CHAPTER IV

MULTIPLICATIVE INTERVAL SEMIGROUPS IN $\mathbb{R}$

In this chapter we are concerned with intervals in $\mathbb{R}$ which are closed under usual multiplication. The purpose is to characterize multiplicative interval semigroups in $\mathbb{R}$ admitting the structure of an AC semiring with zero.

The first proposition gives all of the types of multiplicative interval semigroups in $\mathbb{R}$.

4.1 Proposition. For $S \subseteq \mathbb{R}$, S is a multiplicative interval semigroup in $\mathbb{R}$ if and only if S is one of the following types :

(1) $\mathbb{R}$, (2) $\{0\}$, (3) $\{1\}$, (4) $(0,\infty)$, (5) $[0,\infty)$,

(6) (a,∞) where $a \geq 1$,

(7) $[a,\infty)$ where $a \geq 1$,

(8) $(0,b)$ where $0 < b \leq 1$,

(9) $(0,b]$ where $0 < b \leq 1$,

(10) $[0,b)$ where $0 < b \leq 1$,

(11) $[0,b]$ where $0 < b \leq 1$,

(12) (a,b) where $-1 \leq a < 0 < a^2 \leq b \leq 1$,

(13) $(a,b]$ where $-1 \leq a < 0 < a^2 \leq b \leq 1$,

(14) $[a,b)$ where $-1 < a < 0 < a^2 < b \leq 1$,

(15) $[a,b]$ where $-1 \leq a < 0 < a^2 \leq b \leq 1$.

Proof : Let S be a multiplicative interval semigroup in $\mathbb{R}$.

Let

$$a = \begin{cases} \inf S & \text{if } S \text{ is bounded below,} \\ -\infty & \text{if } S \text{ is not bounded below,} \end{cases}$$

$$b = \begin{cases} \sup S & \text{if } S \text{ is bounded above,} \\ \infty & \text{if } S \text{ is not bounded above.} \end{cases}$$

Since S is an interval in $\mathbb{R}$, $(a,b) \subseteq S$.

<u>Case $a = -\infty$ and $b = \infty$</u>. Then $(a,b) = \mathbb{R}$, so $S = \mathbb{R}$. Thus S is of type (1).

<u>Case $a = -\infty$ and $b < \infty$</u>. Then $(a,b) = (-\infty,b)$, so

$$S = \begin{cases} (-\infty,b) & \text{if } b \notin S , \\ (-\infty,b] & \text{if } b \in S . \end{cases}$$

Since $-(|b|+1) \in S$ and S is closed under multiplication, we have $(-(|b|+1))^2 \in S$. But $(-(|b|+1))^2 = |b|^2 + 2|b| + 1 > 2|b| \geq b$, a contradiction. Hence this case cannot occur.

<u>Case $-\infty < a$ and $b = \infty$</u>. Then $(a,b) = (a,\infty)$, so

$$S = \begin{cases} (a,\infty) & \text{if } a \notin S , \\ [a,\infty) & \text{if } a \in S. \end{cases}$$

If $a < 0$, then $4, \frac{a}{2} \in S$, so $4(\frac{a}{2}) = 2a \in S$, a contradiction since $2a < a$. Then $a \geq 0$. Suppose $0 < a < 1$. Then there exists an element $c \in S$ such that $a \leq c < 1$, so $c^k < a$ for some positive integer k, a contradiction since $c^k \in S$ and $a = \inf S$. Hence $a = 0$ or $a \geq 1$. Thus S is of type (4),(5),(6) or (7).

<u>Case $-\infty < a$ and $b < \infty$</u>. Then

$$S = \begin{cases} (a,b) & \text{if } a \notin S \text{ and } b \notin S, \\ (a,b] & \text{if } a \notin S \text{ and } b \in S, \\ [a,b) & \text{if } a \in S \text{ and } b \notin S, \\ [a,b] & \text{if } a \in S \text{ and } b \in S. \end{cases}$$

Claim that ($-1 \leqslant a < 0$ or $a = 0$ or $a = 1$) and $0 \leqslant b \leqslant 1$. If $a < -1$, then there exists an element x in S such that $a \leqslant x < -1$, so $(x^2)^n > b$ for some positive integer n, a contradiction since $(x^2)^n \in S$ and b = sup S. If $b > 1$, then there exists an element $y \in S$ such that $1 < y \leqslant b$, so $y^m > b$ for some positive integer m, a contradiction since $y^m \in S$ and b = sup S. Hence $-1 \leqslant a \leqslant b \leqslant 1$. If $b < 0$, then there exists an element $c \in S$ such that $c \leqslant b < 0$, so $c^2 \in S$ and $c^2 > b$, a contradiction. If $0 < a < 1$, then there exists an element $d \in S$ such that $a \leqslant d < 1$, so $d^k < a$ for some positive integer k, a contradiction since $d^k \in S$ and a = inf S. Hence we prove the claim. Therefore we have ($a = 0$ and $0 \leqslant b \leqslant 1$) or $a = b = 1$ or $-1 \leqslant a < 0 \leqslant b \leqslant 1$.

<u>Subcase 1 : $a = 0$, $0 \leqslant b \leqslant 1$.</u> If $b = 0$, then $a = b = 0$, so S = {0}, hence S is of type (2). If $0 < b \leqslant 1$, then S is of type (8), (9),(10) or (11).

<u>Subcase 2 : $a = b = 1$.</u> Then S = {1}, so S is of type (3).

<u>Subcase 3 : $-1 \leqslant a < 0 \leqslant b \leqslant 1$, $a \in S$ and $b \in S$.</u> Then S = [a,b]. Since $a \in S = [a,b]$ and S is closed under multiplication, we have $a^2 \leqslant b$, hence $-1 \leqslant a < 0 < a^2 \leqslant b \leqslant 1$. Thus S is of type (15).

<u>Subcase 4 : $-1 \leqslant a < 0 \leqslant b \leqslant 1$, $a \in S$ and $b \notin S$.</u> Then S = [a,b). Since $a \in S = [a,b)$ and S is closed under multiplication, we have $a^2 < b$. Since $b \notin S$, $1 \notin S$, so $a \neq -1$. Hence $-1 < a < 0 < a^2 < b \leqslant 1$. Therefore S is of type (14).

Subcase 5 : $-1 \leq a < 0 \leq b \leq 1$, $a \notin S$ and $b \in S$. Then $S = (a,b]$. To show $a^2 \leq b$, suppose $a^2 > b$. Then $-a = |a| = \sqrt{a^2} > \sqrt{b}$, so $a < -\sqrt{b}$. Then there exists an element $c \in S$ such that $a < c < -\sqrt{b}$ since $a = \inf S \notin S$. Then $c^2 > b$, a contradiction since $c^2 \in S$ and $b = \sup S$. Hence $-1 \leq a < 0 < a^2 \leq b \leq 1$, so S is of type (13).

Subcase 6 : $-1 \leq a < 0 \leq b \leq 1$, $a \notin S$ and $b \notin S$. Then $S = (a,b)$. The proof that $a^2 \leq b$ can be given the same as that of the fifth subcase Hence $-1 \leq a < 0 < a^2 \leq b \leq 1$, so S is of type (12).

The converse follows easily from the property of multiplication in $\mathbb{R}$. #

Next, we shall determine the multiplicative interval semigroups in $\mathbb{R}$ of which types in Proposition 4.1 which admit the structure of an AC semiring with zero. The result is in the next theorem. First we introduce the following four lemmas.

4.2 Lemma. If S is a multiplicative semigroup of nonnegative real numbers, then S admits the structure of an AC semiring with zero.

Proof : Let S be a multiplicative semigroup of nonnegative real numbers. Define $\oplus$ on S by

$$x \oplus y = \max\{x,y\},$$

$$x \oplus 0 = 0 \oplus x = x$$

for all x,y in S. Then $(S^0,\oplus)$ is a commutative semigroup with identity 0. To show the multiplication in S^0 is distributive over $\oplus$, let $x,y,z \in S^0$. If $x = 0$, $y = 0$ or $z = 0$, we clearly obtain $x(y \oplus z) = xy \oplus xz$. Assume $x,y,z \in S$. Then

$$x(y \oplus z) = \begin{cases} xy & \text{if } y \geq z, \\ xz & \text{if } z \geq y. \end{cases}$$

If $y \geq z$, then $xy \geq xz$ since $x \geq 0$, so $xy \oplus xz = xy$. Similarly if $z \geq y$, then $xy \oplus xz = xz$. Hence $x(y \oplus z) = xy \oplus xz$. #

4.3 <u>Lemma</u>. If S is a multiplicative interval semigroup in $\mathbb{R}$ of the form $[a,1]$, where $-1 \leq a < 0 < a^2 \leq 1$, then S does not admit the structure of an AC semiring with zero.

<u>Proof</u> : Suppose S admits the structure of an AC semiring with zero under an addition $\oplus$. Since $-1 \leq a < 0 < a^2 \leq 1$, $[-|a|,|a|] = [a,|a|] \subseteq [a,1] = S$. Claim that $x \oplus (-x) = 0$ for all $x \in [-|a|,|a|]$. Let $x \in [-|a|,|a|]$ and $x \neq 0$. So $x \oplus (-x) = \alpha$ for some $\alpha \in S$. Then $x^2 \oplus (-x^2) = x(x \oplus (-x)) = x\alpha$ and $(-x^2) \oplus x^2 = -x(x \oplus (-x)) = -x\alpha$. Since $x^2 \oplus (-x^2) = (-x^2) \oplus x^2$, $x\alpha = -x\alpha$ which implies that $\alpha = -\alpha$ since $x \neq 0$, so $\alpha = 0$. Hence we have the claim.

Since $\frac{|a|}{2} \in S$, $\frac{|a|S}{2}$ is a subsemiring of S. But $\frac{|a|S}{2} = [\frac{|a|a}{2}, \frac{|a|}{2}] = [\frac{-a^2}{2}, \frac{|a|}{2}]$, so $[\frac{-a^2}{2}, \frac{|a|}{2}]$ is a subsemiring of S containing 0 and $1 \notin \frac{|a|S}{2}$. If $x,y \in (0, \frac{a^2}{2}]$ such that $1 \oplus x = 1 \oplus y$ in S, then $x \oplus x^2 = x \oplus xy$ which implies $x^2 = -x \oplus x \oplus x^2 = -x \oplus x \oplus xy = xy$ since $x \in (0, \frac{a^2}{2}] \subseteq [-|a|,|a|]$, so $x = y$ (since $x \neq 0$). This shows that for $x,y \in (0, \frac{a^2}{2}]$ if $x \neq y$, then $1 \oplus x \neq 1 \oplus y$ in S. It follows that there exists an element c in $(0, \frac{a^2}{2}]$ such that $1 \oplus c \neq a$ and $1 + c \neq 1$. Therefore $-1 \leq a < 1 \oplus c < 1$. So $(1 \oplus c)^n \in [\frac{-a^2}{2}, \frac{|a|}{2}]$ for some positive integer n. But $(1 \oplus c)^n = 1 \oplus \binom{n}{1}^* c \oplus \binom{n}{2}^* c^2 \oplus \ldots$

$\oplus \binom{n}{n-1}^{*} c^{n-1} \oplus \binom{n}{n}^{*} c^{n}$ where $k^{*} c^{i}$ means $c^{i} \oplus c^{i} \oplus \ldots \oplus c^{i}$ k - times, so $(1 \oplus c)^{n} = 1 \oplus cx$ for some $x \varepsilon S$. Since $0 < c \leqslant \frac{a^2}{2}$ and $-1 \leqslant x \leqslant 1$, we have that $-\frac{a^2}{2} \leqslant cx \leqslant \frac{a^2}{2}$, so $cx \varepsilon [-\frac{a^2}{2}, \frac{a^2}{2}] \subseteq \frac{|a|S}{2}$. Then $-cx \varepsilon [-\frac{a^2}{2}, \frac{a^2}{2}] \subseteq \frac{|a|S}{2}$. Thus $1 = 1 \oplus cx \oplus (-cx) = (1 \oplus c)^{n} \oplus (-cx) \varepsilon \frac{|a|S}{2}$ since $(1 \oplus c)^{n}, -cx \varepsilon \frac{|a|S}{2}$ and $\frac{|a|S}{2}$ is a subsemiring of S, a contradiction. #

4.4 <u>Lemma</u>. If S is a multiplicative interval semigroup in $\mathbb{R}$ of the form $[a,b]$, where $-1 \leqslant a < 0 < a^2 \leqslant b \leqslant 1$, then S does not admit the structure of an AC semiring with zero.

<u>Proof</u> : Since $a < 0$, $S = [a,b] = [-|a|,b]$.

<u>Case $|a| \leqslant b$.</u> Suppose there exists an operation $\oplus$ on S such that $(S,\oplus,\cdot)$ is an AC semiring with zero where $\cdot$ is the usual multiplication on S. Let $\bar{a} = \frac{a}{b}$. Then $S = [a,b] = [\bar{a}b,b]$ and $-1 \leqslant \bar{a} < 0 < \bar{a}^2 \leqslant 1$. Define an operation $\oplus'$ on $[\bar{a},1]$ by $x \oplus' y = \frac{bx \oplus by}{b}$ for all $x,y \varepsilon [\bar{a},1]$. To show $\oplus'$ is associative, let x,y and $z \varepsilon [\bar{a},1]$. Then

$$x \oplus' (y \oplus' z) = x \oplus' \frac{(by \oplus bz)}{b}$$

$$= \frac{bx \oplus \frac{b(by \oplus bz)}{b}}{b}$$

$$= \frac{bx \oplus (by \oplus bz)}{b}$$

$$= \frac{(bx \oplus by) \oplus bz}{b} \quad \text{(since } \oplus \text{ is associative on S)}$$

$$= \frac{\frac{b(bx \oplus by)}{b} \oplus bz}{b}$$

$$= \frac{(bx \oplus by)}{b} \oplus' z$$

$$= (x \oplus' y) \oplus' z$$

Next, to show the multiplication is distributive over $\oplus'$ in $[\bar{a},1]$, let x,y and $z \in [\bar{a},1]$. Then

$$x(y \oplus' z) = \frac{x(by \oplus bz)}{b}$$

$$= \frac{bx(by \oplus bz)}{b^2}$$

$$= \frac{(bx)(by) \oplus (bx)(bz)}{b^2}$$ (since the multiplication is distributive over $\oplus$ in S)

$$= \frac{b(bxy) \oplus b(bxz)}{b^2}$$

$$= \frac{b(bxy \oplus bxz)}{b^2}$$

$$= \frac{bxy \oplus bxz}{b}$$

$$= xy \oplus' xz$$

Hence the multiplicative interval semigroup $[\bar{a},1]$ admits the structure of an AC semiring with zero which contradicts to Lemma 4.3.

Case $|a| > b$. Then $aS = [ab,a^2]$ and $|ab| = |a||b| < |a|^2 = a^2$. By the first case, aS does not admit the structure of an AC semiring with zero. If there exists an operation $\oplus$ on S such that $(S,\oplus,\cdot)$ is an AC semiring with zero where $\cdot$ is the multiplication on S, then aS is a subsemiring of $(S,\oplus,\cdot)$ containing 0 which implies that aS admits the structure of an AC semiring with zero. But aS does not admit the structure of an AC semiring with zero, so S does not admit the structure of an AC semiring with zero. #

4.5 Lemma. Let S be a multiplicative interval semigroup in $\mathbb{R}$ of one of the following forms :

(a,b) where $-1 \leqslant a < 0 < a^2 \leqslant b \leqslant 1$,

$(a,b]$ where $-1 \leqslant a < 0 < a^2 \leqslant b \leqslant 1$,

$[a,b)$ where $-1 < a < 0 < a^2 < b \leqslant 1$.

Then S does not admit the structure of an AC semiring with zero.

Proof : Suppose there exists an operation $\oplus$ on S such that $(S,\oplus,\cdot)$ is an AC semiring with zero where $\cdot$ is the multiplication on S. Claim that $[\frac{ab}{2}, \frac{b^2}{2}]$ is a subsemiring of S. Let $d = \min\{b-\frac{b^2}{2}, \frac{ab}{2}-a\}$. Then $d > 0$ and $\frac{b}{2}+\varepsilon \in S$ for every ε such that $0 < \varepsilon < d$. Hence $(\frac{b}{2}+\varepsilon)S$ is a subsemiring of $(S,\oplus,\cdot)$ containing 0 for every ε such that $0 < \varepsilon < d$ which implies that $\bigcap_{0<\varepsilon<d} (\frac{b}{2}+\varepsilon)S$ is a subsemiring of $(S,\oplus,\cdot)$. To show that $\bigcap_{0<\varepsilon<d} (\frac{b}{2}+\varepsilon)S = [\frac{ab}{2}, \frac{b^2}{2}]$, clearly $[\frac{ab}{2}, \frac{b^2}{2}] \subseteq \bigcap_{0<\varepsilon<d} (\frac{b}{2}+\varepsilon)S$ since

$$(\frac{b}{2}+\varepsilon)S = \begin{cases} (\frac{ab}{2}+a\varepsilon, \frac{b^2}{2}+b\varepsilon) & \text{if } S = (a,b), \\ (\frac{ab}{2}+a\varepsilon, \frac{b^2}{2}+b\varepsilon] & \text{if } S = (a,b], \\ [\frac{ab}{2}+a\varepsilon, \frac{b^2}{2}+b\varepsilon) & \text{if } S = [a,b). \end{cases}$$

and $a\varepsilon < 0$, $b\varepsilon > 0$ for all ε such that $0 < \varepsilon < d$. Conversely, let $x \in S \setminus [\frac{ab}{2}, \frac{b^2}{2}]$, so $x < \frac{ab}{2}$ or $x > \frac{b^2}{2}$.

Case $x < \frac{ab}{2}$. Let $\varepsilon_0 = \min\{\frac{1}{2}(\frac{ab}{2}-x), \frac{d}{2}\}$. Then $0 < \varepsilon_0 < d$, and

$$\frac{ab}{2}+a\varepsilon_0 \geqslant \frac{ab}{2}-\varepsilon_0 \geqslant \frac{ab}{2}-(\frac{1}{2}(\frac{ab}{2}-x)) = \frac{ab}{2}-\frac{ab}{4}+\frac{x}{2} = \frac{ab}{4}+\frac{x}{2} > \frac{x}{2}+\frac{x}{2}$$

$= x$ since $-1 \leqslant a$ and $\frac{x}{2} < \frac{ab}{4}$, so $x \notin (\frac{b}{2} + \mathcal{E}_\circ)S$ and thus

$x \notin \bigcap_{0<\mathcal{E}<d} (\frac{b}{2} + \mathcal{E})S$.

Case $x > \frac{b^2}{2}$. Let $\mathcal{E}_\circ = \min\{\frac{1}{2}(x - \frac{b^2}{2}), \frac{d}{2}\}$. Then $0 < \mathcal{E}_\circ < d$ and $\frac{b^2}{2} + b\mathcal{E}_\circ \leqslant \frac{b^2}{2} + \mathcal{E}_\circ \leqslant \frac{b^2}{2} + \frac{1}{2}(x - \frac{b^2}{2}) = \frac{b^2}{2} + \frac{x}{2} - \frac{b^2}{4} = \frac{b^2}{4} + \frac{x}{2} < \frac{x}{2} + \frac{x}{2} = x$ since $0 < b \leqslant 1$ and $\frac{x}{2} > \frac{b^2}{4}$, so $x \notin (\frac{b}{2} + \mathcal{E}_\circ)S$, hence $x \notin \bigcap_{0<\mathcal{E}<d} (\frac{b}{2} + \mathcal{E})S$.

Hence $\bigcap_{0<\mathcal{E}<d} (\frac{b}{2} + \mathcal{E})S = [\frac{ab}{2}, \frac{b^2}{2}]$ is a subsemiring of S which contradicts to Lemma 4.4. #

4.6 Theorem. Let S be a multiplicative interval semigroup in $\mathbb{R}$. Then S admits the structure of an AC semiring with zero if and only if S is one of the following types :

(1) $\mathbb{R}$, (2) $\{0\}$, (3) $\{1\}$, (4) $(0,\infty)$, (5) $[0,\infty)$,

(6) (a,∞) where $a \geqslant 1$,

(7) $[a,\infty)$ where $a \geqslant 1$,

(8) $(0,b)$ where $0 < b \leqslant 1$,

(9) $(0,b]$ where $0 < b \leqslant 1$,

(10) $[0,b)$ where $0 < b \leqslant 1$,

(11) $[0,b]$ where $0 < b \leqslant 1$.

Proof : If S is a multiplicative interval semigroup in $\mathbb{R}$, then from Proposition 4.1, S is of type (1) - (15). If S is of type (15), then from Lemma 4.4 S does not admit the structure of an AC semiring with zero. If S is of type (12), (13) or (14), then from Lemma 4.5, S does not admit the structure of an AC semiring with zero. Hence if S admits the structure of AC semiring with zero, then S is one of type (1) - (11).

The converse holds from Lemma 4.2.