CHAPTER I

REGULAR-STAR SEMIGROUPS AND STAR-REGULAR SEMIGROUPS

Semigroups which are *-regular have been studied by Drazin in [3]. Nordahl and Scheiblich have studied regular-* semigroups in [10]. In this chapter, various general properties of *-semigroups are introduced. In particular, we study regular-* semigroups and *-regular semigroups in general. Many important properties satisfied by both or by one but not the other are introduced. Various examples are also given.

Recall that a map * from a semigroup S into S is an involution of S if

$$(a^*)^* = a \quad \text{and} \quad (ab)^* = b^*a^*$$

for all a, b $\in$ S, and a *-semigroup is a semigroup with an involution. Observe that an involution of a *-semigroup is a one-to-one and onto map on S.

Let S be a *-semigroup. Then for any subset A of S, $(A^*)^* = A$. If A is a subsemigroup of S, then A^* is clearly a subsemigroup of S. If e is an idempotent of S, then $e^* = (ee)^* = e^*e^* \in E(S)$. Thus $(E(S))^* \subseteq E(S)$, so $E(S) = ((E(S))^*)^* \subseteq (E(S))^* \subseteq E(S)$ which implies $(E(S))^* = E(S)$. If S has a zero 0, then for all $a \in S$, $0^*a = (a^*0)^* = 0^*$ and $a0^* = (0a^*)^* = 0^*$, and hence $0^* = 0$. If S has an identity 1, then for all $a \in S$, $1^*a = (a^*1)^* = (a^*)^* = a$ and $a1^* = (1a^*)^* = (a^*)^* = a$, and thus $1^* = 1$.

Let A and B be subsets of a *-semigroup S. Then the following are clearly obtained : $(A \cup B)^* = A^* \cup B^*$ and $(AB)^* = B^*A^*$. Thus we have that $(SA)^* = A^*S$, $(AS)^* = SA^*$, and if $a \in S$, then $(S^1a)^* = (Sa \cup \{a\})^* = a^*S \cup \{a^*\} = a^*S^1$, $(aS^1)^* = (aS \cup \{a\})^* = Sa^* \cup \{a^*\} = S^1a^*$ and $(S^1aS^1)^* = (SaS \cup Sa \cup aS \cup \{a\})^* = Sa^*S \cup a^*S \cup Sa^* \cup \{a^*\} = S^1a^*S^1$. Hence we have the following theorem :

1.1 <u>Theorem</u>. Let S be a *-semigroup and $A \subseteq S$. Then :

(i) If A is a left [right] ideal of S, then A^* is a right [left] ideal of S. If A is the principal left [right] ideal of S generated by $a \in S$, then A^* is the principal right [left] ideal of S generated by a^*.

(ii) If A is an ideal of S, then so is A^*. If A is the principal ideal of S generated by a, then A^* is the principal ideal of S generated by a^*.

The next theorem shows properties of the Green's relations of any *-semigroup.

1.2 <u>Theorem</u>. Let S be a *-semigroup and $a \in S$. Then the following hold :

(i) $R_a^* = L_{a^*}$,

(ii) $L_a^* = R_{a^*}$,

(iii) $H_a^* = H_{a^*}$,

(iv) $D_a^* = D_{a^*}$,

(v) $J_a^* = J_{a^*}$.

Proof : (i) and (ii) have been proved in [10].

(iii) Because $H_x = R_x \cap L_x$ for all $x \in S$, it follows from (i) and (ii) that $H_a^* = H_{a^*}$.

(iv) Let $x \in D_a^*$. Then $x^* \in D_a$, so $(x^*, a) \in \mathcal{D}$. Thus $(x^*, u) \in \mathcal{L}$ and $(u, a) \in \mathcal{R}$ for some $u \in S$. By (ii), $(x, u^*) \in \mathcal{R}$, and by (i), $(u^*, a^*) \in \mathcal{L}$, so $(x, a^*) \in \mathcal{R} \circ \mathcal{L} = \mathcal{D}$ which implies $x \in D_{a^*}$. Thus $D_a^* \subseteq D_{a^*}$. This proves that $D_x^* \subseteq D_{x^*}$ for all $x \in S$. Hence $D_a \subseteq (D_{a^*})^* \subseteq D_{(a^*)^*} = D_a$, and therefore $D_a^* = D_{a^*}$.

(v) Let $x \in J_a^*$. Then $x^* \in J_a$, so $S^1x^*S^1 = S^1aS^1$. Since $S^1xS^1 = (S^1x^*S^1)^* = (S^1aS^1)^* = S^1a^*S^1$, it follows that $x \in J_{a^*}$, hence $J_a^* \subseteq J_{a^*}$. This shows that $J_x^* \subseteq J_{x^*}$ for all $x \in S$. Thus $J_a \subseteq J_{a^*}^* \subseteq J_{(a^*)^*} = J_a$. Therefore $J_a^* = J_{a^*}$. #

For any semigroup S, G is a maximal subgroup of S if and only if $G = H_e$ for some $e \in E(S)$. If S has an identity 1, then H_1 is the unit group of S.

1.3 Corollary. Let S be a *-semigroup. Then the involution * of S preserves the maximal subgroups of S, that is, if G is a maximal subgroup of S, then G* is a maximal subgroup of S. In particular, if S has an identity, then the involution * of S fixes the unit group of S, that is, if G is the unit group of S, then G* = G.

Proof : It follows from the fact that $(E(S))^* = E(S)$, $(H_e)^* = H_{e^*}$ for all $e \in E(S)$ and $1^* = 1$ where 1 is the identity of S. #

Recall that a *-semigroup S is a <u>regular-* semigroup</u> if $a = aa^*a$ for all elements a in S. A *-semigroup S is a <u>proper *-semigroup</u> if for a, b $\in$ S, $a^*a = a^*b = b^*a = b^*b$ implies $a = b$; or equivalently, for a, b $\in$ S, $aa^* = ab^* = ba^* = bb^*$ implies $a = b$. A *-semigroup S is a <u>*-regular semigroup</u> if S is proper and for each $a \in S$, there exists $x \in S$ such that $a = axa$, $x = xax$, $(ax)^* = ax$, $(xa)^* = xa$.

Let S be a semigroup. For $a \in S$, let V(a) denote the set of all inverses of a, that is,

$$V(a) = \{x \in S \mid a = axa \text{ and } x = xax\}.$$

Let S be a *-semigroup and $a \in S$. If $a' \in V(a)$, then $a = aa'a$ and $a' = a'aa'$, so $a^* = a^*(a')^*a^*$ and $(a')^* = (a')^*a^*(a')^*$ which implies $(a')^* \in V(a^*)$. This proves that $(V(a))^* \subseteq V(a^*)$ for all $a \in S$. Hence for $a \in S$, $V(a) \subseteq (V(a^*))^* \subseteq V((a^*)^*) = V(a)$, and thus $(V(a^*))^* = V(a)$ which implies $(V(a))^* = V(a^*)$. If S is an inverse semigroup, then for all $a \in S$, $V(a) = \{a^{-1}\}$, hence $(a^{-1})^* = (a^*)^{-1}$ for all $a \in S$.

Every inverse semigroup is both regular-* and *-regular. Let S be an inverse semigroup. Define $a^* = a^{-1}$ for all a in S. Then S is a *-semigroup since $(a^{-1})^{-1} = a$ and $(ab)^{-1} = b^{-1}a^{-1}$ for all a, b in S. If $a \in S$, then $a = aa^{-1}a = aa^*a$. Hence S is regular-*. If $a \in S$, then $a = aa^{-1}a$, $a^{-1} = a^{-1}aa^{-1}$, $(aa^{-1})^* = (a^{-1})^*a^* = (a^{-1})^{-1}a^{-1} = aa^{-1}$ and $(a^{-1}a)^* = a^*(a^{-1})^* = a^{-1}(a^{-1})^{-1} = a^{-1}a$. Suppose a, b $\in$ S such that $a^{-1}a = a^{-1}b = b^{-1}a = b^{-1}b$. Since $(ab^{-1})(ab^{-1}) = ab^{-1}ab^{-1} = aa^{-1}ab^{-1} = ab^{-1}$, $ab^{-1} \in E(S)$. Thus

$ab^{-1} = aa^{-1}ab^{-1} = ab^{-1}aa^{-1} = aa^{-1}aa^{-1} = aa^{-1}$, so $ba^{-1} = aa^{-1}$. Hence $a = aa^{-1}a = ba^{-1}a = bb^{-1}b = b$. This proves that S is *- regular.

A regular-* semigroup need not be a *-regular semigroup and a *-regular semigroup need not be a regular-* semigroup. They are shown by the following examples :

Example : Let X be a set such that $|X| \geq 2$ and let $S = X \times X$ be the semigroup with the operation defined by $(a, b)(c, d) = (a, d)$ for all $a, b, c, d \in X$. Define the map * on S by $(a, b)^* = (b, a)$ for all $a, b \in X$. Then * is an involution on S. Since for $a, b \in X$, $(a, b)(b, a)(a, b) = (a, b)$, it follows that S is a regular-* semigroup.

Next, we show that S is not a *-regular semigroup. Suppose S is a *-regular semigroup under an involution *. Let $a \in X$. Then $(a, a)^* = (x, y)$ for some $x, y \in X$. Thus $(x, y)^* = (a, a)$ and hence $(x, a)^* = ((x, y)(a, a))^* = (a, a)^*(x, y)^* = (x, y)(a, a) = (x, a)$. Similarly, we can show that $(a, y)^* = (a, y)$. Thus $(a, y) = (a, y)^*(a, y) = (a, y)^*(x, y) = (x, y)^*(a, y) = (x, y)^*(x, y)$. Since S is a proper *-semigroup, $(a, y) = (x, y)$, so $x = a$. Thus $(a, y) = (a, y)^* = (x, y)^* = (a, a)$, so $y = a$. This proves that $(a, a)^* = (a, a)$ for all $a \in X$.

Let a, b be two distinct elements of X. Then $(a, a)^* = (a, a)$ and $(b, b)^* = (b, b)$. Hence $(a, b) = (a, a)(b, b) = (a, a)^*(b, b)^* = ((b, b)(a, a))^* = (b, a)^*$ which implies $(a, b)^* = (b, a)$. Thus $(a, b)^*(a, b) = (a, b)^*(b, b) = (b, b)^*(a, b) =$

$(b, b)^*(b, b) = (b, b)$. Because S is a proper *-semigroup, it follows that $(a, b) = (b, b)$ which implies $a = b$, a contradiction.

Therefore S is not a *-regular semigroup. #

<u>Example</u>. Let I be a set such that $|I| = 2$, $\mathbb{Z}$ the set of integers, and $S = I \times \mathbb{Z} \times I$. Let $P : I \times I \to \mathbb{Z}$ be the map such that

$$(a, b)P = p_{ab} \quad \text{and} \quad p_{ab} = \begin{cases} 0 & \text{if } a = b, \\ 1 & \text{if } a \neq b. \end{cases}$$

Then $p_{ab} = p_{ba}$ for all $a, b \in I$. Define a multiplication on S by $(a, n, b)(c, m, d) = (a, n+p_{bc}+m, d)$. Then S is a semigroup. Define the map * on S by

$$(a, n, b)^* = (b, n, a) \qquad (a, b \in I, n \in \mathbb{Z}).$$

Then the map * is an involution on S, so S is a *-semigroup. To show that S is a *-regular semigroup, suppose $a, b, c, d \in I$, $n, m \in \mathbb{Z}$ such that $(a, n, b)^*(a, n, b) = (a, n, b)^*(c, m, d) = (c, m, d)^*(a, n, b) = (c, m, d)^*(c, m, d)$. Then $(b, n, a)(a, n, b) = (b, n, a)(c, m, d) = (d, m, c)(a, n, b) = (d, m, c)(c, m, d)$ which implies $(b, 2n, b) = (b, n+p_{ac}+m, d) = (d, m+p_{ca}+n, b) = (d, 2m, d)$, so $b = d$, $n = m$ and $n+p_{ac}+m = 2m$. Thus $p_{ac} = 0$ which implies $a = c$. Hence $(a, n, b) = (c, m, d)$. This proves that S is a proper *-semigroup. Because $(a, n, b) = (a, n, b)(b, -n, a)(a, n, b)$, $(b, -n, a) = (b, -n, a)(a, n, b)(b, -n, a)$, $((a, n, b)(b, -n, a))^* = (a, 0, a)^* = (a, 0, a) = (a, n, b)(b, -n, a)$ and $((b, -n, a)(a, n, b))^* = (b, 0, b)^* = (b, 0, b) = (b, -n, a)(a, n, b)$. Hence S is a *-regular semigroup.

Next, to show that S is not a regular-* semigroup, suppose S is a regular-* semigroup under an involution *. Let a, b be two distinct elements of I. Then I = {a, b}. Let $x, y \in I$, $m \in \mathbb{Z}$ such that $(a, 1, b)^* = (x, m, y)$. Then $(a, 1, b) = (a, 1, b)(x, m, y)(a, 1, b)$ $= (a, 1+p_{bx}+m, y)(a, 1, b) = (a, 2+m+p_{bx}+p_{ya}, b)$. It follows that $1 = 2+m+p_{bx}+p_{ya}$, so $m = -1-p_{bx}-p_{ya}$. Thus $(a, 1, b)^* =$ $(x, -1-p_{bx}-p_{ya}, y)$ and hence $(x, -p_{bx}, b)^* =$ $((x, -1-p_{bx}-p_{ya}, y)(a, 1, b))^* = ((a, 1, b)^*(a, 1, b))^* =$ $(a, 1, b)^*(a, 1, b) = (x, -p_{bx}, b)$ and $(a, -p_{ya}, y)^* =$ $((a, 1, b)(x, -1-p_{bx}-p_{ya}, y))^* = ((a, 1, b)(a, 1, b)^*)^* =$ $(a, 1, b)(a, 1, b)^* = (a, -p_{ya}, y)$. Because $(x, -p_{bx}+1-p_{ya}, y) =$ $(x, -p_{bx}, b)(a, -p_{ya}, y) = ((a, -p_{ya}, y)^*(x, -p_{bx}, b)^*)^* =$ $((a, -p_{ya}, y)(x, -p_{bx}, b))^* = (a, -p_{ya}+p_{yx}-p_{bx}, b)^*$ and S is a regular-* semigroup, it follows that $(a, -p_{ya}+p_{yx}-p_{bx}, b) =$ $(a, -p_{ya}+p_{yx}-p_{bx}, b)(a, -p_{ya}+p_{yx}-p_{bx}, b)^*(a, -p_{ya}+p_{yx}-p_{bx}, b) =$ $(a, -p_{ya}+p_{yx}-p_{bx}, b)(x, -p_{bx}+1-p_{ya}, y)(a, -p_{ya}+p_{yx}-p_{bx}, b) =$ $(a, -2p_{ya}+p_{yx}-p_{bx}+1, y)(a, -p_{ya}+p_{yx}-p_{bx}, b) = (a, -2p_{ya}+2p_{yx}-2p_{bx}+1, b)$ which implies $-p_{ya}+p_{yx}-p_{bx} = 2(-p_{ya}+p_{yx}-p_{bx}) + 1$. Hence $p_{ya} - p_{yx} + p_{bx} = 1$ which implies $y = x = a \neq b$, $y = x = b \neq a$, or $x = a \neq b = y$ since I = {a, b}.

<u>Case $y = x = a \neq b$</u>. Then $(a, 1, b)^* = (a, -2, a)$. Hence $(a, 1, b)(a, 1, b)^* = (a, 1, b)(a, -2, a) = (a, 0, a) = (a, 0, a)^*$ and $(a, 1, b)^*(a, 1, b) = (a, -2, a)(a, 1, b) = (a, -1, b) =$ $(a, -1, b)^*$. Thus $(a, 0, a) = (a, -1, b)(a, 0, a) =$ $(a, -1, b)^*(a, 0, a)^* = ((a, 0, a)(a, -1, b))^* = (a, -1, b)^* =$

$(a, -1, b)$, a contradiction.

Case $y = x = b \neq a$. Then $(a, 1, b)^* = (b, -2, b)$, and hence $(a, 1, b)^*(a, 1, b) = (b, -2, b)(a, 1, b) = (b, 0, b) = (b, 0, b)^*$ and $(a, 1, b)(a, 1, b)^* = (a, 1, b)(b, -2, b) = (a, -1, b) = (a, -1, b)^*$. Thus $(b, 0, b) = (b, 0, b)^* = ((b, 0, b)(a, -1, b))^* = (a, -1, b)^*(b, 0, b)^* = (a, -1, b)(b, 0, b) = (a, -1, b)$, a contradiction.

Case $x = a \neq b = y$. Then $(a, 1, b)^* = (a, -3, b)$ and so $(a, 1, b)^*(a, 1, b) = (a, -3, b)(a, 1, b) = (a, -1, b) = (a, -1, b)^*$. From $(a, -3, b) = (a, 1, b)^* = ((a, 0, a)(a, 1, b))^* = (a, 1, b)^*(a, 0, a)^* = (a, -3, b)(a, 0, a)^*$, we have that $(a, 0, a)^* = (a, -1, b)$ or $(b, 0, b)$. If $(a, 0, a)^* = (a, -1, b)$, then $(a, -1, b) = (a, -1, b)^* = (a, 0, a)$, a contradiction. If $(a, 0, a)^* = (b, 0, b)$, then $(a, 0, a)(a, 0, a)^* = (a, 0, a)(b, 0, b) = (a, 1, b) = (a, -3, b)^*$ and thus $(a, -3, b) = (a, 0, a)(a, 0, a)^* = (a, 1, b)$, a contradiction.

This proves that S is not a regular-* semigroup. #

A regular-* semigroup and a *-regular semigroup are regular semigroups. But a regular semigroup S which is a *-semigroup need not be a regular-* semigroup and need not be a *-regular semigroup under the involution of S. They are shown by the following examples :

Example. Let $\mathbb{C}$ be the set of all complex numbers and let $M_2(\mathbb{C})$ be the set of all 2×2 matrices over $\mathbb{C}$. Then $M_2(\mathbb{C})$ is a *-semigroup with matrix multiplication and $A^* = A^t$ (A-transpose). Next, to show $M_2(\mathbb{C})$ is regular, let $A \in M_2(\mathbb{C})$. From [1, Theorem 25] there exist nonsingular matrices P and Q such that $PAQ = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ where $I_r = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Let $A' = Q \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} P$. It is easy to verify that $A = AA'A$. This proves that $M_2(\mathbb{C})$ is regular. Since $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $M_2(\mathbb{C})$ is not a regular-* semigroup under the involution *. #

<u>Example</u>. Let X be a set such that $|X| \geq 2$ and let $S = X \times X$ be the semigroup with an operation defined by $(a, b)(c, d) = (a, d)$ for all $a, b, c, d \in X$. Then the map * on S defined by $(a, b)^* = (b, a)$, $(a, b \in X)$, is an involution on S, so S is a *-semigroup. Since $(a, b) = (a, b)(b, a)(a, b)$ for all $a, b \in X$, S is regular. We have shown that S is not a *-regular semigroup under any involution *. #

A homomorphic image of a regular *-semigroup is not necessarily regular-*. Also, a homomorphic image of a *-regular semigroup need not be *-regular.

<u>Example</u>. Let X be a set such that $|X| \geq 2$ and let $S = X \times X$ be the semigroup with the operation defined by $(a, b)(c, d) = (a, d)$ for all $a, b, c, d \in X$. Define the map * on S by $(a, b)^* = (b, a)$ for all $a, b \in X$. We have shown that S is a regular-* semigroup under the involution *. Let $a \in X$ and let $\psi : S \to S$ be the map such that $(x, y)\psi = (a, y)$, for all $x, y \in X$. The $a \times X$ is a homomorphic image of S. For $x, y \in X$, $(a, x) = (a, y)(a, x)$ and $(a, y) = (a, x)(a, y)$, then $(a, x) \mathcal{R} (a, y)$ in $a \times X$. Thus the semigroup $a \times X$ has only one $\mathcal{R}$-class, so $\mathcal{D} = \mathcal{R}$. Hence the semigroup $a \times X$ has only

one $\mathcal{D}$-class which has only one $\mathcal{R}$-class. For x, y $\in$ X, if $(a, x)\mathcal{L}(a, y)$ in a×X, then (a, x) = (a, u)(a, y) for some u $\in$ X, thus x = y. Hence the cardinality of the set of $\mathcal{L}$-classes in the semigroup a×X is $|X| > 1$. Therefore the semigroup a×X has a $\mathcal{D}$-class D such that the cardinality of the set of $\mathcal{R}$-classes in D and the cardinality of the set of $\mathcal{L}$-classes in D are not equal, so by [10, Corollary 2.4], the semigroup a×X is not a regular-* semigroup. #

<u>Example</u>. Let I be a set such that $|I| = 2$, $\mathbb{Z}$ the set of integers, and $S = I\times\mathbb{Z}\times I$. Let $P : I\times I \to \mathbb{Z}$ be the map such that $(a, b)P = p_{ab}$ and $p_{ab} = \begin{cases} 0 & \text{if } a = b, \\ 1 & \text{if } a \neq b. \end{cases}$ Define a multiplication on S by

$$(a, n, b)(c, m, d) = (a, n+p_{bc}+m, d),$$

and define the map * on S by (a, n, b)* = (b, n, a) $(a, b \in I, n \in \mathbb{Z})$. We have shown that S is a *-regular semigroup under involution *. Let T = I×I be the semigroup with the operation defined by (a, b)(c, d) = (a, d). Then T is a semigroup. Let $\psi : S \to T$ be defined by $(a, n, b)\psi = (a, b)$. Then ψ is a homomorphism from S onto T. But we have shown that T is not a *-regular semigroup. This proves that a homomorphic image of a *-regular semigroup need not be *-regular. #

Let S be a *-regular semigroup. Let a $\in$ S. Then there exists x $\in$ S such that a = axa, x = xax, (ax)* = ax and (xa)* = xa. For each a $\in$ S, such x is unique and it is denoted by $a^{\dagger}$ which is called the <u>Moore-Penrose generalized inverse</u> (or, more briefly, the <u>MP inverse</u>)

of a. To show the uniqueness of x, let $a, x, y \in S$ such that $a = axa = aya$, $x = xax$, $y = yay$, $(ax)^* = ax$, $(ay)^* = ay$, $(xa)^* = xa$ and $(ya)^* = ya$. Then $x = xax = xayax = (xa)^*(ya)^*x = ((ya)(xa))^*x = (ya)^*x = yax = yayax = y(ay)^*(ax)^* = y((ax)(ay))^* = y(ay)^* = yay = y$. Observe that if $e \in E(S)$ such that $e^* = e$ (that is, e is a projection of S), then $e^\dagger = e$.

Let S be a proper *-semigroup. Then S is *-regular if and only if for all $a \in S$, aa^* and a^*a are both regular. Let $a \in S$ and assume that aa^* and a^*a are both regular. Then there exist $x, y \in S$ such that $aa^* = aa^*xaa^*$ and $a^*a = a^*aya^*a$, so $aa^* = aa^*xaa^* = aa^*x^*aa^*$ and $a^*a = a^*aya^*a = a^*ay^*a^*a$. Thus $a^*a = a^*(aya^*a) = (a^*ay^*a^*)a = (a^*ay^*a^*)(aya^*a)$, $a^*a = a^*(ay^*a^*a) = (a^*aya^*)a = (a^*aya^*)(ay^*a^*a)$, $aa^* = a(a^*xaa^*) = (aa^*x^*a)a^* = (aa^*x^*a)(a^*xaa^*)$ and $aa^* = a(a^*x^*aa^*) = (aa^*xa)a^* = (aa^*xa)(a^*x^*aa^*)$. Since S is proper, it follows that $a = aya^*a = ay^*a^*a = aa^*x^*a = aa^*xa$. Let $z = a^*xaya^*$. Claim that $z = a^\dagger$. From $aza = aa^*xaya^*a = aya^*a = a$, $zaz = a^*xaya^*aa^*xaya^* = a^*xaya^*aya^* = a^*xaya^* = z$, $az = aa^*xaya^* = aya^* = ay^*a^*aya^* = (aya^*)^*(aya^*) = (az)^*$ and $za = a^*xaya^*a = a^*xa = a^*xaa^*x^*a = (a^*xa)(a^*xa)^* = (za)^*$, it follows that $z = a^\dagger$. Hence S is *-regular.

A *-semigroup S is *-regular if and only if S is regular and S is proper. If S is regular and S is proper, then for all $a \in S$, aa^*, a^*a are regular, and hence, from the above proof, S is *-regular.

It has been shown by Nordahl and Scheilblich in [10] that in any regular-* semigroup S, the involution * fixes one and only one idempotent per $\mathcal{R}$-class, $a\mathcal{R}b$ implies $aa^* = bb^*$ and each $\mathcal{D}$-class D of S are square; that is, the cardinality of the set $\mathcal{R}$-classes in D and the cardinality of the set $\mathcal{L}$-classes in D are equal. Similar results are true for *-regular semigroups.

1.4 Theorem. Let S be a *-regular semigroup. Then for $a \in S$, $aa^\dagger$ is one and only one idempotent in R_a which is fixed by the involution * .

Proof : Let $a \in S$. Then $a = aa^\dagger a$, $a^\dagger = a^\dagger aa^\dagger$, $(aa^\dagger)^* = aa^\dagger$ and $(a^\dagger a)^* = a^\dagger a$. Thus $aa^\dagger \in E(S)$, $(aa^\dagger)^* = aa^\dagger$ and $aa^\dagger \in R_a$. Suppose $e \in E(S)$, $e \in R_a$ such that $e^* = e$. Then $e = e^* \in (R_a)^* = (R_{aa^\dagger})^* = L_{(aa^\dagger)^*} = L_{aa^\dagger}$, so $e, aa^\dagger \in R_{aa^\dagger} \cap L_{aa^\dagger} = H_{aa^\dagger}$ and hence $e = aa^\dagger$. #

The dual of Theorem 1.4 is as follows :

For a *-regular semigroup S, if $a \in S$, then $a^\dagger a$ is one and only one idempotent in L_a which is fixed by the involution *.

1.5 Corollary. Let S be a *-regular semigroup, Then for $a, b \in S$, $a\mathcal{R}b$ if and only if $aa^\dagger = bb^\dagger$.

Proof : Let $(a, b) \in \mathcal{R}$. Because $(a, aa^\dagger) \in \mathcal{R}$ and $(b, bb^\dagger) \in \mathcal{R}$. It follows that $(aa^\dagger, bb^\dagger) \in \mathcal{R}$. Since $aa^\dagger, bb^\dagger \in E(S)$, $(aa^\dagger)^* = aa^\dagger$ and $(bb^\dagger)^* = bb^\dagger$, it follows by Theorem 1.4 that, $aa^\dagger = bb^\dagger$.

The converse follows from the fact that for all $a \in S$, $a\mathcal{R}aa^\dagger$. #

By the dual of Theorem 1.4, we also have that in a *-regular semigroup S, for a, b ∈ S, $a \mathcal{L} b$ if and only if $a^\dagger a = b^\dagger b$.

1.6 Corollary. For any $\mathcal{D}$-class D of a *-regular semigroup S, the set of $\mathcal{R}$-classes in D and the set of $\mathcal{L}$-classes in D have the same cardinality.

Proof : Let a ∈ S. Let $C = \{e \in D_a \mid e = e^2 = e^*\}$, $A = \{L_x \mid x \in D_a\}$ and $B = \{R_x \mid x \in D_a\}$. Then by Theorem 1.4, $A = \{L_e \mid e \in C\}$, $B = \{R_e \mid e \in C\}$, and $|A| = |B|$ by the map $L_e \longmapsto R_e$ $(e \in C)$. #

It was shown by Nardahl and Scheiblich in [10, Theorem 2.5] that the product of two projections in a regular-* semigroup is an idempotent. However, the product of two projections in a *-regular semigroup need not be an idempotent.

Example. Let I be a set such that $|I| = 2$, $\mathbb{Z}$ the set of integers. Let $S = I \times \mathbb{Z} \times I$ and $P : I \times I \to \mathbb{Z}$ the map such that $(a, b)P = p_{ab}$ and

$$p_{ab} = \begin{cases} 0 & \text{if } a = b, \\ 1 & \text{if } a \neq b. \end{cases}$$

Define a multiplication on S by

$$(a, n, b)(c, m, d) = (a, n+p_{bc}+m, d),$$

and define the map * on S by $(a, n, b)^* = (b, n, a)$ $(a, b \in I, n \in \mathbb{Z})$. We have shown that S is a *-regular semigroup. Let a, b be two distinct elements in I. Then (a, 0, a) and (b, 0, b) are projections of S. But (a, 0, a)(b, 0, b) = (a, 1, b) which is not an idempotent of S. #

In any *-regular semigroup S, it is true that $(a^{\dagger})^{\dagger} = a$ for all $a \in S$, but it is not true that $(ab)^{\dagger} = b^{\dagger}a^{\dagger}$ for all $a, b \in S$. From the above example, $(a, 0, a)(a, 0, a)(a, 0, a) = (a, 0, a)$, $((a, 0, a)(a, 0, a))^* = (a, 0, a)(a, 0, a)$, $(b, 0, b)(b, 0, b)(b, 0, b) = (b, 0, b)$, $((b, 0, b)(b, 0, b))^* = (b, 0, b)(b, 0, b)$, $(a, 1, b) = (a, 1, b)(b, -1, a)(a, 1, b)$, $(b, -1, a) = (b, -1, a)(a, 1, b)(b,-1, a)$, $((a, 1, b)(b, -1, a))^* = (a, 1, b)(b, -1, a)$ and $((b, -1, a)(a, 1, b))^* = (b, -1, a)(a, 1, b)$, we have that $(a, 0, a)^{\dagger} = (a, 0, a)$, $(b, 0, b)^{\dagger} = (b, 0, b)$ and $(a, 1, b)^{\dagger} = (b, -1, a)$. But $(b, -1, a) = (a, 1, b)^{\dagger} = ((a, 0, a)(b, 0, b))^{\dagger}$ and $(b, 1, a) = (b, 0, b)(a, 0, a) = (b, 0, b)^{\dagger}(a, 0, a)^{\dagger}$, so $((a, 0, a)(b, 0, b))^{\dagger} \neq (b, 0, b)^{\dagger}(a, 0, a)^{\dagger}$.

1.7 <u>Theorem.</u> Let S be a *-regular semigroup. Then for $a \in S$,

(i) $(a^*)^{\dagger} = (a^{\dagger})^*$,

(ii) $(a^{\dagger})^{\dagger} = a$,

(iii) $(aa^{\dagger})^{\dagger} = aa^{\dagger}$ and $(a^{\dagger}a)^{\dagger} = a^{\dagger}a$.

<u>Proof</u> : (i) Let $a \in S$. Then $aa^{\dagger}a = a$, $a^{\dagger}aa^{\dagger} = a^{\dagger}$, $(aa^{\dagger})^* = aa^{\dagger}$ and $(a^{\dagger}a)^* = a^{\dagger}a$ and thus $a^* = (aa^{\dagger}a)^* = a^*(a^{\dagger})^*a^*$, $(a^{\dagger})^* = (a^{\dagger}aa^{\dagger})^* = (a^{\dagger})^*a^*(a^{\dagger})^*$, $(a^*(a^{\dagger})^*)^* = a^{\dagger}a = (a^{\dagger}a)^* = a^*(a^{\dagger})^*$ and $((a^{\dagger})^*a^*)^* = aa^{\dagger} = (aa^{\dagger})^* = (a^{\dagger})^*a^*$. Hence $(a^*)^{\dagger} = (a^{\dagger})^*$.

(ii) For $a \in S$, $a^{\dagger}aa^{\dagger} = a^{\dagger}$, $aa^{\dagger}a = a$, $(a^{\dagger}a)^* = a^{\dagger}a$ and $(aa^{\dagger})^* = aa^{\dagger}$, it follows that $(a^{\dagger})^{\dagger} = a$.

(iii) For $a \in S$, $aa^{\dagger}$ and $a^{\dagger}a$ are projections, and hence $(a^{\dagger}a)^{\dagger} = a^{\dagger}a$ and $(aa^{\dagger})^{\dagger} = aa^{\dagger}$. #

By an <u>ordered semigroup</u> we shall mean a semigroup S on which is defined a partial order $\leq$ in such a way that for any a, b of S, $a \leq b$ implies $ax \leq bx$ and $xa \leq xb$ for each element x of S.

Let S be an inverse semigroup. The relation $\leq$ on S defined by $a \leq b \Leftrightarrow aa^{-1} = ab^{-1}$ is a partial order on S and it is called the <u>natural partial order</u> on S. The inverse semigroup S is an ordered semigroup under the natural partial order. If an inverse semigroup S is a *-semigroup, then the natural partial order $\leq$ on S satisfies the property that for a, b $\in$ S, $a \leq b$ implies $a^* \leq b^*$ because $a \leq b$ if and only if $a^{-1} \leq b^{-1}$.

Let P be a partially ordered set. The greatest lower bound of the subset $\{a_i \mid i \in \Lambda\}$ of P, if it exists, is denoted by $\bigwedge_{i\in\Lambda} a_i$ and $\bigvee_{i\in\Lambda} a_i$ denotes the least upper bound of the subset $\{a_i \mid i \in \Lambda\}$ of P, if it exists. For a, b $\in$ P, $a \wedge b$ and $a \vee b$ denote the greatest lower bound of {a, b} and the least upper bound of {a, b}; respectively.

1.8 <u>Theorem</u>. Let S be an ordered *-semigroup with the property that $a \leq b$ implies $a^* \leq b^*$. Let $\{a_i \mid i \in I\}$ be a subset of S. Then the following hold :

(i) If $\bigwedge_{i\in I} a_i$ exists, then $\bigwedge_{i\in I} a_i^*$ exists and $(\bigwedge_{i\in I} a_i)^* = \bigwedge_{i\in I} a_i^*$.

(ii) If $\bigvee_{i\in I} a_i$ exists, then $\bigvee_{i\in I} a_i^*$ exists and $(\bigvee_{i\in I} a_i)^* = \bigvee_{i\in I} a_i^*$.

<u>Proof</u> : (i) Assume that $\bigwedge_{i\in I} a_i$ exists. Because $\bigwedge_{i\in I} a_i \leq a_j$ for all j in I, $(\bigwedge_{i\in I} a_i)^* \leq a_j^*$ for all j in I, so $(\bigwedge_{i\in I} a_i)^*$ is a lower

bound of the set $\{a_i^* \mid i \in I\}$. Let c be a lower bound of the set $\{a_i^* \mid i \in I\}$. Then $c \leq a_j^*$ for all j in I, so $c^* \leq a_j$ for all j in I. Therefore c^* is a lower bound of the set $\{a_i \mid i \in I\}$. Thus $c^* \leq \bigwedge_{i\in I} a_i$, so $c \leq (\bigwedge_{i\in I} a_i)^*$. This proves that $(\bigwedge_{i\in I} a_i)^* = \bigwedge_{i\in I} a_i^*$.

(ii) Assume that $\bigvee_{i\in I} a_i$ exists. Then $a_j \leq \bigvee_{i\in I} a_i$ for all $j \in I$. Thus $a_j^* \leq (\bigvee_{i\in I} a_i)^*$ for all $j \in I$, so $(\bigvee_{i\in I} a_i)^*$ is an upper bound of the set $\{a_i^* \mid i \in I\}$. Let c be an upper bound of the set $\{a_i^* \mid i \in I\}$. Then $a_j^* \leq c$ for all $j \in I$, so $a_j \leq c^*$ for all $j \in I$. Hence c^* is an upper bound of $\{a_i \mid i \in I\}$ which implies $\bigvee_{i\in I} a_i \leq c^*$, and therefore $(\bigvee_{i\in I} a_i)^* \leq c$. This shows that $(\bigvee_{i\in I} a_i)^*$ is the least upper bound of $\{a_i^* \mid i \in I\}$. Hence $(\bigvee_{i\in I} a_i)^* = \bigvee_{i\in I} a_i^*$. #

1.9 <u>Corollary</u>. Let S be an ordered *-semigroup with the property that $a \leq b$ implies $a^* \leq b^*$. Then the following hold :

(i) For $a, b \in S$, if $a \wedge b$ exists, then $a^* \wedge b^*$ exists and $(a\wedge b)^* = a^* \wedge b^*$.

(ii) For $a, b \in S$, if $a \vee b$ exists, then $a^* \vee b^*$ exists and $(a \vee b)^* = a^* \vee b^*$.