CHAPTER II

THE SIERPINSKI GASKET

In this chapter, we introduce the Sierpinski gasket and define harmonic functions on it. Moreover we will discuss some of their properties. See [3] for more details and proofs.

Definition 2.1. Let p_0, p_1 and p_2 be the vertices of an equilateral triangle of unit length in the $\mathbb{R}^2$ plane. For $i \in S = \{0, 1, 2\}$, define $F_i : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F_i(x) = \frac{1}{2}(x - p_i) + p_i = \frac{1}{2}(x + p_i),$$

which is clearly a contraction and has the **fixed point** p_i, i.e., $F_i(p_i) = p_i$. Then the set of contractions $\{F_i,\ i = 0, 1, 2\}$ defines a self-similar set SG, **the Sierpinski gasket.** Then $\mathcal{L} = (SG, S, \{F_i\}_{i \in S})$ is a **post critically finite** self-similar structure. In fact $\mathcal{C}_{\mathcal{L},SG} = \{p_0(1), p_0(2), p_1(2)\}$, $\mathcal{C}_{\mathcal{L}} = \{1\dot{0}, 0\dot{1}, 2\dot{0}, 0\dot{2}, 2\dot{1}, 1\dot{2}\}$ and $\mathcal{P}_{\mathcal{L}} = \{\dot{0}, \dot{1}, \dot{2}\}$. Also $V_0 = \{p_0, p_1, p_2\}$.

Figure (1). The Sierpinski gasket.

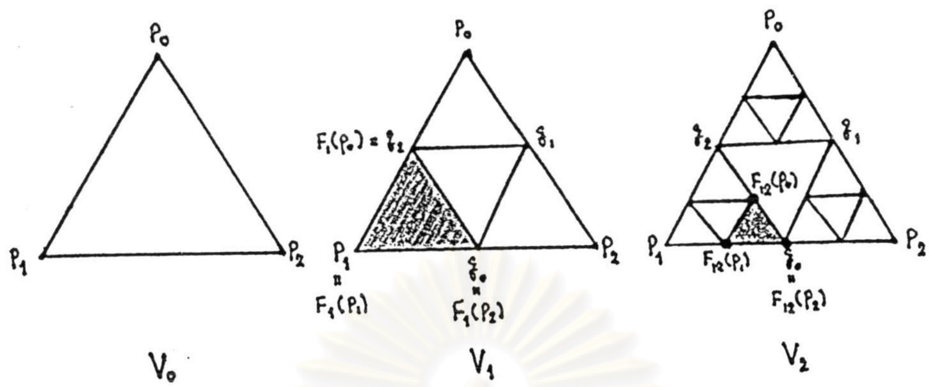

Figure (2). Approximating sequence (V_m, B_m) of the Sierpinski gasket.

Next, we define a sequence (V_m, B_m) of graphs approximating SG, where V_m and B_m denote the set of vertices and the set of edges respectively, see Figure (2).

Definition 2.2. For $V_0 = \{p_0, p_1, p_2\}, B_0 = \{(p_0, p_1), (p_0, p_2), (p_1, p_2)\}$, and $\{F_i\}_{i=0}^{2}$ is the collection of contractions that defines the Sierpinski gasket and each integer $m \geq 1$, we set

$$V_m = \bigcup_{0 \leq i_1, \ldots, i_m \leq 2} F_{i_1} \circ F_{i_2} \circ \ldots \circ F_{i_m}(V_0) = \bigcup_{\omega \in \{0,1,2\}^m} F_\omega(V_0),$$

$$B_m = \{(F_{i_1} \circ \ldots \circ F_{i_m}(p_k), \; F_{i_1} \circ \ldots \circ F_{i_m}(p_l)) \mid 0 \leq i_1, \; i_2, \ldots, i_m \leq 2, \; 0 \leq k < l \leq 2\}.$$

For convenience we write $p_i(\omega) = F_\omega(p_i)$ and set $V_* = \bigcup_{m \geq 0} V_m$. The closure $\overline{V_*}$ is the Sierpinski gasket. Moreover, we consider V_0 as the boundary of SG.

Definition 2.3. Let $\ell(V_m) = \{f \mid f : V_m \to \mathbb{R}\}$, and define a map $H_m : \ell(V_m) \to \ell(V_m)$ by

$$(H_m f)(p) = \sum_{q \in V_{m,p}} (f(q) - f(p)),$$

where $f \in \ell(V_m)$, $p \in V_m$ and $V_{m,p}$ denotes the set

$$V_{m,p} = \{q \mid q \text{ is connected to } p \text{ by an edge in } (V_m, B_m)\}$$

$$= \{q \mid (p, q) \in B_m \text{ or } (q, p) \in B_m\}.$$

In term of matrices we have the following:

$$H_0 = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}, \text{ and } \begin{pmatrix} (H_0f)(p_0) \\ (H_0f)(p_1) \\ (H_0f)(p_2) \end{pmatrix} = H_0 \begin{pmatrix} f(p_0) \\ f(p_1) \\ f(p_2) \end{pmatrix}.$$

If we set

$$T = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}, J = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, X = \begin{pmatrix} -4 & 1 & 1 \\ 1 & -4 & 1 \\ 1 & 1 & -4 \end{pmatrix},$$

we get $H_1 = \begin{pmatrix} T & J^t \\ J & X \end{pmatrix}$, and so

$$\begin{pmatrix} (H_1f)|_{V_0} \\ (H_1f)|_{V_1 \setminus V_0} \end{pmatrix} = \begin{pmatrix} T & J^t \\ J & X \end{pmatrix} \begin{pmatrix} f|_{V_0} \\ f|_{V_1 \setminus V_0} \end{pmatrix}.$$

The maps H_0 and H_1 are related in the following way.

Lemma 2.4.

$$\frac{3}{5}(H_0f)(p_i) = (H_1f)(p_i) + \frac{2}{5}\sum_{j \neq i}(H_1f)(q_i) + \frac{1}{5}(H_1f)(q_i).$$

where $q_i = p_j(k)$ *and* $i \neq j \neq k \in \{0, 1, 2\}$. *See Figure (2).*

Definition 2.5. Denoted by $C(SG)$ the set of the continuous real-valued functions on SG, i.e., $C(SG) = \{f | \ f : SG \to \mathbb{R}, f \text{ is continuous}\}$. Let $f \in C(SG)$. We say that the function f is **harmonic** if f satisfies

$$(H_m f)(p) = 0$$

for every $m \geq 1$ and every p in $V_m \setminus V_0$.

Theorem 2.6. *Given three numbers* α, β, γ, *there exists a unique harmonic function* f *satisfying* $f(p_0) = \alpha, f(p_1) = \beta$ *and* $f(p_2) = \gamma$.

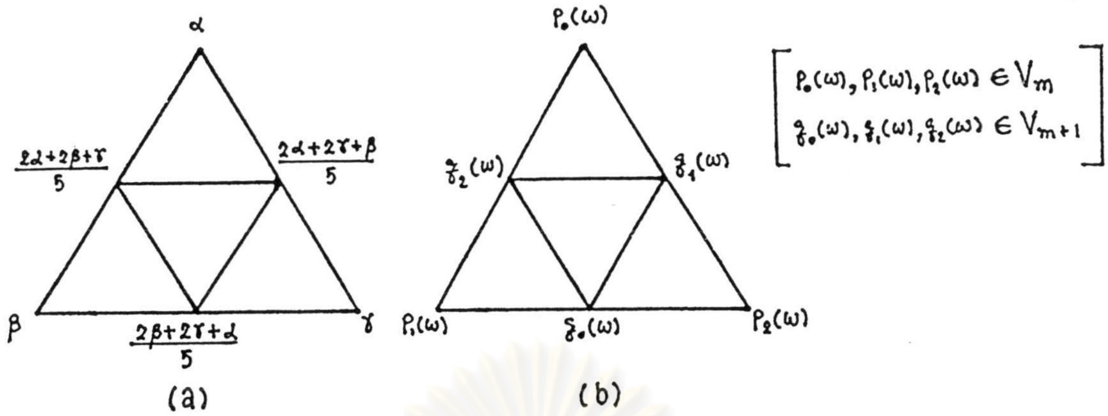

Figure (3). (a) Values of f on V_1; (b) passing from V_m to V_{m+1}

In fact, the unique harmonic function satisfies the matrix form

$$\begin{pmatrix} f(q_0(\omega)) \\ f(q_1(\omega)) \\ f(q_2(\omega)) \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} f(p_0(\omega)) \\ f(p_1(\omega)) \\ f(p_2(\omega)) \end{pmatrix}$$

for every $\omega \in W_m$, $m \in \mathbb{N}_0$ and q_i, $i = 0, 1, 2$ in Figure(3).

Theorem 2.7. *(The Maximum Principle). If a harmonic function defined on the Sierpinski gasket SG attains the maximum value in the interior $SG \setminus V_0$ of SG, then f is constant throughout SG.*

Definition 2.8. For any continuous function u on SG and $p \in V_m \setminus V_0$, write

$$(\Delta_m u(p)) = \frac{3}{2}5^m (H_m u)(p).$$

Suppose for some $\varphi \in C(K)$ we have

$$\max_{p \in V_m \setminus V_0} |(\Delta_m u(p)) - \varphi(p)| \to 0 \text{ as } m \to +\infty.$$

Then we write $\Delta u = \varphi$ and call it the **Laplacian** on the Sierpinski gasket. Furthermore, we denote by $Dom\Delta$, domain of Δ, the set of all $u \in C(K)$ for which there exists some $\varphi \in C(K)$ such that $\Delta u = \varphi$.