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# ภาคผนวก

# ภาคผนวก ก

## ผลงานทางวิชาการที่ได้รับการเผยแพร่

บทความเรื่อง Modification of the Constrained Hung-Turner Beam-Forming Algorithm for improving directional array performance using Phase-Independent Derivative Constraint ได้รับการเผยแพร่ในงานการประชุมวิชาการ World Wireless Congress (WWC) ครั้งที่ 5 ซึ่งจัดโดย CIC/ IEEE ณ เมืองซานฟรานซิสโก ประเทศสหรัฐอเมริกา ระหว่างวันที่ 25-28 พฤษภาคม พ.ศ.2547

# Modification of the Constrained Hung-Turner Beam-Forming Algorithm for improving directional array performance using Phase-Independent Derivative Constraint

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## Abstract

**This paper proposes a modification of the constrained Hung-Turner beamforming algorithm for improving directional array performance especially in the case of an inappropriate reference point selection that can result an undesirably high side lobe beam former response. This modification can be achieved by using a phase-independent derivative constraint. The computer simulations show that the proposed algorithm provides the significant improvement of the directional array performance in comparison with the Hung-Turner and the constrained Hung-Turner algorithm.**

**Index Terms- Hung-Turner algorithm, derivative constraint, phase-independent derivative constraint**

## Introduction

Most array-processing algorithms concern with jammer suppression problem. The Hung-Turner beam forming [1] and an improved version, conventional derivative constrained Hung-Tuner beam forming [2], are the efficient algorithms for jammer suppression. The attractive advantages of involving the conventional derivative constraint with the jammer directions leading to the constrained Hung-Turner algorithm [2]; as a result, the jammer suppression level can be improved more than the conventional Hung-Tuner algorithm.

The suppression level of the constrained Hung-Turner algorithm is well considerable, however, the high side-lobe can occur due to an inappropriate reference point selection of jammer [3]. This problem can be reduced by replacing the conventional derivative constraint with phase-independent derivative constraint, which the high side-lobe problem will be evaluated while the substantial suppression of the jammer is still maintained. In this paper, we explored three schemes, the conventional Hung-Turner, constrained Hung-Turner and the proposed schemes.

## Conventional Hung-Turner and constrained Hung-Turner algorithm

We assume below the array manifold of uniform linear array with the M elements is available and the number of element is larger than the number of jammer. The jammer waveforms are mutually and temporally uncorrelated. In addition, the single desired direction had to be known priori.

For the conventional Hung-Turner algorithm, the optimal weight vector will be found by maximizing the antenna gain toward the desired direction and nulling toward each jammer direction.

$$\max_{\mathbf{w}} \left\| \mathbf{w}^H \mathbf{a}_S \right\|_E^2 \tag{1.1}$$

$$\text{subject to} \quad \mathbf{w}^H \mathbf{a}(\theta_k) = 0 \text{ , for } k = 1, \cdots, L \tag{1.2}$$

$$\text{and} \quad \mathbf{w}^H \mathbf{w} = 1 \tag{1.3}$$

$$\mathbf{a}(\theta) = \left[ 1, \ e^{-j\xi}, \ \cdots \ , e^{-j(M-1)\xi} \right]^T \in \mathbf{C}^{M \times 1} \tag{2}$$

$$\xi \triangleq \frac{2\pi}{\lambda} d \sin(\theta) \tag{3}$$

$$\mathbf{a}_S \triangleq \mathbf{a}(\theta = \theta_S) \tag{4}$$

$$\mathbf{a}(\theta_k) \triangleq \mathbf{a}(\theta = \theta_k) \tag{5}$$

where $\theta_S$ is the desired direction, $\theta_k$ is the jammer direction, H denotes Hermitian transposition, $\mathbf{w}$ is the optimal weight vector, L is the number of jammers, $\mathbf{a}(\theta)$ is the steering vector, T denotes Transposition, d is a distance between two adjacent elements and $\lambda$ is a wavelength. The solution of (1) is given by [2] as follow:

$$\mathbf{w} = \frac{1}{\sqrt{\mathbf{a}_S^H \mathbf{P}_A^{\perp} \mathbf{a}_S}} \mathbf{P}_A^{\perp} \mathbf{a}_S \ \in \mathbf{C}^{M \times 1} \tag{6}$$

$$\mathbf{P}_A^{\perp} = \mathbf{I} - \mathbf{A}(\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \in \mathbf{C}^{M \times M} \tag{7}$$

$$\mathbf{A} = \left[ \mathbf{a}(\theta_1) \ \ \mathbf{a}(\theta_2) \ \ \cdots \ \ \mathbf{a}(\theta_L) \right] \in \mathbf{C}^{M \times L} \tag{8}$$

where **A** is the jammer wave front matrix, $P_A^{\perp}$ is the orthogonal complement of **A** .

The constrained Hung-Turner algorithm is a modification of the conventional Hung-Turner algorithm with conventional derivative constraint. The optimal weight vector provides the sharp null more flat in the directional pattern toward jammer direction. Using the analogy with the conventional Hung-Turner algorithm, the formulation of the constrained Hung-Turner algorithm is given by:

$$\max_{w} \left\| w^H a_s \right\|_E^2 \tag{9.1}$$

$$\text{subject to} \quad w^H a(\theta_k) = 0 \ , \ \text{for } k = 1, \cdots, L \tag{9.2}$$

$$\left. \frac{\partial^n \left\{ w^H a(\theta) \right\}}{\partial \xi^n} \right|_{\theta = \theta_k} = 0 \quad \text{for} \begin{cases} k = 1, & \cdots, L \\ n = 1, & \cdots, N \end{cases} \tag{9.3}$$

$$\text{and} \quad w^H w = 1 \tag{9.4}$$

where N is the number of derivative order. The derivative term of (9.3) can be rewritten as:

$$w^H D^n a(\theta_k) = 0 \tag{10}$$

$$\text{where} \quad D = \text{diag}\{0,1,2,\dots, M-1\} \tag{11}$$

The solution of (9) is given by [2] as follow:

$$w = \frac{1}{\sqrt{a_s^H P_B^{\perp} a_s}} P_B^{\perp} a_s \in C^{M\times 1} \tag{12}$$

$$P_B^{\perp} = I - B(B^H B)^{-1} B^H \in C^{M\times M} \tag{13}$$

$$B = \begin{bmatrix} A & DA & \cdots & D^N A \end{bmatrix} \in C^{M\times L(N+1)} \tag{14}$$

The conventional derivative constraint as (9.3) can provide the flat main-beam response but it may result in the high side-lobe level. This phenomenon comes from the fact that the conventional derivative constraint includes both magnitude and phase of signals to its constraint. The high side-lobe level occurs due to the phase constraint. Because the phase response of the beam former output is controlled by the phase references. If we choose the inappropriate phase references for evaluation, we will encounter this problem automatically, due to the fact that the appropriate phase references cannot be easily obtained and may depend on the signal environment.

An alternative way to control the beam former output response without constraining the phase response is to constraint the derivatives of the power response. Since the power response and its derivative are invariant to the phase references

## Proposed algortithm: Modification of constrained Hung-Turner using phase independent derivative constraint

The proposed algorithm modifies cost function of constrained Hung – Turner by changing from zero to $\varepsilon$.The concept of this modification is that

$$\max_{w} \left\| w^H a_s \right\|_E^2 \tag{15.1}$$

$$\text{subject to} \quad w^H a(\theta_k) = \varepsilon \ , \ \text{for } k = 1, \cdots, L \tag{15.2}$$

$$\left. \frac{\partial^n P(\theta)}{\partial \xi^n} \right|_{\theta = \theta_k} = 0 \ , \ \text{for} \begin{cases} k = 1, & \cdots, L \\ n = 1, & \cdots, N \end{cases} \tag{15.3}$$

$$\text{and} \quad w^H w = 1 \tag{15.4}$$

where $\varepsilon \in R$ and $\varepsilon \to 0$ In (15.3), $P(\theta)$ is the power response of the beam former given by:

$$P(\theta) = \left\| w^H a(\theta) \right\|_E^2 \tag{16}$$

Then the first order derivatives of $P(\theta)$ when $\theta = \theta_k$ can be rewritten as:

$$\left. \frac{\partial P(\theta)}{\partial \xi} \right|_{\theta = \theta_k} = 2\text{Re}\{w^H a_{\xi_k} a^H(\theta_k) w\} = 2\varepsilon \text{Re}\{w^H a_{\xi_k}\} \tag{17}$$

$$\text{where} \quad a_{\xi_k} \triangleq \left. \frac{\partial a(\theta)}{\partial \xi} \right|_{\theta = \theta_k} = [(-j)(m-1)e^{-j(m-1)\xi_k} \Big|_{m = 1,\dots, M}]^T \tag{18}$$

$\text{Re}\{\}$ is real operator . The nonlinear derivative constraint problem in (17) can be converted to a linearly constrained problem by converting the complex valued problem to a real value problem [4]. The new real valued matrices are formed from the original complex valued matrix in the following manner:

$$w_R = \begin{bmatrix} Re\{w\} \\ Im\{w\} \end{bmatrix} \in \mathbf{R}^{2M\times1} \quad (19)$$

$$a_{\xi_k,R} = \begin{bmatrix} Re\{(-j)\ a(\theta_k)\} \\ Im\{(-j)\ a(\theta_k)\} \end{bmatrix} \quad (20)$$

$$G = \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} \in \mathbf{R}^{2M\times2M} \quad (21)$$

where $w_R$ is the real optimal weight vector in real format. $0$ is the MxM zero matrix. With these transformed matrices, it can be shown that the equivalent transformed problem for the original phase-independent derivative constraint from (17) becomes;

$$2\varepsilon Re\{w^H a_{\xi_k}\} = 2\varepsilon w_R^T G\ a_{\xi,R}(\theta_k) = 0 \quad (22)$$

So, if we combine all of (15) , the result via null constraint becomes

$$w_R = \left(I - E(E^T E)^{-1} E^T\right) a_{S,R} + \varepsilon E\left(E^T E\right)^{-1} E^T \quad (23)$$

where $E = \left[ a_R(\theta_k) \ \vdots \ Ga_{\xi,R}(\theta_k) \right]$

If $\varepsilon \to 0$, the last term of the right side of (23) will be neglected. Then,

$$w_R \cong \left(I - E(E^T E)^{-1} E^T\right) a_{S,R} \quad (24)$$

So, for N=1, the result of proposed algorithm for N=1 become

$$w_R = \frac{1}{\sqrt{a_{S,R} P_E^{\perp} a_{S,R}}} P_E^{\perp} a_{S,R} \in \mathbf{R}^{2M\times1} \quad (25)$$

$$P_E^{\perp} = I - E(E^H E)^{-1} E^H \in \mathbf{R}^{2M\times2M} \quad (26)$$

$$E = \left[ A_{\xi,R} \ \vdots \ GA_{\xi,R} \right] \in \mathbf{R}^{2M\times2L} \quad (27)$$

where $A_{\xi,R}$ is the 2MxL jammer wave front matrix in the real format as (20) and $a_{S,R}$ is the 2Mx1 steering response in the desired direction that expressed in the real version as (19). The solution of proposed algorithm is called the phase-independent derivative constraint. This provides that the optimal weight can maximize the main-lobe toward the desired direction and nulling toward each jammer direction but its term of phase dose not varying to each jammer direction.

## SINR Analysis based on data snapshot for moving jammer

We considered the case of two moving jammer and compare each algorithm in term of SINR. The received signal can be defined as

$$r(i) = a_S S_d(i) + A(i) S_j(i) + n(i) \in \mathbf{C}^{M\times1} \quad (28)$$

where $r(i)$ is recived signal , $S_d(i)$ is desired signal, $S_j(i)$ is jammer signal $A(i)$ is moving jammer wave front matrix versus the snapshot index i $A(i) = [a(\theta_1(i)) \ \cdots \ a(\theta_L(i))]$ and $n(i)$ is Additive white gaussian noise .Signal will be modeled as in [1] assumed that signal in each path is independent with another. To identify jammer signal we will consider the maximum strength of signal path. Following to [2] we have:

$$\text{Span}\{r(i-L), \ \cdots \ , r(i-1)\}$$
$$= \text{Span}\{a(\theta_1), \ a(\theta_2), \ \cdots \ , a(\theta_L)\}$$

if desired signal power = 0 and noise power=0 (29)

From (29) the optimal weight vector of our proposed algorithm can be shown as:

$$Q(i) = [r(i-L) \ \ r(i-(L-1)) \ \cdots \ r(i-1)] \in \mathbf{C}^{M\times L} \quad (30)$$

$$Y(i) = [Q_R(i) \ \vdots \ GQ_R(i)] \in \mathbf{R}^{2M\times2L} \quad (31)$$

The transformation of $Q_R$ in (31) is the same form as (19)

$$P_Y^{\perp}(i) = I - Y_2(i)\left(Y_2(i)^T Y_2(i)\right)^{-1} Y_2(i)^T \in \mathbf{R}^{2M\times2M} \quad (32)$$

So that, the optimal weight vector of our Propose algorithm in moving jammer scenario can be shown as:

$$w_R(i) = \frac{1}{\sqrt{a_{S,R}^T P_Y^{\perp}(i) a_{S,R}}} P_Y^{\perp}(i) a_{S,R} \in \mathbf{R}^{2M\times1} \quad (33)$$

and SINR is

$$\mathrm{SINR}(i) = \frac{p_S^2 \left| \mathbf{w}_R^T(i)\mathbf{a}_{S,R} \right|^2}{\mathbf{w}_R^T(i)\mathbf{R}_{rr,R}(i)\mathbf{w}_R(i)} \quad (34)$$

where

$$\mathbf{R}_{rr}(i) = E\left\{ \mathbf{r}(i)\mathbf{r}(i)^H \right\} \quad \text{(Exact)} \quad (35)$$

$$\mathbf{R}_{rr,R}(i) = \begin{bmatrix} \mathrm{Re}\{\mathbf{R}_{rr}(i)\} & -\mathrm{Im}\{\mathbf{R}_{rr}(i)\} \\ \mathrm{Im}\{\mathbf{R}_{rr}(i)\} & \mathrm{Re}\{\mathbf{R}_{rr}(i)\} \end{bmatrix} \in \mathbf{R}^{2M\times 2M} \quad (36)$$

In order to compare the output SINR of each algorithm with optimal output SINR of adaptive array as [2].

$$\mathrm{SINR}_0 = p_S \mathbf{a}_S^H \mathbf{R}_{nn}^{-1} \mathbf{a}_S \quad (37)$$

where $p_S$ is the desired signal power

$$\mathbf{R}_{nn} = E\left\{ \mathbf{n}\mathbf{n}^H \right\} \quad \text{(Exact)} \quad (38)$$

## Simulation result

In all simulation experiments, we assumed an uniform linear array along z-axis with spaced at intervals of half length and no mutual coupling.

Fig.1 represents directional performance corresponding to the 8-element array and two narrowband non moving jammer impinging from the directions -40 and 30 degree. These directions are the appropriate reference point selection in order to the good performance for the both conventional and constrained Hung-Turner.It is shown that our proposed algorithm also given the same level of result as other two algorithms.

Fig.2 represents directional performance corresponding to the 8-element array and the two narrowband non moving jammer impinging from the directions 18 and 20 degree. It is clear that when the given directions are inappropriate reference point selection, our proposed algorithm gives a better result than both the conventional and constrained Hung-Turner algorithm. Note that, these directions were experimentally obtained from the directions which could result the worst directional performance of constrained Hung-Turner.

Fig.3 shown the performance in term of SINR with corresponding to the two moving jammer and 16-element-array where the trajection of the jammer motion versus the snapshot index i are[2]

$$\theta_1(i) = 35^\circ - 0.1^\circ i \quad (39.1)$$

$$\theta_2(i) = 25^\circ + 0.1^\circ i \quad (39.2)$$

The simulation results are obtained by using the same condition as in [2].That is the additive sensor noises will be assumed to be uncorrelated from sensor-to-sensor and with the signals and to have the same power between desired signal power and noise power is 1 watt in each sensor. We also assume equipowered, mutually uncorrelated jammer source with jammer-to-noise power ratio equal to 30 dB in each array sensor for each jammer.

Fig.4 represents instant directional performance at i=50 of fig.3. This result implied to the robustness in the presence of spatial correlation of each algorithm. Because in this scenario, the both of trajections of moving jammer at i=50 are equal to 30 degrees. Therefore, our proposed algorithm which has the better directional performance in this scenario must be the more robust algorithm in the presence of spatial correlation.

In all simulation, we used the derivative order=1 but without averaging over dimension technique.

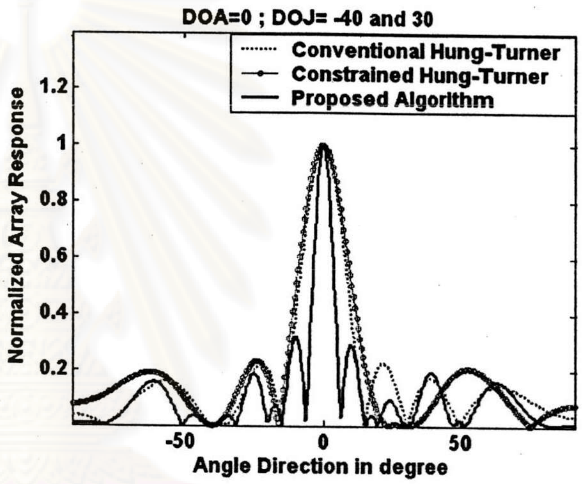


Fig.1 Directional performance in the case of appropriate reference point selection

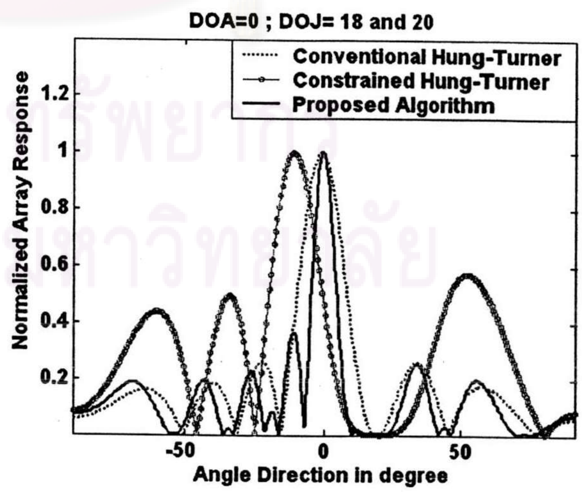


Fig.2 Directional performance in the case of inappropriate reference point selection

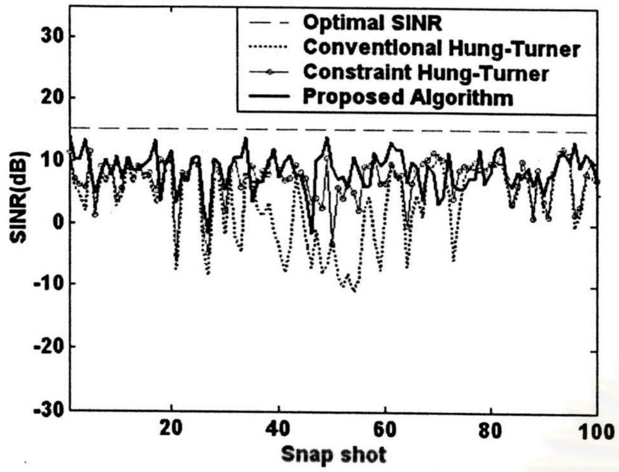


Fig.3 Output SINR with out averaging over dimension technique

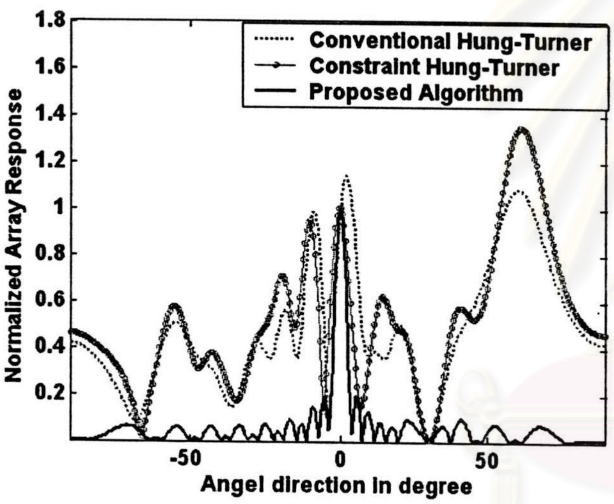


Fig.4 The instant directional performance of SINR in Fig.3 at i= 50

## Conclusion

This paper proposed a modification of the constrained Hung-Tuner algorithm using phase-independent derivative constraint which can improve the array performance such as high side lobe level reduction, reducing beam width, providing the acceptable SINR and robustness in the presence of spatial correlation. Thus, its overall directional performance is better than both Hung-Turner and the constrained Hung-Turner.

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# ภาคผนวก ข

## การวิเคราะห์สมการของ $\mathbf{P}_N(\phi)$

จากสมการสเปคตรัมแบบนัลล์ของขั้นตอนวิธีมิวสิกแบ่งคลื่นแบบที่เราปรับปรุง

$$\begin{aligned} g(\phi) &= \left\| \mathbf{U}_{BnR}^T(\phi)\tilde{\mathbf{A}}_{BR}(\phi) \right\|_F^2 \\ &= trace\left( \mathbf{U}_{BnR}^T(\phi)\tilde{\mathbf{A}}_{BR}(\phi)\tilde{\mathbf{A}}_{BR}^T(\phi)\mathbf{U}_{BnR}(\phi) \right) \\ &= trace\left( \mathbf{A}_{BR}^T(\phi)\mathbf{U}_{BnR}(\phi)\mathbf{U}_{BnR}^T(\phi)\mathbf{A}_{BR}(\phi) \right) \end{aligned} \qquad (ข.1)$$

กำหนดให้

$$\mathbf{P}_N(\phi) = \mathbf{U}_{BnR}(\phi)\mathbf{U}_{BnR}^T(\phi) \qquad (ข.2)$$

โดยที่

$$\mathbf{U}_{BnR}(\phi) = \text{Null space}\left( \begin{bmatrix} \mathbf{U}_{BsR} & \tilde{\mathbf{a}}_{B\gamma_R}(\phi) \end{bmatrix} \right)$$

ทำให้ $g(\phi)$ กลายเป็น

$$g(\phi) = trace\left( \tilde{\mathbf{A}}_{BR}^T(\phi)\mathrm{P}_N(\phi)\tilde{\mathbf{A}}_{BR}(\phi) \right) \qquad (ข.3)$$

จากคุณสมบัติทางคณิตศาสตร์ $\mathbf{P}_N(\phi)$ จะมีความสัมพันธ์คือ

$$\mathbf{P}_N(\phi) = \mathbf{U}_{BnR}(\phi)\mathbf{U}_{BnR}^T(\phi) = \mathbf{I} - \mathbf{P}_S(\phi) \qquad (ข.4)$$

โดยที่

$$\mathbf{P}_S(\phi) = \begin{bmatrix} \mathbf{U}_{BsR} & \tilde{\mathbf{a}}_{B_{\gamma_R}}(\phi) \end{bmatrix} \left[ \begin{bmatrix} \mathbf{U}_{BsR} & \tilde{\mathbf{a}}_{B_{\gamma_R}}(\phi) \end{bmatrix}^T \begin{bmatrix} \mathbf{U}_{BsR} & \tilde{\mathbf{a}}_{B_{\gamma_R}}(\phi) \end{bmatrix} \right]^{-1} \begin{bmatrix} \mathbf{U}_{BsR} & \tilde{\mathbf{a}}_{B_{\gamma_R}}(\phi) \end{bmatrix}^T \qquad (ข.5)$$

พิจารณา

$$\left[ \begin{bmatrix} \mathbf{U}_{BsR} & \tilde{\mathbf{a}}_{B\gamma_R}(\phi) \end{bmatrix}^T \begin{bmatrix} \mathbf{U}_{BsR} & \tilde{\mathbf{a}}_{B\gamma_R}(\phi) \end{bmatrix} \right]^{-1}$$

$$= \left[ \begin{bmatrix} \mathbf{U}_{BsR}^T \\ \tilde{\mathbf{a}}_{B\gamma_R}^T(\phi) \end{bmatrix} \begin{bmatrix} \mathbf{U}_{BsR} & \tilde{\mathbf{a}}_{B\gamma_R}(\phi) \end{bmatrix} \right]^{-1}$$

$$= \begin{bmatrix} \mathbf{U}_{BsR}^T \mathbf{U}_{BsR} & \mathbf{U}_{BsR}^T \tilde{\mathbf{a}}_{B\gamma_R}(\phi) \\ \tilde{\mathbf{a}}_{B\gamma_R}^T(\phi) \mathbf{U}_{BsR} & \tilde{\mathbf{a}}_{B\gamma_R}^T(\phi) \tilde{\mathbf{a}}_{B\gamma_R}(\phi) \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \mathbf{I} & \mathbf{U}_{BsR}^T \tilde{\mathbf{a}}_{B\gamma_R}(\phi) \\ \tilde{\mathbf{a}}_{B\gamma_R}^T(\phi) \mathbf{U}_{BsR} & \tilde{\mathbf{a}}_{B\gamma_R}^T(\phi) \tilde{\mathbf{a}}_{B\gamma_R}(\phi) \end{bmatrix}^{-1} \qquad \text{(ข.6)}$$

จากทฤษฎีบทของการแปลงผกผันเมตริกซ์ (Lemma of inversion matrix)

$$\begin{bmatrix} \mathbf{F}_{11} & \mathbf{F}_{12} \\ \mathbf{F}_{21} & \mathbf{F}_{22} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{F}_{11}^{-1} + \mathbf{F}_{11}^{-1}\mathbf{F}_{12}(\mathbf{F}_{22} - \mathbf{F}_{21}\mathbf{F}_{11}^{-1}\mathbf{F}_{12})^{-1}\mathbf{F}_{21}\mathbf{F}_{11}^{-1} & -\mathbf{F}_{11}^{-1}\mathbf{F}_{12}(\mathbf{F}_{22} - \mathbf{F}_{21}\mathbf{F}_{11}^{-1}\mathbf{F}_{12})^{-1} \\ -(\mathbf{F}_{22} - \mathbf{F}_{21}\mathbf{F}_{11}^{-1}\mathbf{F}_{12})^{-1}\mathbf{F}_{21}\mathbf{F}_{11}^{-1} & (\mathbf{F}_{22} - \mathbf{F}_{21}\mathbf{F}_{11}^{-1}\mathbf{F}_{12})^{-1} \end{bmatrix} \qquad \text{(ข.7)}$$

เมื่อเทียบกับสมการที่ (ข.6) กับสมการที่ (ข.7) จะได้ว่า

$$\mathbf{F}_{11} = \mathbf{I} \qquad \text{(ข.8)}$$

$$\mathbf{F}_{12} = \mathbf{U}_{BsR}^T \tilde{\mathbf{a}}_{B\gamma_R}(\phi) \qquad \text{(ข.9)}$$

$$\mathbf{F}_{21} = \tilde{\mathbf{a}}_{B\gamma_R}^T(\phi) \mathbf{U}_{BsR} \qquad \text{(ข.10)}$$

$$\mathbf{F}_{22} = \tilde{\mathbf{a}}_{B\gamma_R}^T(\phi) \tilde{\mathbf{a}}_{B\gamma_R}(\phi) \qquad \text{(ข.11)}$$

ซึ่งทำให้แปลงสมการที่ (ข.6) ได้เป็น

$$\begin{bmatrix} \mathbf{I} & \mathbf{U}_{BsR}^T\tilde{\mathbf{a}}_{B\gamma_R}(\phi) \\ \tilde{\mathbf{a}}_{B\gamma_R}^T(\phi)\mathbf{U}_{BsR} & \tilde{\mathbf{a}}_{B\gamma_R}^T(\phi)\tilde{\mathbf{a}}_{B\gamma_R}(\phi) \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{G}_{11}(\phi) & \mathbf{G}_{12}(\phi) \\ \mathbf{G}_{21}(\phi) & \mathbf{G}_{22}(\phi) \end{bmatrix} \qquad (ข.12)$$

เมื่อ

$$\mathbf{G}_{11}(\phi) = \mathbf{I} + \mathbf{U}_{BsR}^T\tilde{\mathbf{a}}_{B\gamma_R}(\phi)\left(\tilde{\mathbf{a}}_{B\gamma_R}^T(\phi)\tilde{\mathbf{a}}_{B\gamma_R}(\phi) - \tilde{\mathbf{a}}_{B\gamma_R}^T(\phi)\mathbf{U}_{BsR}\mathbf{U}_{BsR}^T\tilde{\mathbf{a}}_{B\gamma_R}(\phi)\right)^{-1}\tilde{\mathbf{a}}_{B\gamma_R}^T(\phi)\mathbf{U}_{BsR} \qquad (ข.13)$$

$$\mathbf{G}_{12}(\phi) = -\mathbf{U}_{BsR}^T\tilde{\mathbf{a}}_{B\gamma_R}(\phi)\left(\tilde{\mathbf{a}}_{B\gamma_R}^T(\phi)\tilde{\mathbf{a}}_{B\gamma_R}(\phi) - \tilde{\mathbf{a}}_{B\gamma_R}^T(\phi)\mathbf{U}_{BsR}\mathbf{U}_{BsR}^T\tilde{\mathbf{a}}_{B\gamma_R}(\phi)\right)^{-1} \qquad (ข.14)$$

$$\mathbf{G}_{21}(\phi) = -\left(\tilde{\mathbf{a}}_{B\gamma_R}^T(\phi)\tilde{\mathbf{a}}_{B\gamma_R}(\phi) - \tilde{\mathbf{a}}_{B\gamma_R}^T(\phi)\mathbf{U}_{BsR}\mathbf{U}_{BsR}^T\tilde{\mathbf{a}}_{B\gamma_R}(\phi)\right)^{-1}\tilde{\mathbf{a}}_{B\gamma_R}^T(\phi)\mathbf{U}_{BsR} \qquad (ข.15)$$

$$\mathbf{G}_{22}(\phi) = -\left(\tilde{\mathbf{a}}_{B\gamma_R}^T(\phi)\tilde{\mathbf{a}}_{B\gamma_R}(\phi) - \tilde{\mathbf{a}}_{B\gamma_R}^T(\phi)\mathbf{U}_{BsR}\mathbf{U}_{BsR}^T\tilde{\mathbf{a}}_{B\gamma_R}(\phi)\right)^{-1} \qquad (ข.16)$$

นำสมการที่ (ข.12) แทนกลับลงในสมการที่ (ข.6) จากนั้นแทนสมการที่ (ข.6) ลงในสมการที่ (ข.5) จะได้

$$\begin{aligned} \mathbf{P}_S(\phi) &= \begin{bmatrix} \mathbf{U}_{BsR} & \tilde{\mathbf{a}}_{B\gamma_R}(\phi) \end{bmatrix} \begin{bmatrix} \mathbf{G}_{11}(\phi) & \mathbf{G}_{12}(\phi) \\ \mathbf{G}_{21}(\phi) & \mathbf{G}_{22}(\phi) \end{bmatrix} \begin{bmatrix} \mathbf{U}_{BsR} & \tilde{\mathbf{a}}_{B\gamma_R}(\phi) \end{bmatrix}^T \\ &= \begin{bmatrix} \mathbf{U}_{BsR}\mathbf{G}_{11}(\phi) + \tilde{\mathbf{a}}_{B\gamma_R}(\phi)\mathbf{G}_{21}(\phi) & \mathbf{U}_{BsR}\mathbf{G}_{12}(\phi) + \tilde{\mathbf{a}}_{B\gamma_R}(\phi)\mathbf{G}_{22}(\phi) \end{bmatrix} \begin{bmatrix} \mathbf{U}_{BsR}^T \\ \tilde{\mathbf{a}}_{B\gamma_R}^T(\phi) \end{bmatrix} \\ &= \mathbf{U}_{BsR}\mathbf{G}_{11}(\phi)\mathbf{U}_{BsR}^T + \tilde{\mathbf{a}}_{B\gamma_R}(\phi)\mathbf{G}_{21}(\phi)\mathbf{U}_{BsR}^T + \mathbf{U}_{BsR}\mathbf{G}_{12}(\phi)\tilde{\mathbf{a}}_{B\gamma_R}^T(\phi) + \tilde{\mathbf{a}}_{B\gamma_R}(\phi)\mathbf{G}_{22}(\phi)\tilde{\mathbf{a}}_{B\gamma_R}^T(\phi) \end{aligned}$$

(ข.17)

จากสมการที่ (ข.13) - (ข.16) สังเกตว่าหากทำการแทนค่ากลับตรง ๆ จะทำให้สมการ $\mathbf{P}_S(\phi)$ พิจารณา ได้ยากเนื่องจากความไม่กระชับของรูปแบบสมการ ดังนั้นเพื่อให้การพิจารณาสมการดังกล่าว ได้ง่ายขึ้น จึงได้เพิ่มการวิเคราะห์ทางคณิตศาสตร์ที่ช่วยทำให้สมการกระชับขึ้น ซึ่งแสดงได้ดังนี้

พิจารณาทฤษฎีบทการแปลงผกผันเมตริกซ์

$$\left[\mathbf{H}_1 + \mathbf{H}_2\mathbf{H}_3\right]^{-1} = \mathbf{H}_1^{-1} - \mathbf{H}_1^{-1}\mathbf{H}_2\left[\mathbf{I} + \mathbf{H}_3\mathbf{H}_1^{-1}\mathbf{H}_2\right]^{-1}\mathbf{H}_3\mathbf{H}_1^{-1} \qquad (ข.18)$$

เทียบสมการที่ (ข.19) กับสมการที่ (ข.18)

$$\tilde{\mathbf{a}}_{B\gamma_R}(\phi)\left[\tilde{\mathbf{a}}^T_{B\gamma_R}(\phi)\tilde{\mathbf{a}}_{B\gamma_R}(\phi)-\tilde{\mathbf{a}}^T_{B\gamma_R}(\phi)\mathbf{U}_{BsR}\mathbf{U}^T_{BsR}\tilde{\mathbf{a}}_{B\gamma_R}(\phi)\right]^{-1}\tilde{\mathbf{a}}^T_{B\gamma_R}(\phi) \qquad (\text{ข}.19)$$

จะได้ว่า

$$\mathbf{H}_1=\tilde{\mathbf{a}}^T_{B\gamma_R}(\phi)\tilde{\mathbf{a}}_{B\gamma_R}(\phi) \qquad (\text{ข}.19)$$

$$\mathbf{H}_2=-\tilde{\mathbf{a}}^T_{B\gamma_R}(\phi)\mathbf{U}_{BsR} \qquad (\text{ข}.20)$$

$$\mathbf{H}_3=\mathbf{U}^T_{BsR}\tilde{\mathbf{a}}_{B\gamma_R}(\phi) \qquad (\text{ข}.21)$$

ดังนั้นจะแปลงสมการที่ (ข.19) ให้อยู่ในรูปแบบเดียวกับสมการที่ (ข.18) จะได้เป็น

$$\mathrm{P}_{\tilde{\mathbf{a}}_{B\gamma R}}(\phi)+\mathrm{P}_{\tilde{\mathbf{a}}_{B\gamma R}}(\phi)\mathbf{U}_{BsR}\left(\mathbf{I}-\mathbf{U}^T_{BsR}\mathrm{P}_{\tilde{\mathbf{a}}_{B\gamma R}}(\phi)\mathbf{U}_{BsR}\right)^{-1}\mathbf{U}^T_{BsR}\mathrm{P}_{\tilde{\mathbf{a}}_{B\gamma R}}(\phi) \qquad (\text{ข}.22)$$

เมื่อ

$$\mathrm{P}_{\tilde{\mathbf{a}}_{B\gamma R}}(\phi)=\tilde{\mathbf{a}}_{B\gamma_R}(\phi)\left(\tilde{\mathbf{a}}^T_{B\gamma_R}(\phi)\tilde{\mathbf{a}}_{B\gamma_R}(\phi)\right)^{-1}\tilde{\mathbf{a}}^T_{B\gamma_R}(\phi) \qquad (\text{ข}.23)$$

และกำหนดให้



$$\mathbf{P}_n=\mathbf{I}-\mathbf{U}_{BsR}\mathbf{U}^T_{BsR} \qquad (\text{ข}.24)$$



เมื่อนำรูปแบบการแปลงสมการที่ (ข.23) แทนลงในสมการที่ (ข.13) - (ข.16) จากนั้นนำทั้งหมดแทนกลับลงในสมการที่ (ข.17) แล้วจัดรูปสมการ เราจะได้สมการ $\mathbf{P}_S(\phi)$ ในรูปแบบที่กระชับขึ้นดังนี้

$$\mathbf{P}_S(\phi)=\mathbf{U}_{BsR}\mathbf{U}^T_{BsR}+\mathbf{P}_n\left(\mathbf{P}_{\tilde{\mathbf{a}}_{B\gamma R}}(\phi)+\mathbf{P}_{\tilde{\mathbf{a}}_{B\gamma R}}(\phi)\mathbf{U}_{BsR}\left(\mathbf{I}-\mathbf{U}^T_{BsR}\mathbf{P}_{\tilde{\mathbf{a}}_{B\gamma R}}(\phi)\mathbf{U}_{BsR}\right)^{-1}\mathbf{U}^T_{BsR}\mathbf{P}_{\tilde{\mathbf{a}}_{B\gamma R}}(\phi)\right)\mathbf{P}_n \qquad (\text{ข}.25)$$

และ

$$\mathbf{P}_N(\phi) = \mathbf{I} - \mathbf{P}_S(\phi)$$
$$= \mathbf{P}_n - \mathbf{P}_n\left(\mathbf{P}_{\tilde{\mathbf{a}}_{B\gamma R}}(\phi) + \mathbf{P}_{\tilde{\mathbf{a}}_{B\gamma R}}(\phi)\mathbf{U}_{BsR}\left(\mathbf{I} - \mathbf{U}_{BsR}^T\mathbf{P}_{\tilde{\mathbf{a}}_{B\gamma R}}(\phi)\mathbf{U}_{BsR}\right)^{-1}\mathbf{U}_{BsR}^T\mathbf{P}_{\tilde{\mathbf{a}}_{B\gamma R}}(\phi)\right)\mathbf{P}_n \qquad (ข.26)$$

กำหนดให้

$$\mathbf{Z}_1(\phi) = \left(\mathbf{I} - \mathbf{U}_{BsR}^T\mathbf{P}_{\tilde{\mathbf{a}}_{B\gamma R}}(\phi)\mathbf{U}_{BsR}\right)^{-1} \qquad (ข.27)$$

$$\mathbf{Z}_2(\phi) = \mathbf{I} + \mathbf{U}_{BsR}\mathbf{Z}_1(\phi)\mathbf{U}_{BsR}^T\mathbf{P}_{\tilde{\mathbf{a}}_{B\gamma R}}(\phi) \qquad (ข.28)$$

อาศัยการกำหนดจากสมการที่ (ข.27)-(ข.28) ทำให้สามารถกระชับสมการที่ (ข.25) และ (ข.26) ได้ว่า

$$\mathbf{P}_S(\phi) = \mathbf{U}_{BsR}\mathbf{U}_{BsR}^T + \mathbf{P}_n\mathbf{P}_{\tilde{\mathbf{a}}_{B\gamma R}}(\phi)\mathbf{Z}_2(\phi)\mathbf{P}_n \qquad (ข.29)$$

$$\mathbf{P}_N(\phi) = \mathbf{P}_n\left(\mathbf{I} - \mathbf{P}_{\tilde{\mathbf{a}}_{B\gamma R}}(\phi)\mathbf{Z}_2(\phi)\mathbf{P}_n\right) \qquad (ข.60)$$

# ภาคผนวก ค

## การวิเคราะห์สมการ $\mathbf{P}_N^{(1)}(\phi)$

จากสมการ $\mathbf{P}_N(\phi)$

$$\mathbf{P}_N(\phi) = \mathbf{P}_n\left(\mathbf{I} - \mathbf{P}_{\tilde{\mathbf{a}}_{B\gamma_R}}(\phi)\mathbf{Z}_2(\phi)\mathbf{P}_n\right)$$

เมื่อ

$$\mathbf{P}_n = \mathbf{I} - \mathbf{U}_{BsR}\mathbf{U}_{BsR}^T$$

$$\mathbf{P}_{\tilde{\mathbf{a}}_{B\gamma_R}}(\phi) = \tilde{\mathbf{a}}_{B\gamma_R}(\phi)\left(\tilde{\mathbf{a}}_{B\gamma_R}^T(\phi)\tilde{\mathbf{a}}_{B\gamma_R}(\phi)\right)^{-1}\tilde{\mathbf{a}}_{B\gamma_R}^T(\phi)$$

$$\mathbf{Z}_1(\phi) = \left(\mathbf{I} - \mathbf{U}_{BsR}^T\mathbf{P}_{\tilde{\mathbf{a}}_{B\gamma_R}}(\phi)\mathbf{U}_{BsR}\right)^{-1}$$

$$\mathbf{Z}_2(\phi) = \mathbf{I} + \mathbf{U}_{BsR}\mathbf{Z}_1(\phi)\mathbf{U}_{BsR}^T\mathbf{P}_{\tilde{\mathbf{a}}_{B\gamma_R}}(\phi)$$

ข้อสังเกต $\mathbf{P}_n$ และ $\mathbf{U}_{BsR}$ ไม่เป็นฟังก์ชันของ $\phi$ ดังนั้นเมื่อหา $\frac{\partial}{\partial\phi}$ จะให้ค่าเป็นศูนย์

การทำอนุพันธ์ย่อยของ $\mathbf{P}_N(\phi)$ เทียบกับ $\phi$ อันดับที่หนึ่ง แสดงได้ดังนี้

$$\mathbf{P}_N^{(1)}(\phi) = \frac{\partial\left(\mathbf{P}_N(\phi)\right)}{\partial\phi} = \mathbf{P}_n\left(-\mathbf{P}_{\tilde{a}_{B\gamma R}}^{(1)}(\phi)\mathbf{Z}_2(\phi)\mathbf{P}_n - \mathbf{P}_{\tilde{a}_{B\gamma R}}(\phi)\left(\frac{\partial\mathbf{Z}_2(\phi)}{\partial\phi}\right)\mathbf{P}_n\right) \quad (ค.1)$$

เมื่อ

$$\begin{aligned}
\mathbf{P}_{\tilde{a}_{B\gamma R}}^{(1)}(\phi) &= \frac{\partial\,\mathbf{P}_{\tilde{a}_{B\gamma R}}(\phi)}{\partial\phi} \\
&= \frac{\partial}{\partial\phi}\left(\tilde{\mathbf{a}}_{B\gamma_R}(\phi)\left(\tilde{\mathbf{a}}_{B\gamma_R}^T(\phi)\tilde{\mathbf{a}}_{B\gamma_R}(\phi)\right)^{-1}\tilde{\mathbf{a}}_{B\gamma_R}^T(\phi)\right) \\
&= \left(\mathbf{I} - \mathbf{P}_{\tilde{a}_{B\gamma R}}(\phi)\right)\tilde{\mathbf{a}}_{B\gamma_R}^{(1)}(\phi)\left(\tilde{\mathbf{a}}_{B\gamma_R}^T(\phi)\tilde{\mathbf{a}}_{B\gamma_R}(\phi)\right)^{-1}\tilde{\mathbf{a}}_{B\gamma_R}^T(\phi) + \\
&\quad + \tilde{\mathbf{a}}_{B\gamma_R}(\phi)\left(\tilde{\mathbf{a}}_{B\gamma_R}^T(\phi)\tilde{\mathbf{a}}_{B\gamma_R}(\phi)\right)^{-1}\tilde{\mathbf{a}}_{B\gamma_R}^{(1)^T}(\phi)\left(\mathbf{I} - \mathbf{P}_{\tilde{a}_{B\gamma R}}(\phi)\right)
\end{aligned} \quad (ค.2)$$

จากสมการที่ (ค.2) หากกำหนดให้

$$\mathbf{J}(\phi)=\left(\mathbf{I}-\mathbf{P}_{\tilde{a}_{B\gamma_R}}(\phi)\right)\tilde{\mathbf{a}}^{(1)}_{B\gamma_R}(\phi)\left(\tilde{\mathbf{a}}^{T}_{B\gamma_R}(\phi)\tilde{\mathbf{a}}_{B\gamma_R}(\phi)\right)^{-1}\tilde{\mathbf{a}}^{T}_{B\gamma_R}(\phi) \quad (ค.3)$$

จะได้ว่า $\mathbf{P}^{(1)}_{\tilde{a}_{B\gamma R}}(\phi)$ อยู่ในรูปของ

$$\mathbf{P}^{(1)}_{\tilde{a}_{B\gamma R}}(\phi)=\mathbf{J}(\phi)+\mathbf{J}^{T}(\phi) \quad (ค.4)$$

พิจารณา

$$\begin{aligned}\frac{\partial \mathbf{Z}_1(\phi)}{\partial\phi}&=-\left(\mathbf{I}-\mathbf{U}^{T}_{BsR}\mathbf{P}_{\tilde{a}_{B\gamma R}}(\phi)\mathbf{U}_{BsR}\right)^{-1}\left(\frac{\partial}{\partial\phi}\left(\mathbf{I}-\mathbf{U}^{T}_{BsR}\mathbf{P}_{\tilde{a}_{B\gamma R}}(\phi)\mathbf{U}_{BsR}\right)\right)\left(\mathbf{I}-\mathbf{U}^{T}_{BsR}\mathbf{P}_{\tilde{a}_{B\gamma R}}(\phi)\mathbf{U}_{BsR}\right)^{-1}\\&=\mathbf{Z}_1(\phi)\mathbf{U}^{T}_{BsR}\mathbf{P}^{(1)}_{\tilde{a}_{B\gamma R}}(\phi)\mathbf{U}_{BsR}\mathbf{Z}_1(\phi)\end{aligned} \quad (ค.5)$$

นำสมการที่ (ค.5) มาพิจารณาร่วมกับการหา $\frac{\partial \mathbf{Z}_2(\phi)}{\partial\phi}$ จะได้ว่า

$$\begin{aligned}\frac{\partial \mathbf{Z}_2(\phi)}{\partial\phi}&=\mathbf{U}_{BsR}\left(\frac{\partial \mathbf{Z}_1(\phi)}{\partial\phi}\right)\mathbf{U}^{T}_{BsR}\mathbf{P}_{\tilde{a}_{B\gamma R}}(\phi)+\mathbf{U}_{BsR}\mathbf{Z}_1(\phi)\mathbf{U}^{T}_{BsR}\mathbf{P}^{(1)}_{\tilde{a}_{B\gamma R}}(\phi)\\&=\mathbf{U}_{BsR}\mathbf{Z}_1(\phi)\mathbf{U}^{T}_{BsR}\mathbf{P}^{(1)}_{\tilde{a}_{B\gamma R}}(\phi)\mathbf{U}_{BsR}\mathbf{Z}_1(\phi)\mathbf{U}^{T}_{BsR}\mathbf{P}_{\tilde{a}_{B\gamma R}}(\phi)+\mathbf{U}_{BsR}\mathbf{Z}_1(\phi)\mathbf{U}^{T}_{BsR}\mathbf{P}^{(1)}_{\tilde{a}_{B\gamma R}}(\phi)\\&=\mathbf{U}_{BsR}\mathbf{Z}_1(\phi)\mathbf{U}^{T}_{BsR}\mathbf{P}^{(1)}_{\tilde{a}_{B\gamma R}}(\phi)\left(\mathbf{U}_{BsR}\mathbf{Z}_1(\phi)\mathbf{U}^{T}_{BsR}\mathbf{P}_{\tilde{a}_{B\gamma R}}(\phi)+\mathbf{I}\right)\\&=\mathbf{U}_{BsR}\mathbf{Z}_1(\phi)\mathbf{U}^{T}_{BsR}\mathbf{P}^{(1)}_{\tilde{a}_{B\gamma R}}(\phi)\left(\mathbf{I}+\mathbf{U}_{BsR}\mathbf{Z}_1(\phi)\mathbf{U}^{T}_{BsR}\mathbf{P}_{\tilde{a}_{B\gamma R}}(\phi)\right)\\&=\mathbf{U}_{BsR}\mathbf{Z}_1(\phi)\mathbf{U}^{T}_{BsR}\mathbf{P}^{(1)}_{\tilde{a}_{B\gamma R}}(\phi)\mathbf{Z}_2(\phi)\end{aligned} \quad (ค.6)$$

นำสมการที่ (ค.6) แทนลงในสมการที่ (ค.1) จะได้ว่า

$$\begin{aligned}\mathbf{P}^{(1)}_{N}(\phi)&=\mathbf{P}_n\left(-\mathbf{P}^{(1)}_{\tilde{a}_{B\gamma R}}(\phi)\mathbf{Z}_2(\phi)\mathbf{P}_n-\mathbf{P}_{\tilde{a}_{B\gamma R}}(\phi)\left(\mathbf{U}_{BsR}\mathbf{Z}_1(\phi)\mathbf{U}^{T}_{BsR}\mathbf{P}^{(1)}_{\tilde{a}_{B\gamma R}}(\phi)\mathbf{Z}_2(\phi)\right)\mathbf{P}_n\right)\\&=-\mathbf{P}_n\left(\mathbf{P}^{(1)}_{\tilde{a}_{B\gamma R}}(\phi)\mathbf{Z}_2(\phi)\mathbf{P}_n+\mathbf{P}_{\tilde{a}_{B\gamma R}}(\phi)\left(\mathbf{U}_{BsR}\mathbf{Z}_1(\phi)\mathbf{U}^{T}_{BsR}\mathbf{P}^{(1)}_{\tilde{a}_{B\gamma R}}(\phi)\mathbf{Z}_2(\phi)\right)\mathbf{P}_n\right)\\&=-\mathbf{P}_n\left(\left(\mathbf{I}+\mathbf{P}_{\tilde{a}_{B\gamma R}}(\phi)\mathbf{U}_{BsR}\mathbf{Z}_1(\phi)\mathbf{U}^{T}_{BsR}\right)\left(\mathbf{P}^{(1)}_{\tilde{a}_{B\gamma R}}(\phi)\mathbf{Z}_2(\phi)\mathbf{P}_n\right)\right)\end{aligned} \quad (ค.7)$$

เมื่อ $\mathbf{Z}^{T}_1(\phi)=\mathbf{Z}_1(\phi)$ ดังนั้นสมการที่ (ค.7) จึงกลายเป็น

$$\mathbf{P}_N^{(1)}(\phi) = -\mathbf{P}_n \mathbf{Z}_2^T(\phi) \mathbf{P}_{\tilde{a}_{B\gamma R}}^{(1)}(\phi) \mathbf{Z}_2(\phi) \mathbf{P}_n \quad (ค.8)$$

# ประวัติผู้เขียนวิทยานิพนธ์

นาย พรสักก์ หมื่นหาญ สำเร็จการศึกษาปริญญาวิศวกรรมศาสตร์บัณฑิต สาขาวิศวกรรมโทรคมนาคม หลักสูตรต่อเนื่อง ภาควิชาวิศวกรรมโทรคมนาคม คณะวิศวกรรมศาสตร์ สถาบันเทคโนโลยีพระจอมเกล้าเจ้าคุณทหารลาดกระบัง เมื่อปี พศ.2544 เนื่องจากผู้เขียนมีความสนใจเกี่ยวกับงานวิจัยด้าน Smart Antenna System จึงเข้ารับการศึกษาต่อ ในหลักสูตรวิศวกรรมศาสตร์มหาบัณฑิต สาขาวิศวกรรมไฟฟ้าสื่อสาร สังกัดห้องปฏิบัติการวิจัยกรรมวิธีสัญญาณดิจิทัล (DSPRL) ภาควิชาวิศวกรรมไฟฟ้า คณะวิศวกรรมศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย