Chapter 2.

Fundamental Theory

In the past decade many researchers have developed the theories that can be used for design, calculation and prediction of the behaviour of windmills which are presented as follows.

2.1 Rankine-Froude Theory

This theory originated by Rankine W. and R.E. Froude[5], by considering flow past a wind turbine as shown in Fig.2.1. The free stream wind velocity is U which is slowed by a wind device. Applying continuity, momentum and energy equations to the flow, one may determine the thrust and power if the flow is assumed to be entirely axial, with no rotational motion.

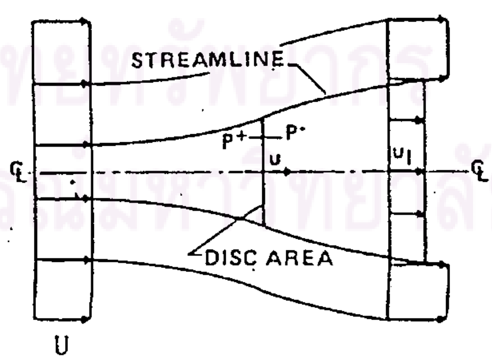

Fig.2.1 One-dimensional flow past a wind turbine

The thrust may be obtained from the following momentum theorem:

$$T = m(U-u_1) = \rho Au(U-u_1) \qquad (2.1)$$

And from consideration of pressure drop caused by the wind machine yields

$$T = A\,\Delta p \quad \text{where} \quad \Delta p = p^{+} - p^{-} \tag{2.2}$$

If the Bernoulli's equation is used between free stream and the upwind side of the turbine, and again between the downwind side of the turbine and the wake, the following equation is obtained

$$T = \rho \frac{A}{2} \left(U^2 - u_1^2 \right) \tag{2.3}$$

From the momentum expression,

$$u = \frac{U - u_1}{2} \tag{2.4}$$

is obtained. This equation shows that, the velocity at the disc is the average of the initial and final velocities. If we denote $U - u = aU$, and the final wake velocity change, $U - u_1$, is twice the velocity change at the disc, $U - u_1 = 2aU$. For the power, from the first law of thermodynamics, by assuming isothermal flow, with $P_1 = P_\infty$:

$$P = \rho A u \left(\frac{U^2}{2} - \frac{u_1^2}{2} \right) = \frac{\rho A u}{2} (U + u_1)(U - u_1)$$

or

$$\frac{P}{1/2\ \rho A U^3} = 4a(1-a)^2 \tag{2.5}$$

which has a maximum when a = 1/3[5,6]

$$\frac{P_{max}}{1/2\ \rho A U^3} = \frac{16}{27} = 0.593 \tag{2.6}$$

2.2 Blade Element Theory

A frequently used method to calculate the performance for propellers and helicopter rotors is to assume that the flow through the rotor occurs in non-interacting circular streamtubes. This method, when used in conjunction with the induced velocities, has been called by a variety of names, including modified blade element theory, vortex theory and strip theory etc. In this method the flow is assumed locally two-dimensional at each radial position.

The element of the wind turbine rotor is shown in Fig.2.2.

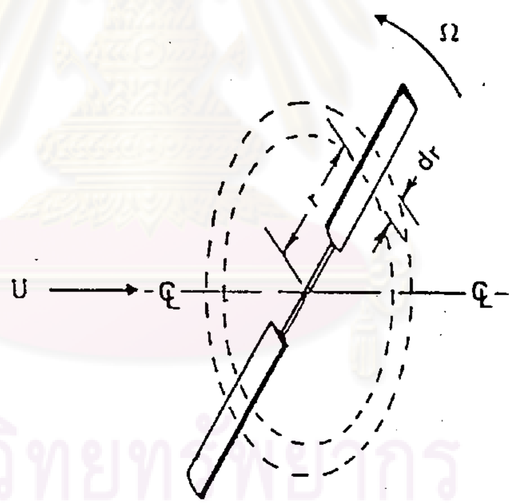

Fig.2.2 Rotor blade element

and the relation of a velocities diagram that acted on the blade element of the windmill is shown in Fig.2.3.

From this diagram, the following trigonometric relations may be verified:

$$\alpha = \phi_2 - \alpha_0 \tag{2.7}$$

$$C_x = C_L \sin\phi_2 - C_D \cos\phi_2 \tag{2.8}$$

$$C_y = C_L \cos\phi_2 + C_D \sin\phi_2 \tag{2.9}$$

$$\phi_2 = \tan^{-1}\left[\frac{r\Omega(1+b)}{U(1-a)}\right] \tag{2.10}$$

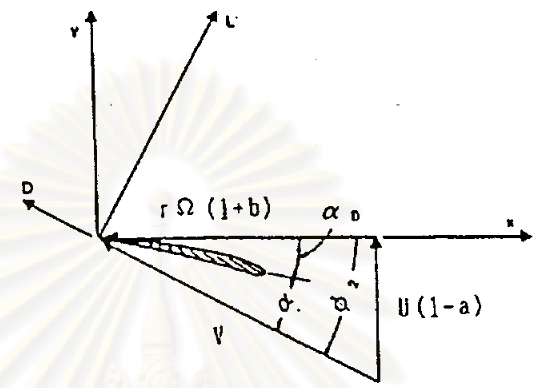

Fig. 2.3 Velocity diagram for a rotor blade element

The relation between the axial interference factor a, and the forces developed on the blade may be obtained by equating the axial force, dT, generated in an annular element of thickness, dr, by considerations of momentum and the axial force predicted from blade element aerodynamic considerations. From the momentum equation

$$dT_M = \rho(2\pi r dr)u(U-u_1) \tag{2.11}$$

and for B blades which have a local chord c

$$dT_B = Bc\ 1/2\rho V^2 C_y\ dr \tag{2.12}$$

one can equate these two expressions to yield

$$\frac{a}{1-a} = \frac{BcC_y}{8\pi r \sin^2\phi_2} \tag{2.13}$$

In similar manner, the torque can be determined from the angular momentum considerations which equate the torque developed by the blade element in an annular differential streamtube. From the momemt of momemtum theorem

$$dQ = \rho\,(2\pi r\,dr)\,ur\,(2b\Omega)\ . \tag{2.14}$$

and torque developed by the blade element

$$dQ = \rho Bc \frac{V^2}{2} C_x dr \tag{2.15}$$

one can combine these relations to yield

$$\frac{b}{1+b} = \frac{BcC_x}{8\pi r \sin\phi_2 \cos\phi_2} \tag{2.16}$$

If suitable airfoil sectional performance data are available, then the local flow condition at given radian position r may be determined.

2.3 Effect of Drag

Effect of profile drag can be found through the effect on torque .

Torque without drag is

$$dQ = \rho B c V^2/2 \cdot C_x \, dr$$

$$= \rho B c V^2/2 \cdot Cl\sin\phi_2 \, rdr \qquad (2.17)$$

Torque with drag is

$$dQ = \rho B c V^2/2 \, (Cl\sin\phi_2 - Cd\cos\phi_2) \, rdr$$

$$= \rho B c V^2/2 \, Cl\sin\phi_2 (1 - Cd\cot\phi_2/Cl) \, rdr \qquad (2.18)$$

In this case

$$Cp = 8/\lambda^2 \int_0^{\lambda} (1 - a) b \, \lambda^3{}_r \, (1 - Cd\cot\phi_2/Cl) \, d\lambda_r$$

$$= Cp_{ideal} - Cw \qquad (2.19)$$

If the angle of attack at a blade section and the blade section do not vary along the blade, then we can write

$$Cw = 8/\lambda^2 \cdot Cd/Cl \int_0^{\lambda} (1 - a) a \, \lambda_r{}^2 \, d\lambda_r \qquad (2.20)$$

under optimum condition (1-a)a is almost constant and equal to 2/9, hence

$$Cw = 8/\lambda^2 \cdot Cd/Cl \cdot 2/9 \int_0^{\lambda} \lambda^2{}_r d\lambda_r$$

$$Cw = 16 \, Cd/27 \, Cl \cdot \lambda \qquad (2.21)$$

2.4 Effect of a Finite Number of Blade

The assumption that the flow through the rotor is rotationally symmetric and the flow at any element is two-dimensional with a finite number of blades obviously dose not hold in actual state of the windmill.

The radial acceleration may be neglected for most wind power machines, while the wake effects can not, due to the finite number of blades, result in performance losses near the tips of the blades.

Tip losses model was analysed by Prandtl. This model was derived for lightly loaded propellers with negligible wake contraction.

Prandtl's model is

$$F = 2/\pi \arccos e^{-f} \tag{2.22}$$

where

$$f = B/2\ (1 - r/R)(1 - \lambda^2)^{1/2} \tag{2.23}$$

$(1 - \lambda)^{1/2}$ came into Prandtl's model when the distance between the vortex sheets in the wake flow is need.

For multiblades propeller, the distance between the sheets is

$$S = 2\pi R/B \cdot \sin\phi_2 \tag{2.24}$$

where ϕ_2 = angle of the vortex sheets as shown in Fig.2.4

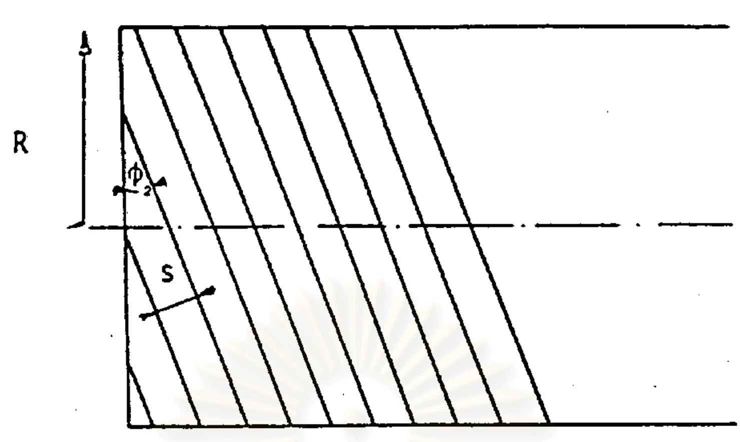

Fig. 2.4 Vortex sheets of lightly loaded multi bladed propeller

In non-contracting wake of Prantl's model.

$$\sin\phi_2 = U/(\Omega R^2 + U^2)$$

$$= 1/(1 - \lambda^2)^{1/2} \quad (2.25)$$

So that Eq.(2.23) becomes

$$f = B/2\ (1 - r/R) \cdot 1/\sin\phi_1 \quad (2.26)$$

In trying to find the effect of the finite number of blades on the maximum power of the propeller, Prandtl derived an effective radius for lightly loaded optimum propellers.

The effective radius may be calculated by

$$R_{eff}/R = 1 - 1.386\ \sin\phi_1/B \quad (2.27)$$

and may be transformed into a factor

$$\eta_B = (1 - 1.386\ \sin\phi_1/B)^2 \quad (2.28)$$

For the ideal windmill $\phi_1 = \phi_2/2$ then the calculated power of infinite number of blades may be corrected by multiplication with the factor η_{B} .