CHAPTER I

INTRODUCTION AND PRELIMINARIES

1.1 Introduction

Many examples of fractal sets such as the Cantor set, the Sierpinski gasket and the Koch curve were known to mathematicians early in the twentieth century as purely mathematical objects. In the 1970s, Mandelbrot introduced the notion of fractals as a new class of mathematical objects which represent nature. For example, some coast lines were not smooth curves and have infinite length like Koch curve.

Although the Laplacian on the Sierpinski gasket and some self-similar sets is now well known, the first order derivative are less studied. In this thesis, we define derivtive of a real-valued function on the Sierpinski gasket that has a simple structure. We also compare our work with the definitions and results in [1] and [2].

This thesis is organized as follows. In the next section we give notation and preliminaries. In chapter 2 we introduce the Sierpinski gasket, harmonic functions and the Laplacian on the Sierpinski gasket. In chapter 3, we explain the Neumann derivative, the normal derivative, the transverse derivative and the derivatives of Strichartz. In chapter 4, we define a derivative of functions on the Sierpinski gasket and study the relation between our definition and results and those in chapter 3.

Throughout this thesis, let $\mathbb{R}$, $\mathbb{R}^*$ and $\mathbb{N}_0$ denote the set of all real numbers, the set of all real numbers without zero and the set of all non-negative intergers, respectively.

1.2 Self-similar structure

In this chapter, we introduce some definitions and the existence and the uniqueness theorem for self-similar sets (see [1],[3] and [4]). To understanding the self-similar structure which will be introduced in the last of this chapter, we need the notations on self-similar sets. These notations are defined by Kigami (see [2] for more details).

Definition 1.1. ([1]) A map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is **Lipschitz continuous** with respect to the Euclidean metric d if there exists constant c, $d(f(x), f(y)) \leq cd(x,y)$ for all $x, y \in \mathbb{R}^2$. This constant c is denoted by $Lip(f)$. In particular, if $c \in (0,1)$, then f is called a **contraction** with contraction ratio $Lip(f)$.

Theorem 1.2. ([1]) *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a contraction with respect to the Euclidean metric d. Then there exists a unique fixed point of f. In other words, there exists a unique solution to the equation $f(x) = x$. Moreover, if x_* is the fixed point of f, then $\{f^n(a)\}_{n\geq 0}$ converges to x_* for all $a \in \mathbb{R}^2$ where f^n is the n-th iteration of f.*

Let $K(\mathbb{R}^2)$ be the sets of all non-empty compact sets in $\mathbb{R}^2$. For an element $A \in K(\mathbb{R}^2)$ and $\epsilon > 0$ we set

$$N_\epsilon(A) = \left\{ x \in \mathbb{R}^2 : \text{dist}(x, A) \equiv \min_{y \in A} d(x,y) < \epsilon \right\},$$

and say that $N_\epsilon(A)$ is the $\epsilon-$neighborhood of A. The set $K(\mathbb{R}^2)$ become an abstract metric space when we give it the following **Hausdorff metric**:

$$d_H(A,B) = \min\{\epsilon \geq 0 : A \subseteq N_\epsilon(B) \text{ and } B \subseteq N_\epsilon(A)\}.$$

The advantage of having a Hausdorff metric implies the following theorem

Theorem 1.3. ([3], [4]) *The Hausdorff metric d_H turns $K(\mathbb{R}^2)$ into a complete metric space.*

Theorem 1.4. ([1]) *If $f_i : \mathbb{R}^2 \to \mathbb{R}^2$ is a contraction with respect to the Euclidean metric d for $i = 1, 2, ..., m$, then there exists a unique non-empty compact subset E of $\mathbb{R}^2$ that* *satisfies*

$$E = \bigcup_{i=1}^{m} f_i(E).$$

E is called the **self-similar set** *with respect to* $\{f_1, f_2, ..., f_m\}$.

Definition 1.5. ([3], [4]) Let N be a natural number and $m \in \mathbb{N}_0$.

(1). $W_m^N = \{1, 2, ..., N\}^m = \{\omega_1\omega_2...\omega_m : \omega_i \in \{1, 2, ..., N\}\}$. When $m = 0$, we define $\{1, 2, ..., N\}^m$ as a singleton set whose element is ϕ; $\{1, 2, ..., N\}^0 = \{\phi\}$ and call ϕ the empty word.

(2). If $f_i : \mathbb{R}^2 \to \mathbb{R}^2$ is a contraction with respect to the Euclidean metric for $i = 1, 2, ..., N$ and that K is the self-similar set with respect to $\{f_1, f_2, ..., f_N\}$. For $\omega = \omega_1\omega_2...\omega_m \in W_m$, we define

$$f_\omega = f_{\omega_1} \circ f_{\omega_2} \circ ... \circ f_{\omega_m}, \quad K_\omega = f_\omega(K).$$

Moreover, $f_\omega(p) = p(\omega)$ for any point $p \in K$. For convenience we make the following convention; $f_\phi =$ the identity map.

(3). The collection of one-sided infinite sequences of symbols $\{1, 2, ..., N\}$ is denoted by $\sum^N$, which is called the **shift space** with $N-$symbol. More precisely,

$$\sum^N = \{1, 2, ..., N\}^{\mathbb{N}} = \{\omega_1\omega_2\omega_3... : \omega_i \in \{1, 2, ..., N\}, \text{for } i \in \mathbb{N}\}.$$

For $k \in \{1, 2, ..., N\}$, define a map $\sigma_k : \sum^N \to \sum^N$ by

$$\sigma_k(\omega_1\omega_2\omega_3...) = k\omega_1\omega_2\omega_3... \ .$$

Also define $\sigma : \sum^N \to \sum^N$ by $\sigma(\omega_1\omega_2\omega_3...) = \omega_2\omega_3\omega_4...$ and σ is called the **shift map**. For ease of notation, we write W_m and $\sum$ instead of W_m^N and $\sum^N$, respectively.

Theorem 1.6. ([1]) *Let f_i be a contraction on $\mathbb{R}^2$ with respect to the Euclidean metric d for $i \in \{1, 2, ..., N\}$ and K be any self-similar set with respect to $\{f_i\}_{i=1}^N$. Then for any $\omega = \omega_1\omega_2\omega_3... \in \sum, \cap_{m\geq1} K_{\omega_1\omega_2...\omega_m}$ contains only one piont. If we define $\pi : \sum \to K$ by $\{\pi(\omega)\} = \cap_{m\geq1} K_{\omega_1\omega_2...\omega_m}$, then π is a continuous surjective map. Moreover, for any $i \in \{1, 2, ..., N\}, \pi \circ \sigma_i = f_i \circ \pi$.*

Definition 1.7. ([1]) Let K be a compact metrizable topological space and let S be a finite set. Also, let F_i be a continuous injection from K to itself for any $i \in S$. Then, $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ is called a **self-similar structure** if there exists a continuous surjection $\pi : \sum \to K$ such that $F_i \circ \pi = \pi \circ \sigma_i$ for every $i \in S$, where $\sum = S^{\mathbb{N}}$ is the one-side shift space and $\sigma_i : \sum \to \sum$ is defined by $\sigma_i(\omega_1\omega_2...) = i\omega_1\omega_2...$ for each $\omega_1\omega_2... \in \sum$.

Proposition 1.8. ([1]) *If $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ is a self-similar structure, then π is unique. In fact*

$$\{\pi(\omega)\} = \bigcap_{m \geq 0} F_{\omega_1\omega_2...\omega_m}(K)$$

for any $\omega = \omega_1\omega_2... \in \sum$.

Definition 1.9. ([1]) Let $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ be a self-similar struture. We define

$$C_{\mathcal{L},K} = \bigcup_{i,j \in S, i \neq j} (F_i(K) \cap F_j(K)),\ \mathcal{C}_{\mathcal{L}} = \pi^{-1}(C_{\mathcal{L},K}) \text{ and } \mathcal{P}_{\mathcal{L}} = \bigcup_{n \geq 1} \sigma^n(\mathcal{C}_{\mathcal{L}}).$$

$\mathcal{C}_{\mathcal{L}}$ is call the **critical set** of $\mathcal{L}$ and $\mathcal{P}_{\mathcal{L}}$ is called the **post critical set** of $\mathcal{L}$. Also we define $V_0(\mathcal{L}) = \pi(\mathcal{P}_{\mathcal{L}})$.

Proposition 1.10. ([1]) *Let $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ be a self-similar struture. For any $p \in K$ and any $m \geq 0$, define*

$$K_{m,p} = \bigcup_{\omega \in W_m : p \in K_\omega} K_\omega.$$

Then $\{K_{m,p}\}_{m \geq 0}$ is a **fundamental system of neighborhoods** *of p.*

Definition 1.11. ([1]) Let $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ be a self-similar structure. $\mathcal{L}$ is said to be **post critically finite** or **p.c.f.** for short if and only if the postcritical set $\mathcal{P}_{\mathcal{L}}$ is a finite set.

Lemma 1.12. ([1]) *Let $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ be postcritically finite and let $p \in K$. If $F_\omega(p) = p$ for some $\omega \in \bigcup_{m \geq 0} W_m$ and $\omega \neq \phi$, then $\pi^{-1}(p) = \{\dot{\omega}\}$.*