λ, and let v^+ be a maximal vector for V. Set $m_1 = \lambda(h_1)$, $m_2 = \lambda(h_2)$, where $h_i = [x_i, y_i]$ for $i \in \{1,2\}$. The set of all elements of V of the form*

$$y_1^{a_4} y_2^{a_3} y_1^{a_2} y_2^{a_1} \cdot v^+$$

where $a_1, a_2, a_3, a_4 \in \mathbb{Z}_0^+$ and

$$0 \le a_1 \le m_2$$

$$0 \le a_2 \le m_1 + a_1$$

$$0 \le a_3 \le \min\{m_1 + a_2, 2a_2\}$$

$$0 \le a_4 \le \min\{m_1, \lfloor a_3/2 \rfloor\}$$

forms a basis of V.