CHAPTER II

EXAMPLES AND GENERAL PROPERTIES

The purpose of this chapter is to examine whether well-known semigroups and groups admit the structure of an AC semiring with zero. Also, some general properties of semigroups admitting such structure are given.

Since a ring is an additively commutative semiring with zero, it follows that every semigroup admitting a ring structure admits the structure of an AC semiring with zero. It was shown in [6] that every zero semigroup admits a ring structure, and a right [left] zero semigroup S admits a ring structure if and only if $|S| = 1$. Then every zero semigroup admits the structure of an AC semiring with zero. The first proposition shows that every right [left] zero semigroup S admits the structure of an AC semiring with zero.

2.1 Proposition. Every right [left] zero semigroup admits the structure of an AC semiring with zero.

Proof : Let S be a right zero semigroup. Then $ab = b$ for all $a,b \in S$. If $|S| = 1$, then S admits the structure of an AC semiring with zero. Assume $|S| > 1$. Then $S \subsetneq S^0$. Let z be a fixed element in S, and define an operation + on S by

$$a + b = \begin{cases} z & \text{if } a \neq b, \\ a & \text{if } a = b. \end{cases}$$

Then $(S,+)$ is a Kronecker semigroup having z as a zero. Extend $+$ on S to S^o by defining $x + 0 = 0 + x = x$ for all $x \in S^o$. Then $(S^o,+)$ is a commutative semigroup. To show the operation of S^o is distributive over $+$, let a,b and $c \in S^o$. If $a = 0$ or $b = 0$ or $c = 0$, it is clearly seen that $a(b+c) = ab + ac$ and $(b+c)a = ba + ca$. If each of a,b and c is not a zero (i.e., $a,b,c \in S$), then $a(b+c) = b + c = ab + ac$ and $(b+c)a = a = a + a = ba + ca$. Hence $(S^o,+,\cdot)$ is an AC semiring with zero, where $\cdot$ is the operation of the semigroup S^o. #

Not every semigroup admits the structure of an AC semiring with zero. It is shown by Proposition 2.2 - Proposition 2.5 .

2.2 <u>Proposition</u>. A Kronecker semigroup S admits the structure of an AC semiring with zero if and only if $|S| \leqslant 2$.

<u>Proof</u> : Let S be a Kronecker semigroup with zero 0. Then for $a,b \in S$, $ab = \begin{cases} 0 & \text{if } a \neq b, \\ a & \text{if } a = b. \end{cases}$ If $|S| = 2$, then S is isomorphic to the multiplicative semigroup $\mathbb{Z}_2$, so it admits a ring structure. Therefore, if $|S| \leqslant 2$, then S admits the structure of an AC semiring with zero.

Conversely, assume that S admits the structure of an AC semiring with zero. Suppose $|S| > 2$. Let a and b be two distinct nonzero elements of S. Then $a + b = c$ for some $c \in S$. If $c = a$, then $b(a+b) = bc = ba = 0$, so $0 = ba + b^2 = b$, a contradiction. If $c \neq a$, then $0 = ac = a(a+b) = a^2 + ab = a + 0 = a$, a contradiction. Hence $|S| \leqslant 2$. #

2.3 Proposition. The Klein's four group does not admit the structure of an AC semiring with zero.

Proof : Let $K = \{1, a, b, c\}$ be the Klein's four group with identity 1. Then $a^2 = b^2 = c^2 = 1$, $ab = ba = c$, $bc = cb = a$, $ca = ac = b$. Suppose K admits the structure of an AC semiring with zero under an addition +. Then $b + c \in K^o$.

Case $b + c \neq 0$. Then $a(b+c) = ab + ac = c + b = b + c$ which implies that $a = 1$, a contradiction.

Case $b + c = 0$. Since $a = a + (b+c) = (a+b) + c$ and $a \neq c$, we have that $a + b \neq 0$. Then $c(a+b) = ca + cb = b + a = a + b$ which implies that $c = 1$, a contradiction. #

2.4 Proposition. For any positive integer $n > 1$, the dihedral group D_n does not admit the structure of an AC semiring with zero.

Proof : Let n be a positive integer, $n > 1$ and D_n the dihedral group with identity 1. Then there exist a,b in D_n such that $a \neq b$ and $D_n = \{1, a, a^2, \ldots, a^{n-1}, b, ab, a^2b, \ldots, a^{n-1}b\}$ where $a^n = b^2 = 1$ and $ba = a^{n-1}b$. If $n = 2$, then D_2 is the Klein's four group which does not admit the structure of an AC semiring with zero. Assume $n > 2$. Suppose D_n admits the structure of an AC semiring with zero under an addition +. Then $a + b \in D_n^o$.

Case $a + b \neq 0$. Then $b(a+b) = ba + b^2 = a^{n-1}b + a^n = a^{n-1}(b+a) = a^{n-1}(a+b)$ which implies $b = a^{n-1}$, a contradiction.

Case $a + b = 0$. Since $a^{n-1} = (a+b) + a^{n-1} = a + (b+a^{n-1})$ and $a \neq a^{n-1}$ (since $n > 2$), we have that $b + a^{n-1} \neq 0$. Then $(b+a^{n-1})b = b^2 + a^{n-1}b = a^n + ba = (a^{n-1}+b)a = (b+a^{n-1})a$ which implies $b = a$, a contradiction. #

Remark that the dihedral group D_1 is the cyclic group of order 2 which admits a ring structure since D_1^o is isomorphic to $(\mathbb{Z}_3, \cdot)$ where $\cdot$ is the multiplication in $\mathbb{Z}_3$, so D_1 admits the structure of an AC semiring with zero.

2.5 Proposition. The quaternion group does not admit the structure of an AC semiring with zero.

Proof : Let G be the quarternion group with identity 1 where $G = \{1, -1, i, -i, j, -j, k, -k\}$, $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. Suppose G admits the structure of an AC semiring with zero under an addition + . Then $1 + (-1) = a$ for some $a \in G^o$. Hence $-a = (-1)a = (-1)(1 + (-1)) = (-1) + 1 = a$, so $a = 0$, which implies $x + (-x) = 0$ for all x in G^o. Let $1 + i = b$ for some b in G^o. Then $b \neq 1$, $b \neq i$, $b \neq 0$ and $b^2 = (1+i)^2 = i + i$ since $x +(-x) = 0$ for all $x \in G^o$. Since $ib = i(1+i) = i + (-1) = -1 + i$, $bib = (1+i)(-1+i) = (-1) + (-1)$. Therefore $-ibib = -i((-1) + (-1)) = i + i = b^2$ which implies that $-ibi = b$. It then follows that $b \neq j$, $b \neq -j$, $b \neq k$ and $b \neq -k$. If $b = -1$, then $j = (-j)(-1) = (-j)b = (-j)(1+i) = -j + k = k + (-j) = k + ik = (1+i)k = bk = (-1)k = -k$, a contradiction. If $b = -i$, then $-k = (-j)(-i) = (-j)b = (-j)(1+i) = -j + k = k + (-j) = k + ik = (1+i)k = bk = -ik = j$, a contradiction. #

Next we show that every cyclic group and every cyclic semigroup admits the structure of an AC semiring with zero.

2.6 Proposition. Every cyclic group admits the structure of an AC semiring with zero.

Proof : Let C be a cyclic group with a generator a. Then $C = \{a^n \mid n \in \mathbb{Z}\}$ where $\mathbb{Z}$ is the set of integers. If C is infinite, then $a^i \neq a^j$ for i,j in $\mathbb{Z}$, $i \neq j$. If C is a finite cyclic group of order m, then $C = \{1, a, \ldots, a^{m-1}\}$ and $a^i \neq a^j$ if $i \neq j$ in $\{0, 1, 2, \ldots, m-1\}$. Let A be a set of integers defined by

$$A = \begin{cases} \mathbb{Z} & \text{if } C \text{ is infinite,} \\ \{0, 1, \ldots, m-1\} & \text{if } C \text{ is finite and } |C| = m. \end{cases}$$

Then $C = \{a^i \mid i \in A\}$ and $a^i \neq a^j$ if $i \neq j$ in A. Define a binary operation + on C^0 by

$$a^i + a^j = a^{\max\{i,j\}},$$

$$0 + a^i = a^i + 0 = a^i$$

for all i,j in A. Then the operation + is commutative on C^0. To show + is associative and the operation of C^0 is distributive over +, let $x,y,z \in C^0$. If at least one of x,y,z is 0, it is clearly seen that $x + (y+z) = (x+y) + z$ and $x(y+z) = xy + xz$. Assume $x,y,z \in C$. Then there exist i,j,k in A such that $x = a^i$, $y = a^j$ and $z = a^k$. Thus

$$\begin{aligned} (x+y) + z &= (a^i + a^j) + a^k \\ &= a^{\max\{i,j\}} + a^k \\ &= a^{\max\{\max\{i,j\},k\}} \\ &= a^{\max\{i,j,k\}} \\ &= a^i + a^{\max\{j,k\}} \\ &= a^i + (a^j + a^k) \\ &= x + (y+z) \end{aligned}$$

$$
\begin{aligned}
x(y+z) &= a^i(a^j + a^k) \\
&= a^i a^{\max\{j,k\}} \\
&= a^{i + \max\{j,k\}} \\
&= a^{\max\{i+j,i+k\}} \\
&= a^{i+j} + a^{i+k} \\
&= a^i a^j + a^i a^k \\
&= xy + xz \,.
\end{aligned}
$$

Hence $(C^0,+,\cdot)$ is an additively commutative semiring with zero where $\cdot$ is the operation on C^0. #

2.7 <u>Proposition</u>. Every cyclic semigroup admits the structure of an AC semiring with zero.

<u>Proof</u> : Let C be a cyclic semigroup with a generator a. Then $C = \{a^n \mid n \in \mathbb{N}\}$ where $\mathbb{N}$ is the set of positive integers. If C is infinite, then $a^i \neq a^j$ for $i,j \in \mathbb{N}$, $i \neq j$. If C is a finite cyclic semigroup of order m, then $C = \{a, a^2, \ldots, a^m\}$ and $a^i \neq a^j$ if $i \neq j$ in $\{1, 2, \ldots, m\}$. Let A be a set of integers defined by

$$
A = \begin{cases} \mathbb{N} & \text{if } C \text{ is infinite,} \\ \{1,2,\ldots,m\} & \text{if } C \text{ is finite and } |C| = m. \end{cases}
$$

Then $C = \{a^i \mid i \in A\}$ and $a^i \neq a^j$ if $i \neq j$ in A. Define a binary operation $+$ on C^0 by

$$
a^i + a^j = a^{\max\{i,j\}} ,
$$

$$
0 + a^i = a^i + 0 = a^i
$$

for all i,j in A. Then the operation + is commutative on C^o. The proof that + is associative and the operation of C^o is distributive over + can be given the same as that of Proposition 2.6. #

The last proposition of this chapter gives a neccessary condition for right [left] group to admit the structure of an AC semiring with zero. The following lemma is required :

2.8 <u>Lemma.</u> Let S be a semigroup with identity 1 and without zero and T a semigroup with a right [left] zero element e. If S×T admits the structure of an AC semiring with zero, then S admits the structure of an AC semiring with zero.

<u>Proof</u> : Assume S×T admits the structure of an AC semiring with zero under an addition + . If $x,y \in S$, then either $(x,e) + (y,e) = 0$ or $(x,e) + (y,e) = (a,t)$ for some $a \in S$, $t \in T$. If $x,y \in S$ such that $(x,e) + (y,e) = (a,t)$, $a \in S$, $t \in T$, then

$$\begin{aligned}(a,e) &= (a,t)(1,e) \\ &= ((x,e) + (y,e))(1,e) \\ &= (x,e)(1,e) + (y,e)(1,e) \\ &= (x,e) + (y,e) \\ &= (a,t) ,\end{aligned}$$

which implies $t = e$. Hence for $x,y \in S$, we have either $(x,e) + (y,e) = 0$ or $(x,e) + (y,e) = (a,e)$ for some $a \in S$. For $x,y \in S$, define $x +' y$ by

$$x +' y = \begin{cases} a & \text{if } (x,e) + (y,e) = (a,e) \text{ , } a \in S, \\ 0 \text{ (the zero of } S^o\text{)} & \text{if } (x,e)+(y,e) = 0 \text{ (the zero of } (S\times T)^o\text{),} \end{cases}$$

and for $x \in S^o$, define $x +' 0 = 0 +' x = x$. Since S has no zero, $S\times T$ has no zero, so $+'$ is well-defined. To show $+'$ is associative, let $x,y,z \in S^o$. If $x = 0$ or $y = 0$ or $z = 0$, it is clear that $(x +' y) +' z = x +' (y +' z)$. Assume $x,y,z \in S$. Then

$$(*) \qquad ((x,e) + (y,e)) + (z,e) = (x,e) + ((y,e) + (z,e))$$

<u>Case $(x,e) + (y,e) = 0$ and $(y,e) + (z,e) = 0$.</u> Then $x +' y = 0$ and $y +' z = 0$. It follows from (*) that $0 + (z,e) = (x,e) + 0$ which implies $(z,e) = (x,e)$, and so $x = z$. Hence $(x +' y) +' z = 0 +' z = 0 +' x = x +' 0 = x +' (y +' z)$.

<u>Case $(x,e) + (y,e) = 0$ and $(y,e) + (z,e) \neq 0$.</u> Then $x +' y = 0$. From (*), we have that $0 + (z,e) = (x,e) + (y +' z,e)$, so $0 \neq (z,e) = (x,e) + (y +' z,e)$. Hence $x +' (y +' z) = z = 0 +' z = (x +' y) +' z$.

<u>Case $(x,e) + (y,e) \neq 0$ and $(y,e) + (z,e) = 0$.</u> The proof of $x +' (y +' z) = (x +' y) +' z$ in this case is similar to that in the second case.

<u>Case $(x,e) + (y,e) \neq 0$ and $(y,e) + (z,e) \neq 0$.</u> Then we obtain from (*) that $(x +' y,e) + (z,e) = (x,e) + (y +' z,e)$. If $(x +' y,e) + (z,e) = 0$, then $(x,e) + (y +' z,e) = 0$, hence $(x +' y) +' z = 0 = x +' (y +' z)$. If $(x +' y,e) + (z,e) \neq 0$, then $(x,e) + (y +' z,e) \neq 0$, hence $((x +' y) +' z,e) = (x +' (y +' z),e)$, so $(x +' y) +' z = x +' (y +' z)$.

Since $+$ is commutative on $(S\times T)^o$, it follows that $+'$ is commutative on S^o. To show that the operation of S^o is distributive over $+'$, let x,y and $z \in S^o$. If $x = 0$ or $y = 0$ or $z = 0$, it is clear that $x(y +' z) = xy +' xz$ and $(y +' z)x = yx +' zx$. Assume x,y and $z \in S$. Then

$$(**) \quad (x,e)((y,e) + (z,e)) = (x,e)(y,e) + (x,e)(z,e) = (xy,e) + (xz,e).$$

<u>Case $(y,e) + (z,e) = 0$.</u> Then $y +' z = 0$. From (**), we have $(x,e)0 = 0 = (xy,e) + (xz,e)$, so $xy +' xz = 0$. Thus $x(y +' z) = x0 = 0 = xy +' xz$.

<u>Case $(y,e) + (z,e) \neq 0$.</u> It follows from (**) that $(x,e)(y +' z,e) = (xy,e) + (xz,e)$. Since S×T has no zero, $(x,e)(y +' z,e) \neq 0$. Then $(xy,e) + (xz,e) \neq 0$, so $(x(y +' z),e) = (x,e)(y +' z,e) = (xy,e) + (xz,e) = (xy +' xz,e)$. Hence $x(y +' z) = xy +' xz$.

The proof of $(y +' z)x = yx +' zx$ is obtained similarly. #

Recall that a semigroup S is called a <u>right</u> [left] <u>group</u> if it is a right [left] simple and left [right] cancellative. A semigroup S is a right [left] group if and only if S is the product of G×E of a group G and a right [left] zero semigroup E [1, Theorem 1.27].

2.9 <u>Proposition.</u> Let S = G×E be a right group where G is a group and E is a right zero semigroup. If S admits the structure of an AC semiring with zero , then G admits the structure of an AC semiring with zero.

<u>Proof</u> : Assume that S admits the structure of an AC semiring with zero. If $|G| = 1$, then G admits the structure of an AC semiring with zero. If $|G| > 1$, then G is a semigroup with identity and without zero, so by Lemma 2.8, G admits the structure of an AC semiring with zero because every element of E is a right zero of E. #