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COHERENT STATE REPRESENTATION APPROACH TO FRACTIONAL QUANTUM HALL EFFECT

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สถานะตัวแทนโคเฮียเรนท์จะใช้ในการศึกษาปรากฏการณ์ควอนตัมฮอลล์เชิงเศษส่วน โดยการใส่พจน์ เชิน-ไซมอน ในลากรานเจียน ซึ่งจะทำให้อิเล็กตรอนในปรากฏการณ์ดังกล่าวจับกับเส้นแรง เชิน-ไซมอน รวมกันเรียกว่า อนุภาคเชิงประกอบ และ ทำให้เกิดการเปลี่ยนแปลงเชิงสถิติของระบบ ในวิทยานิพนธ์นี้ จะศึกษาในระดับสนามเฉลี่ย โดยพบว่า ณ เศษส่วนเติมเต็ม (filling factor) ที่มีค่าเท่ากับ 1/(2p+1) โดย p เป็นจำนวนเต็มใดๆ ที่มีความสัมพันธ์กับสัมประสิทธิ์หน้าพจน์ เชิน-ไซมอน และ สามารถอธิบายปรากฏการณ์ในแบบของปรากฏการณ์ควอนตัมเชิงจำนวนเต็มได้ และ พบว่าสภาพนำไฟฟ้าจะเป็นค่าของเศษส่วนเติมเต็มคูณกับประจุของอิเล็กตรอน และ ค่าคงที่ของแพลงค์

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SARUN PHIBANCHON: COHERENT STATE REPRESENTATION APPROACH TO FRACTIONAL QUANTUM HALL EFFECT. THESIS ADVISOR: DR. RUJIKORN DHANAWITTAYAPOL, THESIS COADVISOR : DR. PORNTHEP NISAMANEEPHONG, 76 pp.
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The fractional quantum Hall effect is studied by using the coherent path integral technique. By introducing the Chern-Simons term into the Lagrangian, the electrons in the system become attached to the Chern-Simons flux quanta resulting in a system of composite particles and the phenomenon of statistical transmutation. In this thesis, we use the mean field approach in which the method of integer quantum Hall effect can be applied only when the filling factor takes the form 1/(2p+1) with p being an arbitrary positive integer related to the coefficient of the Chern-Simons term in the Lagrangian. It is also found that the Hall conductivity is given by 1/(2p+1) multiplied by the electron charge and Planck's constant.

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CHAPTER I

INTRODUCTION

There are various remarkable phenomena in condensed matter systems such as superconductivity that can be explained theoretically by the so-called phenomenological approach in which the relevant free energy is expanded as a series of the appropriate order parameter. The phenomenological approach is therefore a way to understand the system from the macroscopic point of view. The other phenomenon is the quantum Hall effect (QHE) which was discovered by Von Klitzing et al. [1] in 1980. They performed their experiment using a two-dimensional system in a high magnetic field ($\approx$ 1-10 Tesla). What they found their experiment is that the Hall resistance varies stepwise with changes in the magnetic field (Fig 3.6), this is not an expected result from the classical Hall effect (fig 3.2). Each step is called a plateau. Von Klitzing and his colleagues concluded that at each plateau the Hall conductance equals an integral value multiplied by the electric charge and Planck's constant. This integer is called the filling factor, ν, which can be defined as the number of electrons in a sample divided by the number of magnetic flux quanta penetrating it. This phenomenon, however, can be described by solving a one-particle Schrödinger equation in an external magnetic field which has discrete energy eigenvalues called the Landau levels. Two years later, Tsui et al. [2] discovered that filling factors can also be fractional numbers. These phenomena with the fractional and integral filling factors, therefore, are respectively called the Fractional Quantum Hall Effect (FQHE) and the Integer Quantum Hall Effect (IQHE). Their experiment was performed using the cleaner sample, in a higher magnetic field ($\approx$ 30 Tesla) and at lower temperature ($\approx$ 150 mK) than what Von Klitzing and his colleagues have done. The clean system implies that electrons can move through the sample more freely than those in the dirty system before they are scattered by impurities in the sample. To explain this result one should consider the interactions between electrons. The Hamiltonian of the FQH system can be written as

$$H = \sum_i \frac{(p_i + eA(x_i))^2}{2m} + \sum_{i \neq j = 1} V(x_i - x_j). \tag{1.1}$$

If we neglect the second term on the right hand side of Eq. (1.1), the Hamiltonian will describe the system of the IQHE. In 1983, Laughlin [7] proposed the wave function for explaining the ground state of the FQHE as

$$\psi_m(z_i) = \prod_{i \neq j} (z_i - z_j)^m e^{-\sum_k \frac{|z_k|^2}{4l^2}} \tag{1.2}$$

where z_i is the complex coordinate for the i$^{\text{th}}$ electron, m is an odd number $(3, 5, 7, \ldots)$ and l^2, the magnetic length, equals $1/eB$ with B being the magnetic field. This wave function describes a uniform distribution of (fluid-like) electrons and describes only at ν equal to $1/m$. The meaning of the filling factor according to Laughlin is that if ν equal to one, there is one flux quanta per electron while if ν is equal to 1/3, there are three flux quanta per electron. Later on in 1989, Jain [8] proposed another picture for the explanation of FQH system. In his picture, the FQHE is just the IQHE of composite fermions; each composite fermion being formed by attaching an even number of flux quanta to an electron. Thus the case $v = 1/3$ corresponds to the situation that one flux is attached to a composite fermion. This approach, however, does not give the explanation for all the filling factors. The common thing of the two approaches above is that they try to find the wave function. Presently, there are many works in the field of theoretical explanation of the FQH system. Among these is the Ginzburg-Landau-Chern-Simons approach (GLCS) [13, 14] which successfully interprets a variety of the properties of the FQH system from a phenomenological point of view.

The organization of this thesis is as follows. In Chapter 2, we give a short review of coherent states. The detailed discussion of the Quantum Hall effect is then given in Chapter 3. In Chapter 4, the coherent state representation is applied to the fractional quantum Hall effect. Finally, the conclusion and discussion is made in Chapter 5.

CHAPTER II

COHERENT STATES

In 1926, Schrödinger proposed the concept of what is now called the coherent states [12] in connection with the classical states of the quantum harmonic oscillator. Thus the coherent states were invented immediately after the birth of quantum mechanics. However, between 1926 and 1963, activities of this field remained dormant. But thirty-five years after Schrödinger's pioneering paper, the first modern and specific application was made by Glauber and Sudarshan [9, 12] and launched this fruitful and important field of study. Glauber constructed the eigenstates of the annihilation operator of the harmonic oscillator in order to study the electromagnatic correlation functions. At the same time roughly as Glauber and Sudarshan, Klauder [10] developed a set of continuous states in which the basic ideas of coherent states for arbitrary Lie groups were contained. This chapter will follow the Glauber's definitions of coherent states in the study of the single particle state. The many-particle states will be constructed in the same way but will be more carefully considered in case of the indistinguishable many particles whose treatment depends on whether the partcles are bosons or fermions.

2.1 The Glauber's Definitions of Coherent States

The single-mode coherent states can be constructed starting from any one of three mathematical definitions.

Definition 1 The coherent states $\mid \alpha >$ are eigenvalues of the harmonic oscillator annihilation operator $\widehat{a}$

$$\widehat{a} \mid \alpha >= \alpha \mid \alpha > \tag{2.1}$$

where α is a complex number.

Definition 2 The coherent states $\mid \alpha >$ can be obtained by applying a displacement operator $\widehat{D}(\alpha)$ on the vacuum state of the harmonic oscillator,

$$\mid \alpha >= \widehat{D}(\alpha) \mid 0 > \tag{2.2}$$

where $\mid 0 >$ is the vacuum state of the oscillator. The displacement operator $\widehat{D}(\alpha)$ is defined as

$$\widehat{D}(\alpha) = e^{\alpha \widehat{a}^\dagger - \alpha^* \widehat{a}}. \tag{2.3}$$

Definition 3 The coherent state $\mid \alpha >$ is the quantum state with a minimum uncertainty relation,

$$(\Delta p)^2 (\Delta q)^2 = \left(\frac{1}{2}\right)^2 \tag{2.4}$$

where the coordinate and momentum operators $(\widehat{q}, \widehat{p})$ are defined as

$$\widehat{q} = \frac{1}{\sqrt{2}}(\widehat{a} + \widehat{a}^\dagger) \tag{2.5}$$

$$\widehat{p} = \frac{1}{i\sqrt{2}}(\widehat{a} - \widehat{a}^\dagger) \tag{2.6}$$

and

$$(\Delta f)^2 \equiv < \alpha \mid (\widehat{f} - < \widehat{f} >)^2 \mid \alpha > \tag{2.7}$$

with $< \widehat{f} > \equiv < \alpha \mid \widehat{f} \mid \alpha >$. $\widehat{a}$ and $\widehat{a}^\dagger$ satisfy the commutation relation (see Appendix A)

$$[\hat{a}, \hat{a}^\dagger] = 1. \tag{2.8}$$

Consider the coherent states in the Hilbert space

$$\begin{aligned} \mid \alpha > &= \widehat{D}(\alpha) \mid 0 > \\ &= e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} \mid 0 > . \end{aligned}$$

By using the Baker-Campbell-Hausdorff formula (proved in Appendix A), it can be expressed as

$$\begin{aligned} \mid \alpha > &= e^{\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}} e^{-[\alpha\hat{a}^\dagger, -\alpha^*\hat{a}]/2} \mid 0 > \\ &= e^{\frac{-\alpha\alpha^*}{2}} e^{\alpha\hat{a}^\dagger} \mid 0 > \end{aligned} \tag{2.9}$$

where we have used

$$\begin{aligned} e^{\alpha^*\hat{a}} \mid 0 > &= \sum_{n=0}^{\infty} \frac{(\alpha^*\hat{a})^n}{n!} \mid 0 > \\ &= \mid 0 > \quad \text{(because } \hat{a} \mid 0 >= 0). \end{aligned}$$

The coherent state has the following properties (see Appendix A for proofs):

$$1. \qquad \frac{1}{\pi} \int d^2\alpha \mid \alpha >< \alpha \mid = 1; \tag{2.10}$$

$$2. \qquad < \beta \mid \alpha >= e^{\beta^*\alpha} e^{\frac{-|\beta|^2}{2}} e^{\frac{-|\alpha|^2}{2}}. \tag{2.11}$$

All of the above is based on the single-particle state. We next consider the many-particle case following [11]. It is, however, not different from the single-particle case much. We first introduce the creation and annihilation operators for the many-particle case.

<u>Boson</u>

$$\hat{a}^\dagger_{\alpha_i} \mid n_{\alpha_1}, \ldots, n_{\alpha_i}, \ldots\rangle = \sqrt{n_{\alpha_i} + 1} \mid n_{\alpha_1}, \ldots, n_{\alpha_i} + 1, \ldots\rangle \tag{2.12}$$

$$\hat{a}_{\alpha_i} \mid n_{\alpha_1}, \ldots, n_{\alpha_i}, \ldots\rangle = \sqrt{n_{\alpha_i}} \mid n_{\alpha_1}, \ldots, n_{\alpha_i} - 1, \ldots\rangle \tag{2.13}$$

<u>Fermion</u>

$$\hat{a}^\dagger_{\alpha_i} \mid n_{\alpha_1}, \ldots, n_{\alpha_i}, \ldots\rangle = \begin{cases} 0 & \text{if} \quad n_{\alpha_i} = 1 \\ \mid n_{\alpha_1}, \ldots, n_{\alpha_i} + 1, \ldots\rangle & \text{if} \quad n_{\alpha_i} = 0 \end{cases} \tag{2.14}$$

and

$$\hat{a}_{\alpha_i} \mid n_{\alpha_1}, \ldots, n_{\alpha_i}, \ldots\rangle = \begin{cases} \mid n_{\alpha_1}, \ldots, n_{\alpha_i} - 1, \ldots\rangle & \text{if} \quad n_{\alpha_i} = 1 \\ 0 & \text{if} \quad n_{\alpha_i} = 0 \end{cases} \tag{2.15}$$

The boson operators satisfy the commutation relation, whereas the fermion operators satisfy an anti-commutation relation namely,

$$[\hat{a}_\lambda, \hat{a}^\dagger_\eta]_{-\gamma} = \delta_{\lambda\eta} \tag{2.16}$$

where $\gamma = +1$ and -1 for the bosonic and fermionic case respectively. Let ϕ denote the collection of numbers $(\phi_{\alpha_1}, \phi_{\alpha_2}, \ldots)$. We define the many-particle coherent state $\mid \phi\rangle$ by $\hat{a}_{\alpha_i} \mid \alpha\rangle = \phi_{\alpha_i} \mid \alpha\rangle$. We now consider the cases of boson and fermion coherent states separately.

2.1.1 Boson Coherent States

It is convenient to expand a boson coherent state in an occupation number representation,

$$\mid \phi >= \sum_{n_{\alpha_1}, n_{\alpha_2}, \ldots, n_{\alpha_p}, \ldots} \phi_{n_{\alpha_1} n_{\alpha_2} \ldots n_{\alpha_p} \ldots} \mid n_{\alpha_1}, n_{\alpha_2}, \ldots, n_{\alpha_p}, \ldots > \tag{2.17}$$

where

$$\mid n_{\alpha_1}, n_{\alpha_2}, \ldots, n_{\alpha_p} >= \frac{(\hat{a}^\dagger_{\alpha_1})^{n_{\alpha_1}}}{\sqrt{n_{\alpha_1}!}} \frac{(\hat{a}^\dagger_{\alpha_2})^{n_{\alpha_2}}}{\sqrt{n_{\alpha_2}!}} \ldots \frac{(\hat{a}^\dagger_{\alpha_p})^{n_{\alpha_p}}}{\sqrt{n_{\alpha_p}!}} \ldots \mid 0 > \tag{2.18}$$

and $\phi_{n_{\alpha_1} n_{\alpha_2}, \ldots} = \langle n_{\alpha_1} \ldots \mid \phi\rangle$. From $\hat{a}_\alpha \mid \alpha >= \phi_\alpha \mid \alpha >$,

$$\begin{aligned} \hat{a}_{\alpha_i} \mid \phi > &= \sum_{n_{\alpha_1} n_{\alpha_2} \ldots n_{\alpha_p} \ldots} \mid n_{\alpha_1} n_{\alpha_2} \ldots n_{\alpha_p} \ldots >< n_{\alpha_1} n_{\alpha_2} \ldots n_{\alpha_p} \ldots \mid \hat{a}_{\alpha_i} \mid \phi > \\ &= \sum_{n_{\alpha_1} n_{\alpha_2} \ldots n_{\alpha_p} \ldots} \phi_\alpha \phi_{n_{\alpha_1} n_{\alpha_2} \ldots} \mid n_{\alpha_1} n_{\alpha_2} \ldots\rangle \end{aligned}$$

so that

$$\phi_{\alpha_i} \phi_{n_{\alpha_1} n_{\alpha_2} \ldots n_{\alpha_p} \ldots} = < n_{\alpha_1} n_{\alpha_2} \ldots n_{\alpha_p} \ldots \mid \hat{a}_{\alpha_i} \mid \phi > .$$

Using $\langle n_{\alpha_1} n_{\alpha_2} \ldots n_{\alpha_i} \ldots \mid \hat{a}_{\alpha_i} = \langle n_{\alpha_1} n_{\alpha_2} \ldots n_{\alpha_i} + 1 \ldots \mid \sqrt{n_{\alpha_i} + 1}$, we get

$$\begin{aligned} \phi_{\alpha_i} \phi_{n_{\alpha_1} n_{\alpha_2} \ldots n_{\alpha_i} \ldots} &= \sqrt{n_{\alpha_i} + 1} \langle n_{\alpha_1} n_{\alpha_2} \ldots n_{\alpha_i} + 1 \ldots \mid \phi\rangle \\ &= \sqrt{n_{\alpha_i} + 1} \phi_{n_{\alpha_1} n_{\alpha_2} \ldots n_{\alpha_i} + 1 \ldots} . \end{aligned} \tag{2.19}$$

which implies,

$$\phi_{n_{\alpha_1} n_{\alpha_2} \ldots n_{\alpha_i} \ldots n_{\alpha_p} \ldots} = \frac{\phi_{\alpha_1}^{n_{\alpha_1}}}{\sqrt{n_{\alpha_1}!}} \frac{\phi_{\alpha_2}^{n_{\alpha_2}}}{\sqrt{n_{\alpha_2}!}} \ldots \frac{\phi_{\alpha_p}^{n_{\alpha_p}}}{\sqrt{n_{\alpha_p}!}} \ldots \tag{2.20}$$

where we have set $\phi_{00\ldots0\ldots} = 1$. Substituting (2.10) into (2.17), we get

$$\begin{aligned} \mid \phi > &= \sum_{n_{\phi_1} n_{\phi_2} \ldots n_{\phi_p} \ldots} \frac{(\phi_1 \hat{a}^\dagger_{\alpha_1})^{n_{\alpha_1}}}{n_{\alpha_1}!} \frac{(\phi_2 \hat{a}^\dagger_{\alpha_2})^{n_{\alpha_2}}}{n_{\alpha_2}!} \ldots \frac{(\phi_p \hat{a}^\dagger_{\alpha_p})^{n_{\alpha_p}}}{n_{\alpha_p}!} \ldots \mid 0 > \\ &= e^{\sum_\alpha \phi_\alpha \hat{a}^\dagger_\alpha} \mid 0 > \end{aligned} \tag{2.21}$$

and consequently

$$< \phi \mid = < 0 \mid e^{\sum_\alpha \phi^* \hat{a}_\alpha}. \tag{2.22}$$

Note that this result agrees with Eq. (2.10) up to a multiplicative constant. With this compact form of $\mid \phi >$, the creation operator $\hat{a}^\dagger_\alpha$ can be represented by a differential operator $\partial / \partial \phi_\alpha$ when acting on $\mid \alpha >$, that is,

$$\begin{aligned} \hat{a}^\dagger_\alpha \mid \phi > &= \hat{a}^\dagger_\alpha e^{\sum_\alpha \phi_\alpha \hat{a}^\dagger_\alpha} \mid 0 > \\ &= \frac{\partial}{\partial \phi_\alpha} e^{\sum_\alpha \phi_\alpha \hat{a}^\dagger_\alpha} \mid 0 > \\ &= \frac{\partial}{\partial \phi_\alpha} \mid \phi > . \end{aligned} \tag{2.23}$$

Consider the overlap of two coherent states,

$$< \phi \mid \phi^{'} >= \sum_{n_{\alpha_1} n_{\alpha_2} \ldots n_{\alpha_p}} \sum_{n_{\alpha_1^{'}} n_{\alpha_2^{'}} \ldots n_{\alpha_p^{'}}} \frac{(\phi^*_{\alpha_1})^{n_{\alpha_1}}}{\sqrt{n_{\alpha_1}!}} \ldots \frac{(\phi^*_{\alpha_p})^{n_{\alpha_p}}}{\sqrt{n_{\alpha_p}!}} \frac{(\phi^{'}_{\alpha_1^{'}})^{n_{\alpha_1}}}{\sqrt{n_{\alpha_1^{'}}!}} \ldots \frac{(\phi^{'}_{\alpha_p^{'}})^{n_{\alpha_p}}}{\sqrt{n_{\alpha_p^{'}}!}} < n_{\alpha_1} \ldots n_{\alpha_p} \mid n_{\alpha_1^{'}} \ldots n_{\alpha_p^{'}} > .$$

Since $< n_{\alpha_1} \ldots n_{\alpha_p} \mid n_{\alpha_1^{'}} \ldots n_{\alpha_p^{'}} >= \delta_{n_{\alpha_1} n_{\alpha_1^{'}}} \ldots \delta_{n_{\alpha_p} n_{\alpha_p^{'}}}$, then

$$\begin{aligned} < \phi \mid \phi^{'} > &= \sum_{n_{\alpha_1} n_{\alpha_2} \ldots n_{\alpha_p}} \frac{(\phi^*_{\alpha_1} \phi^{'}_{\alpha_1})^{n_{\alpha_1}}}{n_{\alpha_1}!} \ldots \frac{(\phi^*_{\alpha_p} \phi^{'}_{\alpha_p})^{n_{\alpha_p}}}{n_{\alpha_p}!} \\ &= e^{\sum_\alpha \phi^*_\alpha \phi^{'}_\alpha}. \end{aligned} \tag{2.24}$$

The closure relation can be written similarly to that of the single-particle case,

$$\frac{1}{\pi}\int\prod_{\alpha} d^2\phi_\alpha\, e^{-\sum_\alpha \phi^*_\alpha\phi_\alpha} \mid \phi_\alpha >< \phi_\alpha \mid = 1. \tag{2.25}$$

The derivation of this result is similar to the one given in Appendix A for the single-particle coherent states.

2.1.2 Fermion Coherent States

In the case of many-fermion system, we need to take into account the Pauli's exclusion principle, that is, no two fermions can be in the same state. An important consequence of this is that the parameters used to parameterize the fermion coherent state have to be anticommuting numbers known as the Grassmann numbers. Let ξ_α and ξ_β be Grassmann numbers, then

$$\{\xi_\alpha, \xi_\beta\} = 0 \tag{2.26}$$

and consequently

$$\xi_\alpha^2 = 0. \tag{2.27}$$

Due to the nilpotency of the Grassmann number ξ, any function of ξ can be expressed as

$$f(\xi) = f_0 + f_1\xi \tag{2.28}$$

where f_0 and f_1 are independent of ξ. Similarly

$$A(\xi,\xi^*) = a_0 + a_1\xi^* + a_1^*\xi + a_{12}\xi^*\xi. \tag{2.29}$$

The differentiation and integration involving Grassmann numbers are defined as follows:

Differentiation

$$\begin{aligned}\frac{\partial\xi^*\xi}{\partial\xi} &= -\frac{\partial\xi\xi^*}{\partial\xi} \\ &= -\xi^*. \end{aligned}\tag{2.30}$$

Integration

$$
\begin{aligned}
\int d\xi &= \int d\xi^* = 0, \\
\int d\xi\xi &= \int d\xi^*\xi^* = 1.
\end{aligned}
$$

We are now ready to define the fermion coherent state. Let ξ denote a collection of Grassmann numbers ξ_α ($\alpha = 1, 2, 3, \ldots$). The fermion coherent state $\mid \xi\rangle$ parametrized by ξ is defined by $\hat{a}_\alpha \mid \xi\rangle = \xi_\alpha \mid \xi\rangle$ where $\hat{a}_\alpha$ and $\hat{a}^\dagger_\alpha$ satisfy the algebra

$$
\begin{aligned}
\{\hat{a}_\alpha, \hat{a}_\beta\} = \{\hat{a}^\dagger_\alpha, \hat{a}^\dagger_\beta\} &= 0 \\
\{\hat{a}^\dagger_\alpha, \hat{a}_\beta\} &= \delta_{\alpha\beta}
\end{aligned}
$$

and anti-commute with any Grassmann number, that is, $\{\hat{a}_\alpha, \xi\} = \{\hat{a}^\dagger_\alpha, \xi\} = 0$ but commute with commuting numbers.

Similar to the bosonic case, the explicit form of $\mid \xi\rangle$ is found to be

$$
\mid \xi >= e^{-\sum_\alpha \xi_\alpha \hat{a}^\dagger_\alpha} \mid 0 > \tag{2.31}
$$

and consequently,

$$
< \xi \mid = < 0 \mid e^{\sum_\alpha \xi^*_\alpha \hat{a}_\alpha}. \tag{2.32}
$$

Also, the creation operator $\hat{a}^\dagger_\alpha$ when acting on $\mid \xi\rangle$ can be represented by a differential operator, $\hat{a}^\dagger_\alpha \mid \xi\rangle = -\dfrac{\partial}{\partial \xi} \mid \xi\rangle$. The closure relation takes the form

$$
\int \prod_\alpha d^2\xi_\alpha e^{-\sum \alpha \xi^*_\alpha \xi_\alpha} \mid \xi >< \xi \mid = 1. \tag{2.33}
$$

The expansion of the states and operators in terms of coherent states will be considered in the next two sections.

2.2 Expansion of Arbitrary States in Terms of Coherent States

We now derive the expansion of an arbitrary state in terms of coherent states. We begin by recalling that, since the set of one-dimensional harmonic oscillator eigenstates $| n\rangle$ forms a complete set in Hilbert space, we can expand an arbitrary state $| f\rangle$ in terms of them, namely,

$$\begin{aligned} | f > &= \sum_n | n >< n | f > \\ &= \sum_n < n | f > \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} | 0 > . \end{aligned} \tag{2.34}$$

Imposing the normalization condition,

$$< f | f >= 1, \tag{2.35}$$

we find

$$\begin{aligned} 1 &= < f | f > \\ &= \sum_n < f | n >< n | f > \\ &= \sum_n |< n | f >|^2 . \end{aligned} \tag{2.36}$$

An arbitrary state $| f\rangle$, therefore, can be expressed in the form

$$| f >= f(\hat{a}^\dagger) | 0 > \tag{2.37}$$

where $f(\hat{a}^\dagger) = \sum_n \langle n | f\rangle \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}$. To express $| f\rangle$ in terms of coherent states, we use the completeness relation (2.10),

$$\begin{aligned} | f > &= \frac{1}{\pi}\int d^2\alpha | \alpha >< \alpha | f > \\ &= \frac{1}{\pi}\int d^2\alpha | \alpha >< \alpha | f(\hat{a}^\dagger) | 0 > \\ &= \frac{1}{\pi}\int d^2\alpha | \alpha >< \alpha | 0 > f(\alpha^*) \\ &= \frac{1}{\pi}\int d^2\alpha | \alpha > e^{-\frac{|\alpha|^2}{2}} f(\alpha^*) \end{aligned} \tag{2.38}$$

where we have used $\langle\alpha \mid f(\hat{a}^\dagger) = f(\alpha^*)\langle\alpha \mid$ in going from the second line to the third line and $\langle\alpha \mid 0\rangle = e^{-|\alpha|^2/2}$ (see Appendix A for its derivation). Consider the inner product between the coherent state $\langle\beta \mid$ with $\mid f\rangle$, using Eq. (2.38), we find

$$\begin{aligned} <\beta \mid f> &= \frac{1}{\pi}\int d^2\alpha <\beta \mid \alpha> e^{-\frac{|\alpha|^2}{2}} f(\alpha^*) \\ &= \frac{1}{\pi}\int d^2\alpha e^{\beta^*\alpha} e^{-\frac{|\alpha|^2}{2}} e^{-\frac{|\beta|^2}{2}} f(\alpha^*) e^{-\frac{|\alpha|^2}{2}} \\ &= \frac{e^{-\frac{|\beta|^2}{2}}}{\pi}\int d^2\alpha e^{\beta^*\alpha} e^{-|\alpha|^2} f(\alpha^*). \end{aligned} \quad (2.39)$$

But if we use the form of $\mid f\rangle$ in Eq. (2.37), we find

$$\begin{aligned} \langle\beta \mid f\rangle &= \langle\beta \mid f(\hat{a}^\dagger) \mid 0\rangle \\ &= f(\beta^*)\langle\beta \mid 0\rangle \\ &= f(\beta^*) e^{-|\beta|^2/2}. \end{aligned} \quad (2.40)$$

Thus

$$f(\beta^*) = \frac{1}{\pi}\int d^2\alpha e^{\beta^*\alpha} e^{-|\alpha|^2} f(\alpha^*). \quad (2.41)$$

Now consider another state $\mid g\rangle$,

$$< g \mid = \frac{1}{\pi}\int d^2\alpha < \alpha \mid e^{-\frac{|\alpha|^2}{2}} g^*(\alpha^*). \quad (2.42)$$

The inner product of the two states $< g \mid$ and $\mid f >$ may then be expressed as

$$\begin{aligned} < g \mid f > &= \frac{1}{\pi^2}\int d^2\beta d^2\alpha <\beta \mid \alpha> e^{-\frac{|\beta|^2}{2}} e^{-\frac{|\alpha|^2}{2}} g^*(\beta^*) f(\alpha^*) \\ &= \frac{1}{\pi^2}\int d^2\beta d^2\alpha e^{\beta^*\alpha} e^{-\frac{|\beta|^2}{2}} e^{-\frac{|\alpha|^2}{2}} e^{-\frac{|\beta|^2}{2}} e^{-\frac{|\alpha|^2}{2}} g^*(\beta^*) f(\alpha^*) \\ &= \frac{1}{\pi^2}\int d^2\beta d^2\alpha e^{\beta^*\alpha} e^{-|\beta|^2} e^{-|\alpha|^2} g^*(\beta^*) f(\alpha^*) \end{aligned} \quad (2.43)$$

where we have used $\beta \mid \alpha\rangle = e^{-|\beta|^2/2} e^{-|\alpha|^2/2} e^{\beta^*\alpha}$ derived in Appendix A. Using Eq. (2.41), we finally obtain

$$< g \mid f > = \frac{1}{\pi}\int d^2\beta e^{-|\beta|^2} g^*(\beta^*) f(\beta^*). \quad (2.44)$$

2.3 Expansion of Operators in Terms of Coherent States

Consider a general quantum mechanical operators A which may be expressed as

$$\begin{aligned} A &= \sum_{n,m} \mid n > A_{nm} < m \mid \\ &= \sum_{n,m} A_{nm} \frac{(\hat{a}^\dagger)^n \mid 0 >< 0 \mid (\hat{a})^m}{\sqrt{n!m!}} \end{aligned} \tag{2.45}$$

in the harmonic oscillator basis. Here, $A_{nm} = \langle n \mid A \mid m \rangle$. To rewrite it in the coherent state basis, we use the completeness relation (2.10),

$$\begin{aligned} A &= \frac{1}{\pi^2} \int \mid \alpha >< \alpha \mid A \mid \beta >< \beta \mid d^2\alpha d^2\beta \\ &= \frac{1}{\pi^2} \int \mid \alpha > \sum_{n,m} A_{nm} \frac{(a^*)^n (\beta)^m}{\sqrt{n!m!}} < \alpha \mid 0 >< 0 \mid \beta >< \beta \mid d^2\alpha d^2\beta \\ &= \frac{1}{\pi^2} \int \mid \alpha > \sum_{n,m} A_{nm} \frac{(a^*)^n (\beta)^m}{\sqrt{n!m!}} e^{-\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2}} < \beta \mid d^2\alpha d^2\beta. \end{aligned} \tag{2.46}$$

This is the expansion of the operator A in terms of coherent states.

CHAPTER III

QUANTUM HALL EFFECTS

In this chapter, we will discuss the quantum Hall effects in details. In Section 3.1, the classical Hall effect which can be seen at room temperature is discussed. The two-dimensional systems exhibiting the quantum Hall effects are next discussed in Section 3.2. Finally, Sections 3.3 and 3.4 are respectively devoted to the explaination of the integer and fractional quantum Hall effects.

3.1 Classical Hall Effect

The classical Hall effect was discovered in 1879 by E. Hall [4]. Consider a conducting material with a rectangular cross section of size $d \times b$ in a uniform field $\vec{B} = B\hat{z}$ as shown in Fig. 3.1.

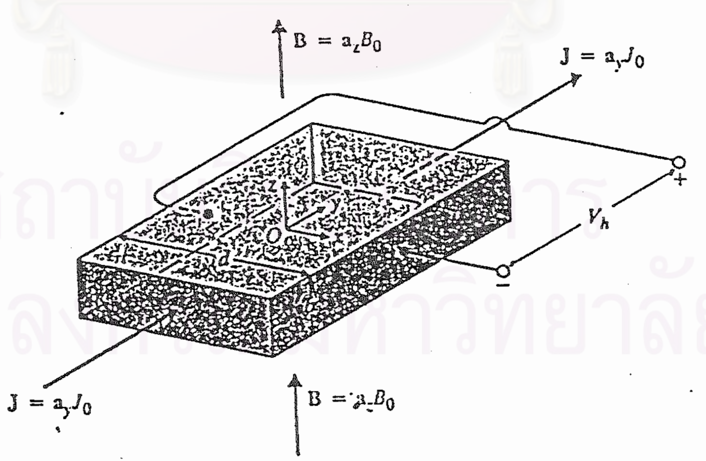

Figure 3.1: The Hall geometry.

The experiment was performed at room temperature and the magnetic field used was increased up to about one Tesla. What Hall found is that the voltage V_h, known as the Hall voltage, varied linearly with the magnetic field as shown in Fig. 3.2.

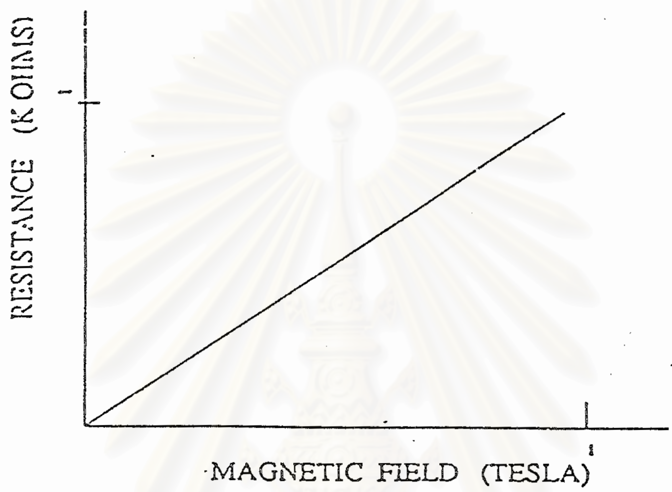

Figure 3.2: The linear Hall voltage at moderate fields and room temperature.

This phenomenon can be explained as follows. In the presence of both electric and magnetic fields, a charge carrier of charge q in the sample is acted on by the force

$$\vec{F} = q(\vec{E} + \vec{u} \times \vec{B}). \quad (3.1)$$

Suppose a uniform direct current flowing in the y-direction,

$$\begin{aligned} \vec{J} &= J_0\hat{y} \\ &= Nq\vec{u}, \end{aligned} \quad (3.2)$$

where N is the number of charge carriers per unit volume and $\vec{u}$ is the velocity of the charge carriers each having charge q, is applied to the system. Note that if the material is a conductor, the charge carriers are electrons and q is negative. At the beginning, the Lorentz force tends to push the charge carriers along the x-direction, causing the electrons to accumulate on one side and the positive ion excess to establish on the opposite side of the sample. These surface charges then produce a transverse electric field that tends to cancel the Lorentz force. When the steady state is reached, the number of surface charges is large enough that the transverse electric field $\vec{E}_H$, known as the Hall field, precisely cancels the Lorentz force,

$$\vec{E}_H + \vec{u} \times \vec{B} = 0 \tag{3.3}$$

so that

$$\vec{E}_H = -\vec{u} \times \vec{B}. \tag{3.4}$$

This is known as the Hall effect. In our case, $\vec{J} = -J_0\hat{y}$ (with $J_0 = qu_0N$) and $\vec{B} = B_0\hat{z}$, so that

$$\begin{aligned} \vec{E}_H &= -(J_0\hat{y}) \times (B_0\hat{z}) \\ &= -J_0B_0\hat{x}. \end{aligned} \tag{3.5}$$

A transverse potential V_h thus appears and has the value

$$\begin{aligned} V_h &= -\int_0^d \vec{E}_H \cdot d\vec{r} \\ &= \int_0^d J_0B_0\, dx \\ &= J_0B_0d. \end{aligned} \tag{3.6}$$

Thus $V_H \propto B_0$ in agreement with Hall's result.

3.2 Two-Dimensional Electron Systems

Since the experiments which demonstrate the quantum Hall effects have to be performed in two dimensions, the creation of the systems requires a surface of an object or an interface between two substances and a force to keep things there. Therefore the electrons are confined in an interface between a semiconductor

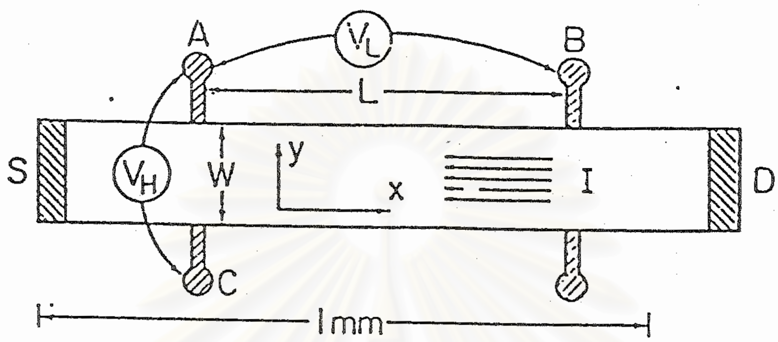

Figure 3.3: A top view of the inversion layer. This picture shows the trajectory of charge carriers in the surface of the system. The Hall voltage is measured between A and C while the transverse voltage is measured between A and B.

and an insulator or the interface between two semiconductors (these systems is sometimes called the inversion layer). The Integer Quantum Hall Effect (IQHE) [1], to be discussed in Section 3.3, was first seen in the experiment using the silicon MOSFET (metal-oxide-semiconductor-field effect) and the experimental setup is shown in Fig. 3.3. As for the Fractional Quantum Hall Effect (FQHE) which will be discussed in Section 3.4, the experiment was done using the sample created from two different semiconductors ($Al_xGa_{1-x}As - GaAs$) or heterojunction [2] as shown in Fig. 3.5. These two types of system are created from different substances. The electron mobilities of these two systems [4] are also different as shown in a Table 3.1.

3.3 Integer Quantum Hall Effect (IQHE)

When the quantum Hall effect was discovered in 1980, the properties of electrons in this extraordinary state differ fundamentally from those in all other known

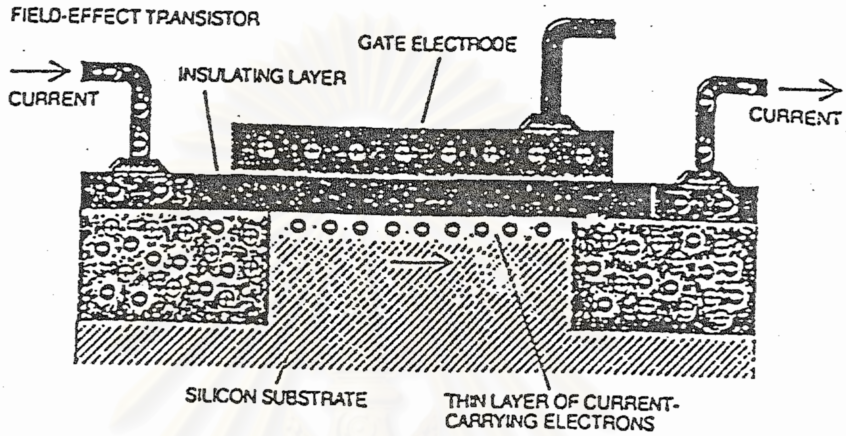

Figure 3.4: A side view of a si-MOSFET. The electric field is used to confine the electrons on the surface between an insulator and a semiconductor.

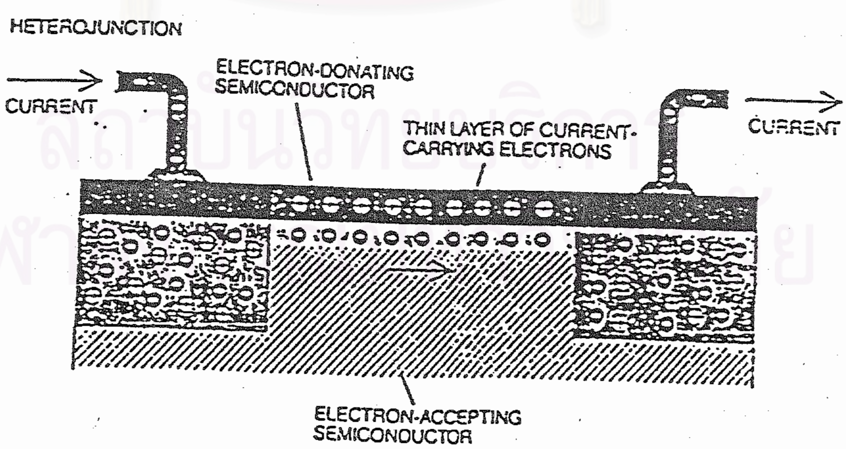

Figure 3.5: A side view of a heterojunction.

Table 3.1: Various types of the two-dimensional systems. Here, μ is the electron mobility in the absence magnetic field and τ_0 is the relaxation time.

Interface constituents	2D gas	m(m_e)	μ ($10^3 cm^2$ /Vs)	$\tau_0(10^{-12}$ s)
Si-SiO_2	e or (h)	0.19	10	1.1
GaAs-$Al_{0.29}Ga_{0.71}$As	e or (h)	0.068(0.38)	100	3.9
$In_{0.53}Ga_{0.47}$As-InP	e	0.080	30	1.4
InAs-GaSb	e or (h)	0.023(0.36)	170	2.2
InP-$Al_{0.48}In_{0.52}$As	e	~ 0.08	10	0.5
$Ga_{0.25}In_{0.75}As_{0.50}P_{0.50}$-InP	e	0.058	13	0.4
$In_{0.53}Ga_{0.47}$As-$In_{0.48}Al_{0.52}$As	e	0.05	90	2.6
$Hg_{0.78}Cd_{0.22}$Te-HgCdTe oxides	e	0.006	90	0.3

states of matter. The latest explorations in this field, however, have uncovered a striking relation between the quantum Hall effect and the more familiar phenomenon of superconductivity.

In 1980, Klaus von Klitzing, then at the high magnetic field laboratory of the Max Planck institute in Grenoble, Michael Pepper and Gerhardt Dordda discovered that, under special circumstances, the Hall effect does not obey the usual rules. When they chilled the trapped electrons to within a degree or two degree Kelvin, they found that the Hall voltage did not rise smoothly as the strength of the magnetic field is increased. The experimental result is shown in Fig. 3.6 [17] where it is seen that the Hall voltage rose in steps, with the values that did not vary at all ever in a small range of magnetic field strengths. In addition, the longitudinal voltage, that is, the voltage necessary to maintain the flow of current, nearly vanished when these plateaus in the Hall voltage were reached. In other words, the electrons became perfectly conducting. Perhaps more astonishing is a quantity called the Hall conductance which is the ratio of the current and the Hall voltage. Von Klitzing and his colleagues found that at each plateau the Hall conductance equaled an integer multiple of the quantum of conductance which is defined as $e^2/\hbar$, this integer is called the filling factor.

To understand how the Hall conductance adopt these values, we first

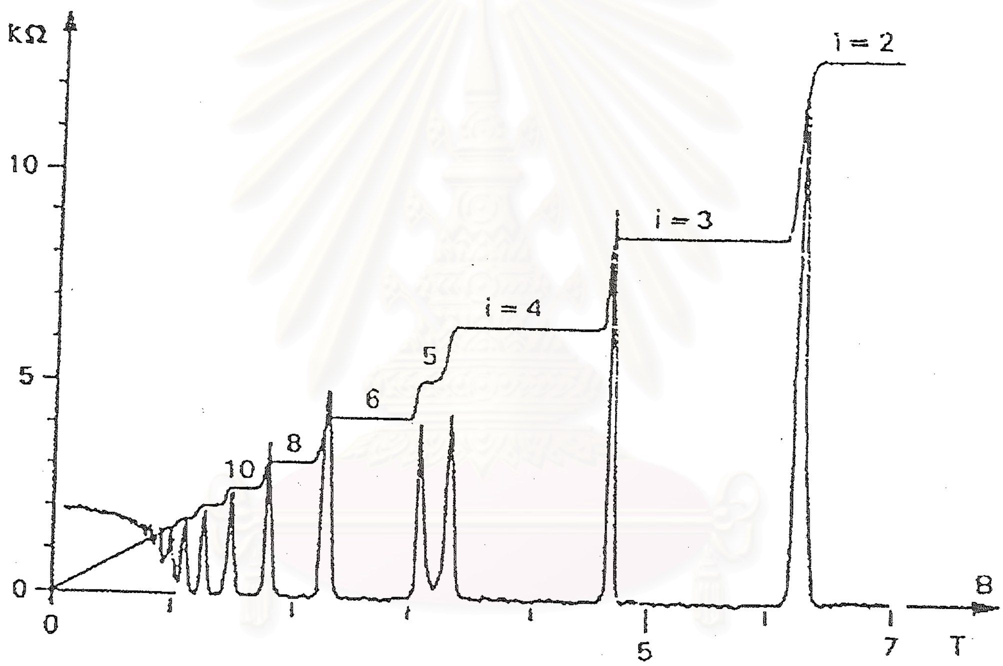

Figure 3.6: The Hall resistance varies stepwise with changes in high magnetic fields at low temperature [17].

introduce two important quantities. The first one is the magnetic flux quantum which serves as a fundamental unit for measuring the magnetic field strengths. The second important quantity is the filling factor defined as the number of the number of electrons in a sample divided by the number of magnetic flux quanta penetrating it. When the filling factor is one, this is one magnetic flux quantum per electron. When the filling factor is 1/3, there are three flux quanta for each electron.

The IQHE can be considered as a one-body problem. Consider an electron moving in a magnetic field before it is scattered from other electrons or impurities. Let τ be the time before an electron scattered and $\vec{P}$ be the electron's momentum, then its classical equation of motion takes the form

$$\frac{d}{dt}\vec{P} = \vec{F}_{ext} - \frac{\vec{P}}{\tau} \tag{3.7}$$

with $\vec{F}_{ext}$ being an external force. If m is the electron mass and $\vec{F}_{ext}$ is due to the applied electric and magnetic fields, then Eq. (3.7) becomes

$$m\left(\frac{d}{dt} + \frac{1}{\tau}\right)\vec{v} = -e(\vec{E} + \vec{v} \times \vec{B}). \tag{3.8}$$

If $\vec{B} = B\hat{z}$, then

$$m\left(\frac{d}{dt} + \frac{1}{\tau}\right)v_x = -e(E_x + Bv_y), \tag{3.9}$$

$$m\left(\frac{d}{dt} + \frac{1}{\tau}\right)v_y = -e(E_y - Bv_x), \tag{3.10}$$

$$m\left(\frac{d}{dt} + \frac{1}{\tau}\right)v_z = -eE_z. \tag{3.11}$$

In the steady state, the time derivatives are zero so that

$$v_x = -\frac{e\tau}{m}E_x - \frac{e\tau}{m}Bv_y, \tag{3.12}$$

$$v_y = -\frac{e\tau}{m}E_y + \frac{e\tau}{m}Bv_x, \tag{3.13}$$

$$v_z = -\frac{e\tau}{m}E_z. \tag{3.14}$$

Let n be the electron density, then the current $\vec{j} = -ne\vec{v}$. Using Eqs. (3.12)-(3.14), the components of the currents take the tensorial form of the Ampere's law,

$$j_x = \sigma_{xx} E_x + \sigma_{xy} E_y \tag{3.15}$$

$$j_y = \sigma_{yx} E_x + \sigma_{yy} E_y \tag{3.16}$$

$$j_z = \sigma_{zz} E_z \tag{3.17}$$

where the components of the conductivity tensor σ_{ij} are

$$\sigma_{xx} = \sigma_{yy} = \frac{\sigma_0}{1 + (\omega_c \tau)^2} \tag{3.18}$$

$$\sigma_{xy} = -\sigma_{yz} = -\frac{\sigma_0 \omega_c \tau}{1 + (\omega_c \tau)^2} \tag{3.19}$$

$$\sigma_{zz} = \sigma_0 \tag{3.20}$$

with $\sigma_0 = e^2 \tau n/m$ and $\omega_c = eB/m$. In the two-dimensional system of our interest, $j_z = E_z = 0$ and so we can neglect σ_{zz} in what follows. When $\omega_c \tau \gg 1$ corresponding to the situation in which the electrons can move freely,

$$\begin{aligned} \sigma_{xx} &\approx 0, \\ \sigma_{xy} &\approx \frac{en}{B}. \end{aligned} \tag{3.21}$$

The above analysis is based on the classical picture. We now treat this problem quantum mechanically. It will be seen that the energy of an electron in a magnetic field is quantized and these quantized energy levels are called the Landau levels.

Consider the Schrödinger equation for an electron in a uniform magnetic field

$$\frac{1}{2m} \left(-i\hbar \vec{\nabla} - e\vec{A} \right)^2 \Psi = E\Psi. \tag{3.22}$$

When $\vec{B} = B\hat{z}$, we can choose the Landau gauge where

$$A_x = -yB, \tag{3.23}$$

$$A_y = A_z = 0. \tag{3.24}$$

Our system is a two-dimensional system in the xy-plane of size $L_x \times L_y$, so the Schrödinger equation takes the form

$$\left(-\frac{1}{2m}\frac{\partial^2}{\partial y^2} + \frac{1}{2m}\left(-i\frac{\partial}{\partial x} - eBy\right)^2\right)\Psi(x,y) = E\Psi(x,y) \tag{3.25}$$

in the unit $\hbar = 1$. Let $\Psi(x,y) = e^{ikx}f(y)$, then Eq. (3.25) becomes

$$\left(\frac{1}{2}m\left(\frac{eB}{m}\right)^2\left(y - \frac{k}{eB}\right)^2 - \frac{1}{2m}\frac{\partial^2}{\partial y^2}\right)f(y) = Ef(y). \tag{3.26}$$

This is nothing but the one-dimensional harmonic oscillator equation along the y-axis with the frequency $\omega = eB/m$ and the equilibrium position $y_0 = k/eB$. The energy E thus takes the quantized form

$$\epsilon_n = \frac{eB}{m}\left(n + \frac{1}{2}\right) \tag{3.27}$$

where $n = 0, 1, 2, \ldots$. Each energy level is called a Landau level. Imposing the periodic boundary condition along the x-axis, then we find that k takes the quantized values

$$k_m = \frac{2\pi m}{L_x} \tag{3.28}$$

where $m = 0, \pm 1, \pm 2, \ldots$. But since $y_0 = k/eB$ must be bounded between 0 and L_y, then the maximum possible value of m is $eBA/2\pi$ with $A = L_xL_y$ being the area of the system. Thus the number of degeneracy of each Landau level is

$$N = \frac{A}{2\pi l^2} \tag{3.29}$$

where $l^2 = 1/eB$ is called the magnetic length, or equivalently the number of states per unit area of a full Landau level is

$$n_B = \frac{1}{2\pi l^2} = \frac{eB}{2\pi}. \tag{3.30}$$

The filling factor is given by

$$\nu = \frac{n}{n_B}. \tag{3.31}$$

If the Landau level is full, then ν is an integer and the Fermi level must lie in the gap between occupied levels. It is plausible that there is no scattering of electrons. Substituting Eq. (3.31) into Eq. (3.21), we get

$$\begin{aligned}\sigma_{xy} &= \frac{ne}{B} \\ &= \frac{\nu n_B e}{B} \\ &= \frac{\nu e^2}{2\pi}.\end{aligned} \tag{3.32}$$

Restoring $\hbar$ to our result, we find $\sigma_{xy} = \nu e^2/h$. In this case, plateaus occur whenever the filling factor, ν, is equal to an integer or whenever an integer number of Landau levels are fully occupied. This can be explained as follows. As the magnetic field is increased, the filling factor is varied according to Eqs. (3.30) and (3.31). If ν is not an integer, there are unoccupied states available in the Landau level and the electrons can move into these states causing the conductivity (resistivity) to increase (decrease). But if ν is an integer, then all Landau levels are either full or empty so that the electrons have to pass the energy gap between two successive Landau levels before they can conduct the current. However, there also exist the localized states in the gap (such as impurity in the sample), in which the electrons are localized and do not contribute to the conductivity. This causes the plateaus.

We end this section by presenting an alternative way for obtaining the Landau level based on the algebraic approach. We start with the Hamiltonian.

$$H = \sum\left(\frac{1}{2m}(p_x + eA_x)^2 + \frac{1}{2m}(p_y + eA_y)^2\right).$$

By introducing the covariant momentum, $\vec{\sqcap}$, with components,

$$\sqcap_x = -i\partial_x + eA_x, \tag{3.33}$$

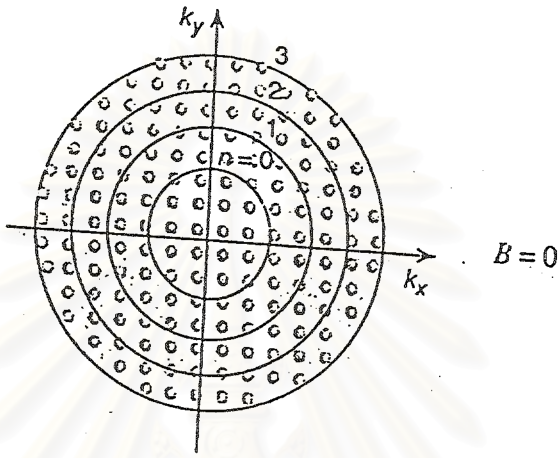

Figure 3.7: Quantization of free electrons in the absence of the magnetic field.

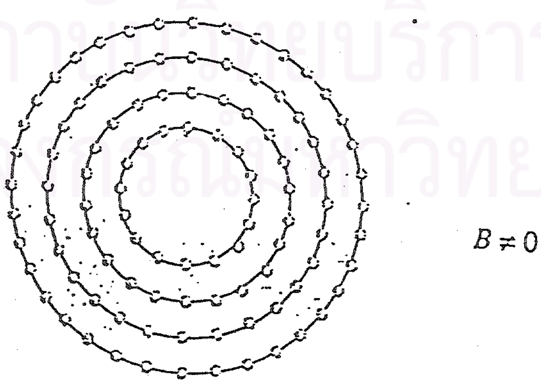

Figure 3.8: Quantization of free electrons in a magnetic field.

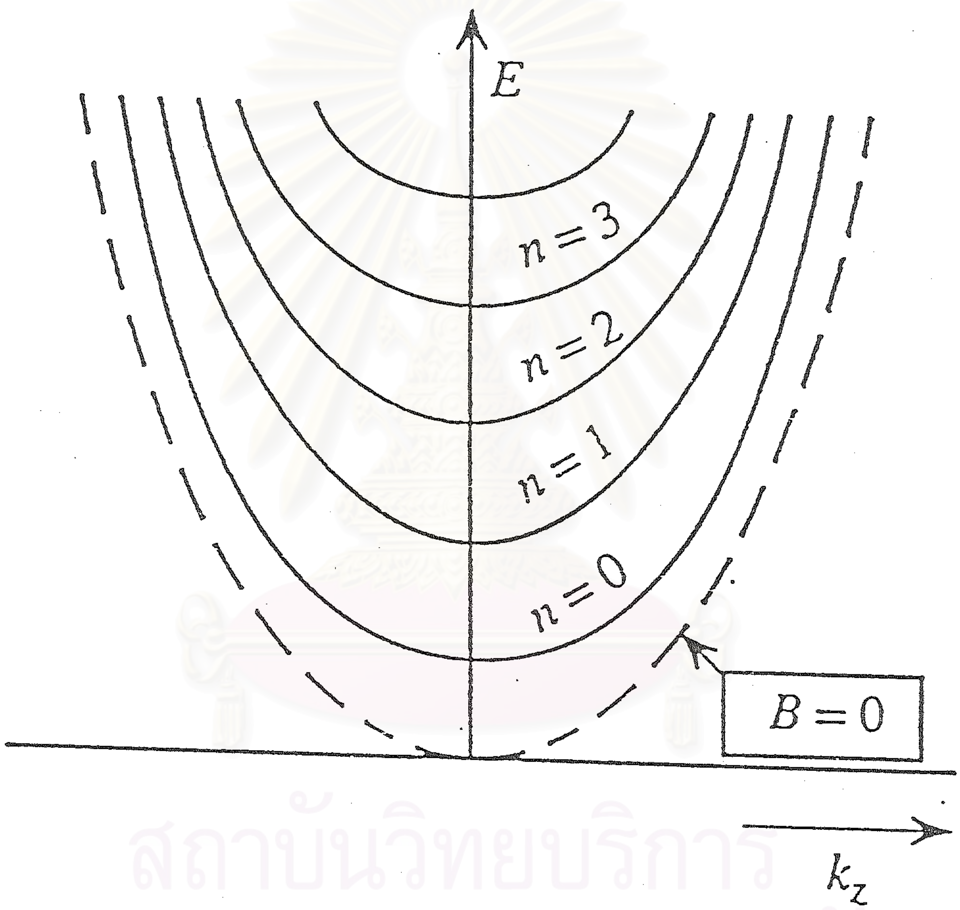

Figure 3.9: The magnetic field causes quantization in the x-y plane, leading to Landau levels. The dashed curve is for zero field.

$$\sqcap_y = -i\partial_y + eA_y, \tag{3.34}$$

and the guiding-center coordinates,

$$X = x + \frac{\sqcap_y}{eB}, \tag{3.35}$$

$$Y = y - \frac{\sqcap_x}{eB}, \tag{3.36}$$

then they satisfy the commutation relations,

$$[\sqcap_x, \sqcap_y] = \frac{i}{l^2}, \tag{3.37}$$

$$[\sqcap_x, x] = -i, \tag{3.38}$$

$$[\sqcap_y, y] = -i, \tag{3.39}$$

$$[X, Y] = -il^2, \tag{3.40}$$

where $l^2 = 1/eB$ is the magnetic length. Next introduce the annihilation operators

$$a = \frac{l}{\sqrt{2}}(\sqcap_x - i\sqcap_y), \tag{3.41}$$

$$b = \frac{1}{l\sqrt{2}}(X - iY), \tag{3.42}$$

together with the corresponding creation operators,

$$a^\dagger = \frac{1}{\sqrt{2}}(\sqcap_x + i\sqcap_y), \tag{3.43}$$

$$b^\dagger = \frac{1}{l\sqrt{2}}(X + iY). \tag{3.44}$$

They satisfy the commutation relations,

$$[a, a^\dagger] = 1,$$

$$[b, b^\dagger] = 1.$$

Note that a and b do not commute each others. The Hamiltonian is now rewritten in terms of the creation and annihilation operators,

$$H = \omega_c\left(a^\dagger a + \frac{1}{2}\right) \tag{3.45}$$

where $\omega_c = \frac{1}{ml^2}$ is called the cyclotron frequency. It is easily seen that the eigenvalues of this Hamiltonian are precisely the Landau Levels.

3.4 Fractional Quantum Hall Effect (FQHE)

Two years after discovery of the integer quantum Hall effect, Tsui, Stormer and Gossard [2] discovered what is now called the Fractional Quantum Hall Effect (FQHE). The sample they used was created from the heterojunction mentioned in Section 3.2 and the experiment was done in a high magnetic field (up to 30 Tesla) and at low temperatures. The experimental result shown in Fig. 3.10 is similar to that of the IQHE except that, at the plateaus, the filling factors are not integers but instead are fractional numbers. What is needed for a similar understanding of the fractional quantum Hall effect is some mechanism that selects rational filling factors, ν, of the Landau level electron density. It is obvious that such mechanism requires interactions among the electrons. Consider the Hamiltonian of a system of interacting electrons given by

$$H = \sum_i \frac{(p_i + eA(x_i))^2}{2m} + \sum_{i \neq j} v(x_i - x_j) \tag{3.46}$$

where the second term on the right hand side denotes the inter-electron Coulomb repulsion. This Hamiltonian cannot be solved exactly, being a highly non-trivial many-body problem. Laughlin [6] solved the above equation but only for a system of three electrons, and this idea led to the general many-body wave function in his following paper [7] in which he proposed the trial wave function to describe the ground state of the FQHE,

$$\psi(\{z_i\}) = N_m \prod_{i<j} (z_i - z_j)^m e^{-\frac{1}{4l^2}\sum_k |z_k|^2} \tag{3.47}$$

where z_i is the complex coordinate for the i$^{\text{th}}$ electron, N_m is the normalization constant, m is an odd number (3,5,7,9,...) and $l^2 = 1/eB$ is the magnetic length.

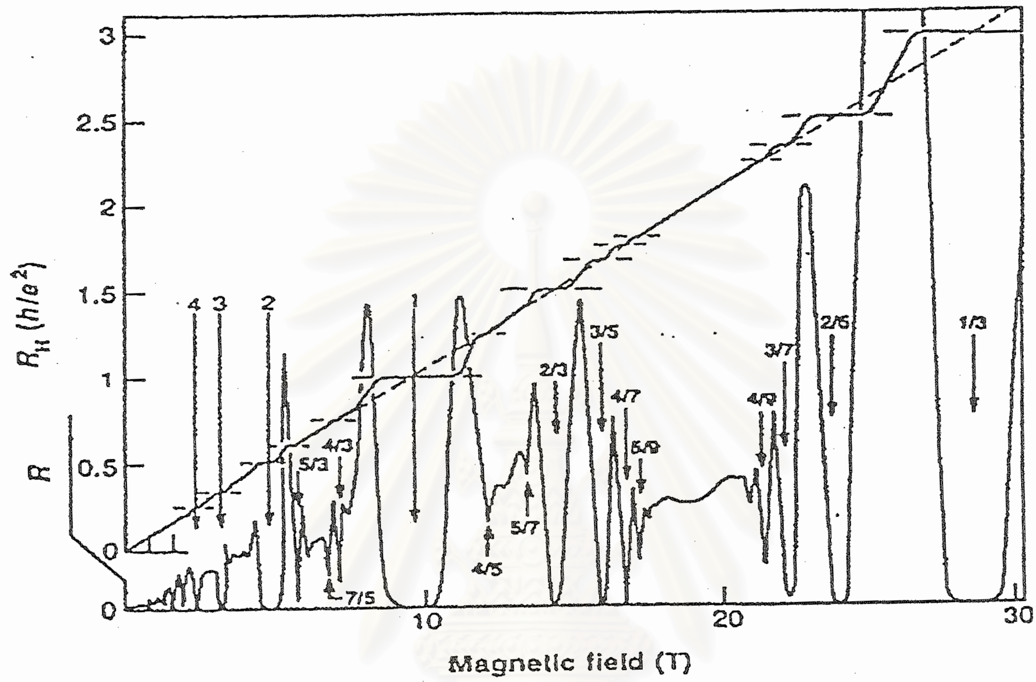

Figure 3.10: The Hall resistance varies stepwise with changes in the magnetic fields at low temperatures.

Clearly, this wave function is totally antisymmetric and describes a uniform distribution of (fluid-like) electrons.

In 1989, Jain [8] proposed the idea of composite fermions which serves as an alternative way of explaining the FQHE. In his theory, the FQHE of strongly interacting fermions (electrons) is just the IQHE of composite fermions. Before explaining his idea, it is appropriate to pictorially illustrate the idea of IQHE and Laughlin's idea of FQHE respectively in Figs. 3.11 and 3.12 [17]. The $\nu = 1$ IQHE case (Fig. 3.11) corressponds to the situation that one magnetic flux quantum is attached to each electron, while in the $\nu = 1/3$ FQHE case

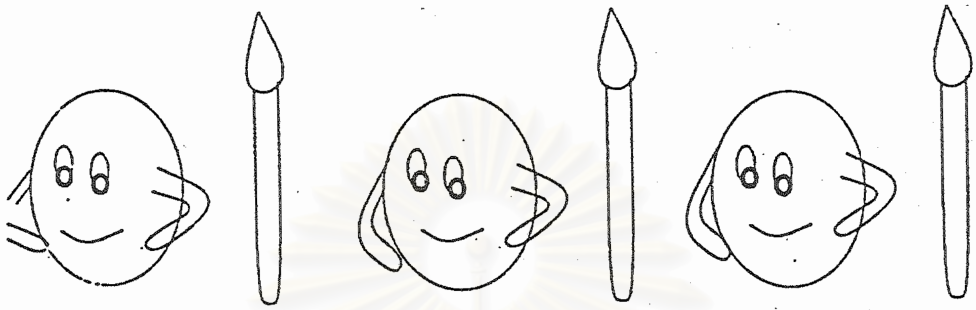

Figure 3.11: IQHE at $\nu = 1$. Electrons are depicted as balls and flux quanta are depicted as tubes. There is one flux quanta per electron [17].

(Fig. 3.12), each electron is attached to three flux quanta. In the pictures, the electron's hand represents the interaction with other electrons. Jain proposed the composite fermion as the fermion with even number of flux quanta attached (Fig. 3.13) [17]. Observe that composite electrons are no longer interacting. Hence, the FQHE is analogous to the IQHE of composite fermions. However, this picture could not accommodate all the Laughlin fractions $\nu = 1/(\text{odd numbers})$ but well explains the other cases such as $\nu = 2/5, 3/7, \ldots$. For example, the case $\nu = 2/5$ is just the IQHE of composite fermions at $\nu = 2$. In the case that each electron is attached to three fictitious flux quanta, the electrons see no net field attached to them, and they can form Bose condensate (Fig. 3.14) [17].

What we have been discussed so far are the examples of the explaination of the FQHE based on finding the wave function. At the fraction values of the filling factor, the tranverse resistence nearly vanishes. This implies that the electrons can move without energy loses. So, the explaination of the FQHE is analogous to the explaination of superconductivity (Landau-Ginzburg theory [19]). Zhang et al. [13] and Zhang [14] took this approach by analogous to composite bosons. The microscopic Hamiltonian (3.46) describing a system of fermions is written in

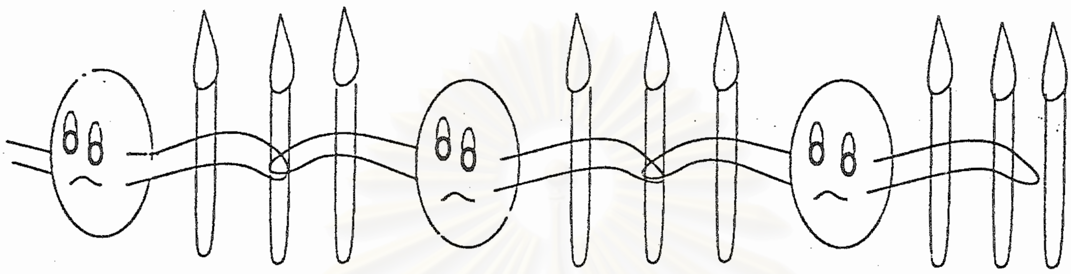

Figure 3.12: FQHE at $\nu = 1/3$. Electron's holding hands imply strong interactions. On the average, there are three flux quanta per electron [17].

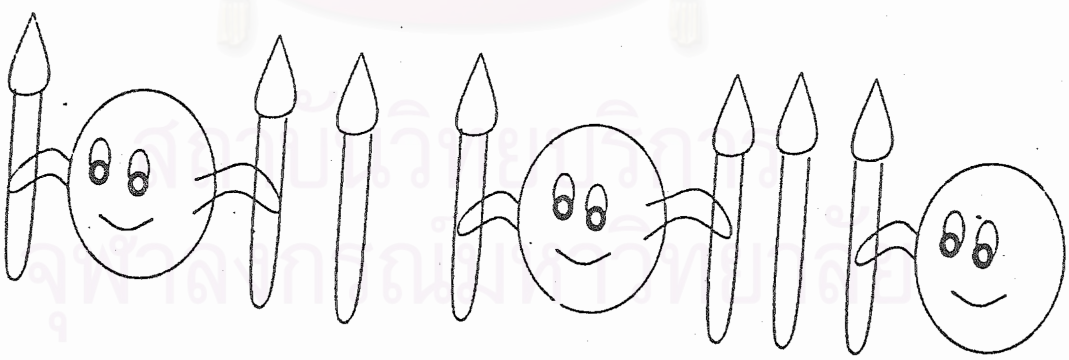

Figure 3.13: FQHE at $\nu = 1/3$. Each electron has two flux quanta attached resulting in one flux quanta per composite electron on the average [17].

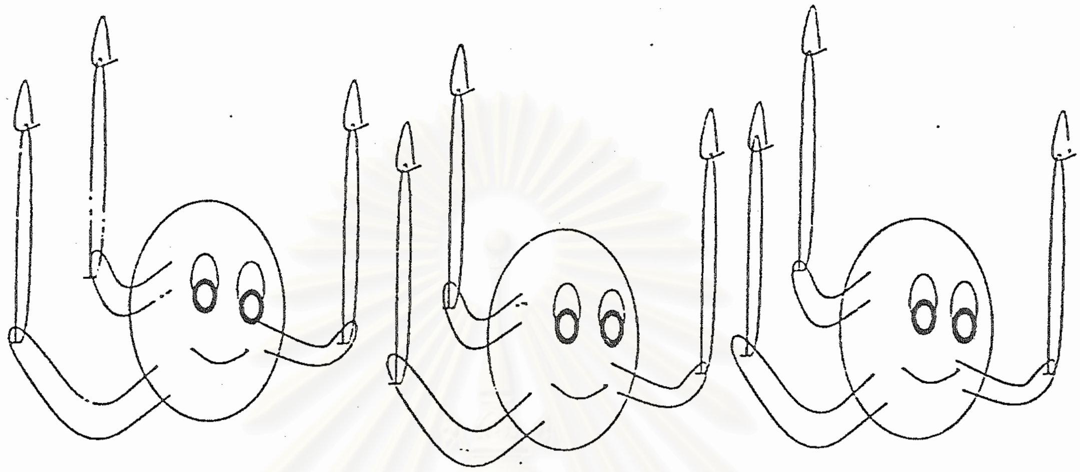

Figure 3.14: FQHE at ν=1/3. Each electron with three flux quanta attached forms a composite boson [17].

the second quantization form as,

$$\begin{aligned} H &= \int d^2x \hat{\phi}^\dagger(x) \frac{(p+eA)^2}{2m} \hat{\phi}(x) \\ &\quad + \frac{1}{2} \int d^2x d^2y \hat{\phi}^\dagger(x) \hat{\phi}^\dagger(y) v(|\, x-y\,|) \hat{\phi}(x) \hat{\phi}(y). \end{aligned} \tag{3.48}$$

By introducing the auxiliary gauge field, called the statistical gauge field, into the system, the statistics of the system can be changed. To avoid the confusion, the electromagnetic gauge field is denoted by A and the statistical gauge field is denoted by a. The statistical gauge field has the explicit form,

$$a(x_i) = \frac{\phi_0}{2\pi} \frac{\theta}{\pi} \sum_{i \neq j} \nabla \alpha_{ij} \tag{3.49}$$

where $\phi_0 = h/e$ is the unit of flux quantum, θ is a parameter corresponding to the statistical transmutation and α_{ij} is the angle between i^{th} and j^{th} particles. With this statistical gauge, the Hamiltonian now take the form,

$$H' = \int d^2x \hat{\phi}^\dagger(x) \frac{(p - eA - ea)^2}{2m} \hat{\phi}(x), + \frac{1}{2} \int d^2x d^2y \hat{\phi}^\dagger(x) \hat{\phi}^\dagger(y) v(|\, x - y \,|) \hat{\phi}(x) \hat{\phi}(y). \quad (3.50)$$

Introducing the unitary operator

$$u = e^{-i \sum_{i \neq j} \frac{\theta}{\pi} \alpha_{ij}}, \quad (3.51)$$

and defining

$$\tilde{\phi}(x) = u\phi(x), \quad (3.52)$$

where $\theta = (2k+1)\pi$, $k = 0, 1, 2, 3, \ldots$. ϕ and $\tilde{\phi}$ then satisfy

$$H\phi = E\phi.$$

and

$$H'\tilde{\phi} = E'\tilde{\phi}.$$

The partition function takes the form

$$Z = \int D[\phi^*] D[\phi] D[a] e^{\int d^3x (\phi^*(x) D_0 \phi(x) - H(\phi^*, \phi) + L_{cs})}, \quad (3.53)$$

where

$$S_{cs} = -\frac{1}{2\theta} \int d^3x \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho, \quad (3.54)$$

is called the Chern-Simons action [20, 21] and $D_\mu = \partial_\mu - A_\mu - a_\mu$ with $\mu = 0, 1, 2$. The minimum energy solution corresponds to the constant field configuration,

$$\phi = \sqrt{\rho}$$

$$a_i = -A_i$$

$$a_0 = 0$$

where $\rho = \phi^*\phi$ is the particle density that can be treated as the order parameter of the system analogous to the expansion of order parameter in the free energy of Landau-Ginzburg theory in the phenomenological approach of the superconductivity. If L_{cs} is extremized, substitute above solutions into equation of motion, we get

$$\nu = \frac{1}{2k+1}.$$

CHAPTER IV

COHERENT STATE REPRESENTATION APPLIED TO FRACTIONAL QUANTUM HALL EFFECT

In the first few chapters, we have discussed about the fundamental concepts of quantum Hall effects. We are now going to derive the microscopic fractional quantum Hall effect by using the coherent state representation. Of course, as being mentioned before, the Chern-Simons term will play a crucial role in the statistical transmutation.

We will first derive the essential physics of the Laughlin's mean field. We then derive the fluctuation terms by using the standard technique called the *Saddle Point Approximation* (SPA). Our starting point is the Hamiltonian describing a system of many electrons in the presence of electromagnetic interaction,

$$H = \sum_i \left(\frac{(\vec{p_i} + e\vec{A}(x_i))^2}{2M} + eA_0 \right) + \sum_{i \neq j} \frac{e^2}{\mid x_i - x_j \mid^2}. \tag{4.1}$$

This Hamiltonian can be written in a second quantized form

$$\begin{aligned} H &= \int d^2x \hat{\psi}^\dagger(x) \left(\frac{(\vec{p} + e\vec{A}(x))^2}{2M} + eA_0 \right) \hat{\psi}(x) \\ &\quad + \frac{1}{2} \int d^2x d^2y \hat{\psi}^\dagger(x) \hat{\psi}^\dagger(y) v(x-y) \hat{\psi}(x) \hat{\psi}(y), \end{aligned} \tag{4.2}$$

by using the field operators

$$\hat{\psi}(x) = \sum_n \psi_n(x) \hat{a} \tag{4.3}$$

and

$$\hat{\psi}^\dagger(x) = \sum_n \psi_n^*(x) \hat{a}^\dagger, \tag{4.4}$$

where the field operators in Eqs. (4.3) and (4.4) satisfy the commutation relation,

$$[\hat{\psi}(x), \hat{\psi}^\dagger(y)] = \delta(x-y).$$

In the field theoretical approach, a composite-particle is introduced to describe the FQHE. This picture is essential for understanding various aspect of QH effect. A composite-particle is obtained by attaching m quanta of the Chern-Simons flux to an electron : It is a composite boson when a m = 2p+1 = odd, and a composite fermion when m=2p=even. It acquires a physical reality by trading the Chern-Simons flux for the magnetic flux in the quantum Hall state.

Let $\psi(x)$ be the electron field. A composite-particle field $\phi(x)$ is defined by an operator phase transformation,

$$\phi(x) = e^{-im\Theta(x)}\psi(x). \tag{4.5}$$

The phase field $\Theta(x)$ is defined by

$$\Theta(x) = \int d^2y\theta(x-y)\rho(y), \tag{4.6}$$

where m is an integer and $\theta(x-y)$ is the angle between the the x-axis. The commutation relations among the above quantities can be derived as follows,

$$\begin{aligned}\phi(x)\phi(y) &= e^{-im\Theta(x)}\psi(x)e^{-im\Theta(y)}\psi(y) \\ &= e^{-im\Theta(x)}e^{-im\theta(y-x)}e^{-im\Theta(y)}\psi(x).\end{aligned} \tag{4.7}$$

Consider

$$e^A B e^{-A} = A + [A,B] + \frac{1}{2!}[A,[A,B]] + ... \tag{4.8}$$

So,

$$e^{im\Theta(y)}\psi(x)e^{-im\Theta(y)} = \psi(x) + (im)[\Theta(y),\psi(x)] + \frac{(im)^2}{2!}[\Theta(y),[\Theta(y),\psi(x)]] + ... \tag{4.9}$$

Commutation relation between $\Theta(y)$ and $\psi(x)$ is then,

$$\begin{aligned}[\Theta(y),\psi(x)] &= [\int d^2z\theta(y-z)\rho(z),\psi(x)] \\ &= \int d^2z\theta(y-z)[\rho(z),\psi(x)].\end{aligned} \tag{4.10}$$

Using the identity $[ab,c] = a\{b,c\} - \{a,c\}b$, the right hand side of (4.10) is now

$$\begin{aligned} [\rho(z), \psi(x)] &= -\{\psi^\dagger(z), \psi(x)\}\psi(z), \\ &= -\delta(z-x)\psi(x). \end{aligned} \tag{4.11}$$

By substituting Eq.(4.11) into Eq.(4.10) we obtain

$$\begin{aligned} [\Theta(y), \psi(x)] &= -\int d^2z\theta(y-z)\delta(z-x)\psi(z), \\ &= -\theta(y-x)\psi(x). \end{aligned} \tag{4.12}$$

Eq.(5) and Eq.(4.9), we obtain

$$\begin{aligned} e^{im\Theta(y)}\psi(x)e^{-im\Theta(y)} &= \psi(x) + (im)[\Theta(y), \psi(x)] + \frac{(im)^2}{2!}[\Theta(y), [\Theta(y), \psi(x)]] + ..., \\ &= \psi(x) + (-im\theta(y-x)\psi(x) + \frac{(-im\theta(y-x))^2}{2!}\psi(x) + ..., \\ &= (1 + (-im\theta(y-x)) + \frac{(-im\theta(y-x))^2}{2!} + ...)\psi(x), \\ &= e^{-im\theta(y-x)}\psi(x). \end{aligned} \tag{4.13}$$

Eq.(4.7) can be rewritten as

$$\phi(x)\phi(y) = e^{-im\Theta(x)}e^{-im\theta(y-x)}e^{-im\Theta(y)}\psi(x)\psi(y). \tag{4.14}$$

Thus,

$$\begin{aligned} \phi(y)\phi(x) &= e^{-im\Theta(y)}\psi(y)e^{-im\Theta(x)}\psi(x), \\ &= e^{-im\Theta(y)}e^{-im\Theta(x)}e^{im\Theta(x)}\psi(y)e^{-im\Theta(x)}\psi(x), \\ &= e^{-im\Theta(y)}e^{-im\Theta(x)}e^{-im\theta(x-y)}\psi(y)\psi(x). \end{aligned} \tag{4.15}$$

Consider commutation relation of $[\Theta(x), \Theta(y)]$,

$$\begin{aligned} [\Theta(y), \Theta(x)] &= [\int d^2z\theta(y-z)\rho(z), \int d^2b\theta(x-b)\rho(b)], \\ &= \int d^2zd^2b\theta(y-z)\theta(x-b)[\rho(z), (b)]. \end{aligned} \tag{4.16}$$

A commutation relation on the RHS of Eq.(4.16) is then,

$$\begin{aligned}[\rho(z),\rho(b)] &= \psi^\dagger(z)[\psi(z),\rho(b)] + [\psi^\dagger(z),\rho(b)]\psi(z),\\ &= \psi^\dagger(z)(\{\psi(z),\psi^\dagger(b)\}\psi(b)) - \psi^\dagger(b)(\{\psi^\dagger(z),\psi(b)\})\psi(z),\\ &= \delta(z-b)\psi^\dagger(z)\psi(b) - \delta(z-b)\psi^\dagger(b)\psi(z). \qquad (4.17)\end{aligned}$$

We substitute Eq.(4.17) into Eq.(4.16), then

$$[\Theta(y),\Theta(x)] = 0. \qquad (4.18)$$

By Eqs. (4.18),(4.15) and (4.14), then

$$\begin{aligned}\phi(x)\phi(y) &= e^{-im\Theta(y)}e^{-im\theta(y-x)}e^{-im\Theta(x)}\psi(y)\psi(x),\\ &= -e^{-im\Theta(y)}e^{-im\theta(x-y)-im\pi}e^{-im\Theta(x)}\psi(y)\psi(x),\\ &= e^{-im\pi}\phi(y)\phi(x),\\ \longrightarrow\quad & \phi(y)\phi(x) - (-1)^{m+1}\phi(x)\phi(y) = 0. \qquad (4.19)\end{aligned}$$

When m is an odd number above equation is the commutation relation, while it is the anticommutation relation if m is even,

$$[\phi(y),\phi(x)] = 0, \qquad (4.20)$$

$$\{\phi(y),\phi(x)\} = 0. \qquad (4.21)$$

Similarly, we can compute the commutation relation between $\phi^\dagger(x)$ and $\phi^\dagger(y)$:

$$\begin{aligned}e^{im\Theta(y)}\psi^\dagger(x)e^{-im\Theta(y)} &= \psi^\dagger(x) + (im)[\Theta(y),\psi^\dagger(x)] + \frac{(im)^2}{2!}[\Theta(y),[\Theta(y),\psi^\dagger(x)]] + ...,\\ \Theta(x)\psi^\dagger(y) - \psi^\dagger(y)\Theta(x) &= \int d^2z\theta(y-z)[\rho(z),\psi^\dagger(y)],\\ &= \int d^2z\theta(y-z)\psi^\dagger(z)\delta(z-x), =\\ \theta(y-x)\psi^\dagger(x), (4.22)\end{aligned}$$

(4.23)

So,

$$\begin{aligned} e^{im\Theta(y)}\psi^\dagger(x)e^{-im\Theta(y)} &= \psi^\dagger(x) + (im\psi^\dagger(x)) + (im\psi^\dagger(x))^2 + ..., \\ &= (1 + (im\theta(y-x)) + (im\theta(y-x))^2 + ...)\psi^\dagger(x), \\ &= e^{im\theta(y-x)}\psi^\dagger(x). \end{aligned} \tag{4.24}$$

Thus,

$$\begin{aligned} \phi^\dagger(x)\phi^\dagger(y) &= \psi^\dagger(x)e^{im\Theta(x)}\psi^\dagger(y)e^{im\Theta(y)}, \\ &= \psi^\dagger(x)e^{im\theta(x-y)}\psi^\dagger(y)e^{im\Theta(x)}e^{im\Theta(y)}. \end{aligned} \tag{4.25}$$

$$\begin{aligned} \phi^\dagger(y)\phi^\dagger(x) &= \psi^\dagger(y)e^{im\Theta(y)}\psi^\dagger(x)e^{im\Theta(x)}, \\ &= \psi^\dagger(y)e^{im\theta(y-x)}\psi^\dagger(x)e^{im\Theta(y)}e^{im\Theta(x)}, \\ &= -\psi^\dagger(x)e^{im\theta(x-y)+im\pi}\psi^\dagger(y)e^{im\Theta(x)}e^{im\Theta(y)}, \\ &= (-1)^{m+1}\phi^\dagger(x)\phi^\dagger(y). \end{aligned} \tag{4.26}$$

We thus arrive at a similar result as Eq.(4.20 and Eq.(4.21)), namely,

$$[\phi^\dagger(y), \phi^\dagger(x)] = 0, \tag{4.27}$$

$$\{\phi^\dagger(y), \phi^\dagger(x)\} = 0. \tag{4.28}$$

The commutation relation between $\phi(x)$ and $\phi^\dagger(y)$ can be calculated as follows:

$$\begin{aligned} \phi(x)\phi^\dagger(y) &= e^{-im\Theta(x)}\psi(x)\psi^\dagger(y)e^{im\Theta(y)}, \\ &= e^{-im\Theta(x)}\delta(x-y)e^{im\Theta(y)} \\ &\quad - e^{-im\Theta(x)}\psi^\dagger(y)\psi(x)e^{im\Theta(y)}, \\ &= \delta(x-y) - e^{-im\Theta(x)}\psi^\dagger(y)e^{\theta(y-x)}e^{im\Theta(y)}\psi(x), \\ &= \delta(x-y) + (-1)^{m+1}\phi^\dagger(y)\phi(x). \end{aligned}$$

$$\longrightarrow \quad \phi(x)\phi^\dagger(y) - (-1)^{m+1}\phi^\dagger(y)\phi(x) = \delta(x-y). \tag{4.29}$$

If m is an odd number,

$$[\phi(x), \phi^{\dagger}(y)] = \delta(x-y). \tag{4.30}$$

If m is an even number,

$$\{\phi(x), \phi^{\dagger}(y)\} = \delta(x-y). \tag{4.31}$$

Following these commutation relations, we conclude that the composite particle field ϕ is bosonic when m is an odd number and is fermionic when m is an even number. Next we will find an eigenstate for the composite particle. Let $\phi_{cp}(x)$ denotes composite particle wave function,

$$\phi_{cp}(x) \;=\; <0 \mid \phi(x_1)...\phi(x_n) \mid \phi> . \tag{4.32}$$

The eigenstate $\mid \phi(x) >$ can be constructed by using Eq.(4.32) then,

$$\begin{aligned} \mid \phi > &= \int D[x]\phi_{cp}(x)\phi^{\dagger}(x_n)...\phi^{\dagger}(x_1) \mid 0 >, \\ &= \int D[x]\phi_{cp}(x)\psi^{\dagger}(x_n)e^{im\Theta(x_n)}...\psi^{\dagger}(x_1)e^{im\Theta(x_1)} \mid 0 > . \end{aligned} \tag{4.33}$$

But

$$\begin{aligned} e^{im\Theta(y)}\psi^{\dagger}(x)e^{-im\Theta(y)} &= e^{\theta(x-y)}\psi^{\dagger}(x), \\ e^{im\Theta(x_1)}\psi^{\dagger}(x_2)e^{im\Theta(x_2)} &= \psi^{\dagger}(x_2)e^{\theta(x_1-x_2)}e^{im\Theta(x_1)}e^{im\Theta(x_2)}, \end{aligned}$$

then

$$\begin{aligned} \mid \phi > = \int D[x]\phi_{cp}(x)e^{im\sum_{i<j}\theta(x_i-x_j)} \\ \psi^{\dagger}(x_n)...\psi^{\dagger}(x_1) \mid 0 > . \end{aligned} \tag{4.34}$$

Consider now the path integral for the coherent states,

$$\hat{a} \mid z >= z \mid z >, \tag{4.35}$$

where z is a complex number and $\hat{a}$ is an annihilation operator. The propagator can be written as,

$$K(x_f, T; x_i, 0) =< f \mid e^{-iT\hat{H}} \mid i > . \tag{4.36}$$

Time, T, is divided into N intervals, each of length $\epsilon = T/N$. Then the propagator is rewritten as,

$$\begin{aligned} < f \mid e^{-iT\hat{H}} \mid i > &= < f \mid (e^{-i\epsilon\hat{H}})^N \mid i >, \\ &= lim_{\epsilon\to 0, N\to\infty} < f \mid (1 - i\epsilon T\hat{H})^N \mid i >, \end{aligned} \tag{4.37}$$

where $\mid i >$ and $\mid f >$ are initial and final states respectively. Inserting the closure relation, $\int \frac{d\alpha^* d\alpha}{2\pi i} e^{-|\alpha|^2} \mid \alpha >< \alpha \mid = 1$, at each time interval to the propagator, we get

$$\begin{aligned} < f \mid (1 - i\epsilon\hat{H})^N \mid i > &= \int \frac{d\alpha_N^* d\alpha_N}{2\pi i} e^{-|\alpha_N|^2} \ldots \\ & \int \frac{d\alpha_1^* d\alpha_1}{2\pi i} e^{-|\alpha_1|^2} < f \mid (1 - i\epsilon\hat{H} \mid \alpha_N > \ldots \\ & < \alpha_1 \mid (1 - i\epsilon\hat{H}) \mid i > . \end{aligned}$$

Since

$$< \alpha_{i+1} \mid (1 - i\epsilon\hat{H} \mid \alpha_i >=< \alpha_{i+1} \mid \alpha_i > -i\epsilon < \alpha_{i+1} \mid \hat{H} \mid \alpha_i >,$$

then

$$\begin{aligned} < f \mid (1 - i\epsilon\hat{H})^N \mid i >= \int \prod_i \frac{d\alpha_i^* d\alpha_i}{2\pi i} e^{-|\alpha_i|^2} < f \mid (1 - i\epsilon\hat{H}) \mid \alpha_N >^* \\ \prod_i (< \alpha_{i+1} \mid \alpha_i > -i\epsilon < \alpha_{i+1} \mid \hat{H} \mid \alpha_i >) \\ < \alpha_1 \mid (1 - i\epsilon\hat{H}) \mid i >, \end{aligned}$$

$$
\begin{aligned}
< f \mid (1-i\epsilon\hat{H})^N \mid i >= & \int \prod_i \frac{d\alpha_i^* d\alpha_i}{2\pi i} e^{-|\alpha_i|^2} < f \mid (1-i\epsilon\hat{H}) \mid \alpha_N >^* \\
& \prod_i (< \alpha_{i+1} \mid \alpha_i > -i\epsilon H(\alpha_{i+1}^*, \alpha_i) < \alpha_{i+1} \mid \alpha_i >) \\
& < \alpha_1 \mid (1-i\epsilon\hat{H}) \mid i >, \\
= & \int \prod_i \frac{d\alpha_i^* d\alpha_i}{2\pi i} e^{-|\alpha_i|^2} < f \mid (1-i\epsilon\hat{H}) \mid \alpha_N >^* \\
& e^{\sum_i \alpha_{i+1}^* \alpha_i} \prod_i (1 - i\epsilon H(\alpha_{i+1}^*, \alpha_i) < \alpha_1 \mid (1-i\epsilon\hat{H}) \mid i > .
\end{aligned} \tag{4.38}
$$

The initial and final states can be expanded in terms of the coherent states as,

$$
< f \mid = \int \frac{d\alpha_f^* d\alpha_f}{2\pi i} e^{-|\alpha_f|^2} < f \mid \alpha >< \alpha \mid, \tag{4.39}
$$

$$
\mid i >= \int \frac{d\alpha_f^* d\alpha_i}{2\pi i} e^{-|\alpha_i|^2} \mid \alpha >< \alpha \mid i > . \tag{4.40}
$$

Using Eqs.(4.38),(4.39) and (4.40), the propagator can be expressed in the form,

$$
\begin{aligned}
< f \mid e^{iT\hat{H}} \mid i >= & \; lim_{\epsilon\to 0, N\to\infty} \int \prod_i \frac{d\alpha_i^* d\alpha_i}{2\pi i} e^{-|\alpha_i|^2} \\
& e^{\sum_i \alpha_{i+1}^* \alpha_i} \prod_i (1 - i\epsilon H(\alpha_{i+1}^*, \alpha_i)) \\
& \int \frac{d\alpha_f^* d\alpha_f}{2\pi i} e^{-|\alpha_f|^2} \frac{d\alpha_f^* d\alpha_i}{2\pi i} e^{-|\alpha_i|^2} \\
& \phi_f^*(\alpha_f) < \alpha_f \mid \alpha_N >^* \phi_i(\alpha_i^*) < \alpha_1 \mid \alpha_i > \\
& (1 - i\epsilon H(\alpha_f^*, \alpha_N))(1 - i\epsilon H(\alpha_i^*, \alpha_N)), \\
< f \mid e^{iT\hat{H}} \mid i >= & \int D[\alpha^*] D[\alpha] e^{i\int_0^T dt (\frac{1}{2i}(\alpha\partial_t\alpha^* - \alpha^*\partial_t\alpha) - H(\alpha^*,\alpha))} \\
& e^{|\alpha_i|^2 + |\alpha_f|^2} \phi_f^*(\alpha_f)\phi_i(\alpha_i^*),
\end{aligned} \tag{4.41}
$$

where

$$
L = \frac{1}{2i}(\alpha\partial_t\alpha^* - \alpha^*\partial_t\alpha) - H(\alpha^*, \alpha). \tag{4.42}
$$

If the system is described by field configurations, it can be replaced with the coherent state representation,

$$
\begin{aligned}
< f \mid e^{-iT\hat{H}} \mid i >= & \int D[\alpha^*] D[\alpha] e^{i\int_0^T dt (\frac{1}{2i}(\alpha\partial_t\alpha^* - \alpha^*\partial_t\alpha) - H(\alpha^*,\alpha))} \\
& e^{\int (|\alpha_f|^2 + |\alpha_i|^2)} \int dx (\alpha_f^*(\alpha(x,t_f))\alpha_i(\alpha(x,t_i))).
\end{aligned} \tag{4.43}
$$

If $| f >$ and $| i >$ are the coherent states, the propagator will be written as

$$< \alpha_f \mid e^{-iT\hat{H}} \mid \alpha_i >= \int D[\alpha^*]D[\alpha]e^{i\int_0^T dt(\frac{1}{2i}(\alpha\partial_t\alpha^*-\alpha^*\partial_t\alpha)-H(\alpha^*,\alpha))}. \qquad (4.44)$$

From Eqs. (4.2) and (4.5), the Hamiltonian is written in terms of $\phi(x)$ and $\phi^\dagger(x)$ as

$$\begin{aligned} H &= \int d^2x\hat{\phi}^\dagger(x)\left(\frac{(\vec{p}+e\vec{A}(x)-e\vec{a})^2}{2M}+eA_0\right)\hat{\phi}(x) \\ &+\frac{1}{2}\int d^2xd^2y\hat{\phi}^\dagger(x)\hat{\phi}^\dagger(y)v(x-y)\hat{\phi}(x)\hat{\phi}(y), \end{aligned} \qquad (4.45)$$

where $\vec{a}$ takes the explicit form

$$\begin{aligned} \vec{a}_k &= \frac{m}{e}\partial_k\Theta(x), \\ &= \frac{m}{e}\int d^2z, \partial_k\theta(x-z)\rho(z). \end{aligned} \qquad (4.46)$$

$$\begin{aligned} \epsilon_{jk}\partial_j a_k &= \frac{m}{e}\epsilon_{jk}\int d^2z\partial_j\partial_k\theta(x-z)\rho(z), \\ &= \frac{m}{e}\epsilon_{jk}\partial_j\partial_k\Theta(x). \end{aligned} \qquad (4.47)$$

Let $\omega(z) = \omega_r(z) + \omega_i(z) =| \omega | e^{i\chi}$, we can calculate $\epsilon_{jk}\partial_j a_k$ by,

$$\begin{aligned} \frac{\partial}{\partial z^*} &= \frac{1}{2}(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}), \\ \frac{\partial\omega(z)}{\partial z^*} &= \frac{1}{2}(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}) \mid \omega \mid e^{i\chi}, \\ &= \frac{1}{2}(\mid \omega \mid i\frac{\partial\chi}{\partial x}e^{i\chi}+\frac{\partial \mid \omega \mid}{\partial x}e^{i\chi} \\ &\quad + i\{\mid \omega \mid i\frac{\partial\chi}{\partial y}e^{i\chi}+\frac{\partial \mid \omega \mid}{\partial y}e^{i\chi}\}), \\ &= \frac{1}{2}(i\frac{\partial\chi}{\partial x}+\frac{i}{\mid \omega \mid}\frac{\partial \mid \omega \mid}{\partial y} \\ &\quad + \frac{i}{\mid \omega \mid}\frac{\partial \mid \omega \mid}{\partial x}-\frac{\partial\chi}{\partial y}) \mid \omega \mid e^{i\chi}, \\ &= \frac{1}{2}(i(\frac{\partial ln \mid \omega \mid}{y}+\frac{\partial\chi}{\partial x})+(\frac{\partial ln \mid \omega \mid}{x}-\frac{\partial\chi}{\partial y})) \mid \omega \mid e^{i\chi}. \\ \frac{\partial \ln(\omega)}{\partial x} &= \frac{\partial\chi}{\partial y}, \\ \frac{\partial \ln(\omega)}{\partial y} &= -\frac{\partial\chi}{\partial x}. \end{aligned}$$

Two equations above can be combined as,

$$\partial_i \ln(\omega) = \epsilon_{ij}\partial_j \chi. \tag{4.48}$$

If $\omega = r$ and $\chi = \theta$,

$$\begin{aligned}
\epsilon_{ij}\partial_i\partial_j\theta &= \partial_i\partial_i \ln(r), \\
&= \bigtriangledown^2 \ln(r), \\
&= 2\pi\delta(x).
\end{aligned} \tag{4.49}$$

The last step above equation came from the Green's function in 2-dimensional space. From Eqs. (4.31) and (4.33),

$$\begin{aligned}
\epsilon_{jk}\partial_j a_k &= \frac{m}{e}\epsilon_{jk}\int d^2z\partial_j\partial_k\theta(x-z)\rho(z), \\
&= 2\pi\frac{m}{e}\int d^2z\delta(x-z)\rho(z), \\
&= \frac{2\pi m}{e}\rho(x).
\end{aligned} \tag{4.50}$$

In the unit $\hbar = 1$ or $h = 2\pi$

$$\epsilon_{jk}\partial_j a_k = \frac{hm}{e}\rho(x), \tag{4.51}$$

or in terms of the Dirac flux quantum, $\Phi_D = h/e$,

$$\epsilon_{jk}\partial_j a_k = \Phi_D m\rho(x). \tag{4.52}$$

Consider the partition function using the coherent states as its basis,

$$\begin{aligned}
Z &= Tre^{-\beta(H-\mu N)}, \\
&= \int d\alpha\alpha < \alpha \mid e^{-\beta(H-\mu N)} \mid \alpha > .
\end{aligned} \tag{4.53}$$

Comparing the propragator Eq.(4.44) with the partition function Eq.(4.53) and integrating by parts the Lagrangian Eq.(4.42), we can identify

$$\begin{aligned}
\beta &= it. \\
L &= \frac{1}{2i}(\alpha\partial_t\alpha^* - \alpha^*\partial_t\partial) - H, \\
&= i\alpha^*\partial_t\alpha - H.
\end{aligned}$$

Changing variable $t = -i\tau$, the action in the imaginary time is

$$i\int_0^{-i\beta} dtL = \int_0^{\beta} Ld\tau.$$

By means of a Lagrange multiplier field, a_0, the Lagrangian is formulated from the Hamiltonian of the partition function,

$$\begin{aligned} Z &= \int D[\phi,\phi^*]exp(-\int d\tau[d^2x[\phi^\dagger(x)\partial_\tau\phi(x) \\ &+ \frac{1}{2M}\phi^\dagger(x)(\vec{p}+e\vec{A}+\vec{a})^2\phi(x) + eA_0\phi^\dagger(x)\phi(x)]] \\ &+ \frac{1}{2}\int d^2xd^2y(\phi^*(x)\phi(x))v(\mid x-y\mid)\phi^*(y)\phi(y) \\ &- \mu\phi^*(x)\phi(x) + a_0(\frac{e^2\epsilon_{jk}\partial_j a_k}{2\pi m} - e\phi^*(x)\phi(x)). \end{aligned} \tag{4.54}$$

Therefore,

$$\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho = \epsilon^{0ij}a_0\partial_i a_j - \epsilon^{i0j}a_i\partial_0 a_j + \epsilon^{ij0}a_i\partial_j a_0.$$

The action Eq.(4.54) can be written as

$$\begin{aligned} S &= \int d^3x[\phi^*(x)(\partial_\tau + eA_0 - ea_0 - \mu)\phi(x) \\ &+ \frac{1}{2M}\phi^*(x)(\vec{p}+e\vec{A}-e\vec{a})^2\phi(x) \\ &+ \frac{1}{2}\int d^3xd^3y(\phi^*(x)\phi(x)v(\mid x-y\mid)\phi^*(y)\phi(y) \\ &+ \frac{e^2}{2m\pi}\int d^3x\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho, \end{aligned} \tag{4.55}$$

When the background charge, $\bar{\rho}$, is added to the system, the action is now rewritten as

$$\begin{aligned} S &= \int d^3x[\phi^*(x)(\partial_\tau + eA_0 - ea_0 - \mu)\phi(x) \\ &+ \frac{1}{2M}\phi^*(x)(\vec{p}+e\vec{A}-e\vec{a})^2\phi(x) \\ &+ \frac{1}{2}\int d^3xd^3y(\phi^*(x)\phi(x)-\bar{\rho})v(\mid x-y\mid)(\phi^*(y)\phi(y)-\bar{\rho}) \\ &+ \frac{e^2}{2m\pi}\int d^3x\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho. \end{aligned} \tag{4.56}$$

The last term of the Lagrangian is called Chern-Simons action [11]. The action is extremized with respect to a_0, we obtain the equation of motion,

$$\begin{aligned}
\partial_\alpha \frac{\partial L}{\partial(\partial_\alpha a_0)} &= \frac{\partial L}{\partial a_0}, \\
-\frac{e}{4m\pi}\epsilon^{ij}\partial_i a_j &= -\phi^*(x)\phi(x) + \frac{e}{4m\pi}\epsilon^{ij}\partial_i a_j, \\
\phi^*(x)\phi(x) &= \frac{e}{2m\pi}\epsilon^{ij}\partial_i a_j.
\end{aligned}$$

Let $A^{eff} = A - a$,

$$\begin{aligned}
\epsilon^{ij}\partial_i A_j^{eff} &= \epsilon^{ij}\partial_i A_j - \epsilon^{ij}\partial_i a_j, \\
B^{eff} &= B - b, \\
b &= \frac{2m\pi}{e}\rho(x).
\end{aligned}$$

By introducing the auxiliary field for the interaction term, we can write (see Appendix B)

$$e^{-\int d^2x d^y \frac{1}{2}(\phi^*(x)\phi(x)-\bar{\rho})v(\phi^*(y)\phi(y)-\bar{\rho})} = \int D[\lambda] e^{\int dx\lambda(x)(\phi^*(x)\phi(x)-\bar{\rho})} e^{\frac{1}{2}\int d^2x d^2y \lambda(x) v^{-1}\lambda(y)}. \tag{4.57}$$

The action can be rewritten as

$$\begin{aligned}
S &= \int d^2x d\tau \{\phi^*(x)(\partial_\tau + eA_0 - ea_0 - \mu - \lambda(x))\phi(x) + \frac{e^2}{2m\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho\} \\
&\quad + \int d^2x d\tau \{\phi^*(x)\frac{(\vec{p} + e\vec{A} - e\vec{a})^2}{2M}\phi(x)\} + \int d^2x d\tau \lambda(x)\bar{\rho} \\
&\quad - \frac{1}{2}\int d^2x d^2y d\tau \{\lambda(x) v^{-1}\lambda(y)\}.
\end{aligned}$$

Now fields can be integrated out by using the Gaussian integral (Appendix B). The action is remain in the gauges and the Hubbard fields.

$$\begin{aligned}
S^{eff} &= -Trln(\partial_\tau + eA_0 - ea_0 - \mu - \lambda(x) + \frac{(\vec{p} + e\vec{A} - e\vec{a})^2}{2M}) \\
&\quad + \int d^2x d\tau \frac{e^2}{2m\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho \\
&\quad + \int d^2x d\tau \lambda(x)\bar{\rho} - \frac{1}{2}\int d^2x d^2y d\tau \{\lambda(x) v^{-1}\lambda(y)\}.
\end{aligned}$$

Consider the first order derivative with respect to $\lambda(x)$ and $a_\mu(x) = A_\mu - A_\mu^{eff}$,

$$\begin{aligned}\frac{\partial S^{eff}}{\partial\lambda(x)} &= 0,\\ \frac{\partial S^{eff}}{\partial a_\mu(x)} &= 0.\end{aligned}$$

The equations of motion follow from above extremization. Let $\bar{a}$ and $\bar{\lambda}(x)$ are the values that correspond to the equations of motion

$$< j_0 > -\bar{\rho} + \int d^3y v^{-1}\lambda(y) = 0, \quad (4.58)$$

$$< j_\mu(x) > -\frac{e^2}{\pi m}\epsilon^{\mu\nu\rho}(< \partial_\nu A_\rho > - < \partial_\nu A_\rho^{eff} >) = 0. \quad (4.59)$$

The condition for the uniform liquid state is obtained $\lambda(x) = 0$ namely,

$$< j_0 >= \bar{\rho}.$$

The Chern-Simons statistical magnetic field is given by,

$$\begin{aligned}\vec{b} &= \vec{\nabla} \times \vec{a},\\ &= \vec{B} - \vec{B}^{eff}.\end{aligned} \quad (4.60)$$

Setting $\mu = 0$ in the equation of motion, we find the uniform average statistical magnetic field,

$$< b > = \frac{m\pi}{e^2}\bar{\rho}. \quad (4.61)$$

Recall that the filling factor, ν, is a ratio of total number of applied (external) flux quantum to total number of electrons,

$$\begin{aligned}\nu &\equiv \frac{1}{2p+1},\\ &= \frac{\Phi_D\rho_0}{B},\end{aligned}$$

where p is an interger.

$$\begin{aligned} B^{eff} &= B - < b >, \\ &= B - (2p+1)\frac{B}{2p+1}, \\ &= 0. \end{aligned}$$

This implies that the composite boson (m is an odd number), in this case, sees no magnetic field but if m is an even number, it is called a composite fermion, a new particle will sees the magnetic field,

$$\begin{aligned} B^{eff} &= B - (2p)\frac{B}{2p+1}, \\ &= \frac{B}{2p+1}. \end{aligned}$$

Due to the nonvanished magnetic field in the case of composite fermions, the new filling factor will be determined by,

$$\begin{aligned} \nu_{eff} &= \frac{\Phi_D \rho_0}{B^{eff}}, \\ &= \frac{\Phi_D \rho_0}{B}(2p+1), \\ &= (2p+1)\nu. \end{aligned}$$

The FQH state of electrons at $\nu = 1/(2p+1)$ is the IQH state of composite fermions at $\nu_{eff} = 1$.

Consider the fluctuation of the system around mean field by using the Saddle Point Approximation (SPA). We keep terms up to the second order, $\lambda(x) \to \lambda(x) + \delta\lambda$ and $A^{eff} \to A^{eff} + \delta A^{eff}$,

$$\begin{aligned} S^{eff} &= S^{eff}|_{\bar{a},\bar{\lambda}} + \frac{\partial S^{eff}}{\partial A^{eff}_\mu}|_{\bar{a},\bar{\lambda}} \\ &\quad + \frac{\partial^2 S^{eff}}{\partial A^{eff}_\mu A^{eff}_\nu}|_{\bar{a},\bar{\lambda}} + ..., \end{aligned} \tag{4.62}$$

where the first order derivative of the action is zero because of the SPA. Now we will consider the determinant of the first term above. We substitute $\lambda(x) \rightarrow \lambda(x) + \delta\lambda$ and $A^{eff} \rightarrow A^{eff} + \delta A^{eff}$ into the action above. Consider the potential term,

$$\begin{aligned} \frac{\partial^2 S^{eff}}{\partial\lambda_1\partial\lambda_2} &= \frac{1}{2}\int d^3x d^3y \frac{\partial^2}{\partial\lambda(x)\partial\lambda(y)}\{\lambda(x)v^{-1}\lambda(y)\}, \\ &= v^{-1}(|x-y|). \end{aligned} \tag{4.63}$$

The potential term when substitute the fluctuation can be written as,

$$\frac{1}{2}\int d^3x d^3y \delta\lambda(x) v^{-1}\delta\lambda(y).$$

The partition function is now written in the fluctuation term,

$$\begin{aligned} Z(A) &= Z(\bar{a},\bar{\lambda}) e^{\frac{1}{2}\int d^3x d^3y \delta A^{eff}_\mu \prod_{\mu\nu} \delta A^{eff}_\nu + \frac{\alpha}{2}\int d^3x \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho} \\ & e^{\frac{\alpha}{2}\int d^3\lambda(x) b(x) + \frac{1}{2}\int d^3x d^3y \delta\lambda(x) v^{-1}\delta\lambda y}, \end{aligned} \tag{4.64}$$

where $\prod_{\mu\nu}(x,y) = \frac{\partial^2 S^{eff}}{\partial A^{eff}_\mu(x)\partial A^{eff}_\nu(y)}$. $\delta\lambda$ is integrated out by using the Gaussian integral,

$$\int D[\delta\lambda] e^{\frac{1}{2}\int d^3 d^3 \delta\lambda(x) v^{-1}\delta\lambda(y) + \frac{\alpha}{2}\int d^3x \delta\lambda(x) b(x)} = e^{-\frac{\alpha^2}{2}\int d^3x d^3y b(x) v(|x-y|) b(y)}$$

$$\begin{aligned} Z(A) &= \int D[\delta a] e^{\frac{1}{2}\int d^3x d^3y \delta(A_\mu(x)-a_\mu(x)) \prod_{\mu\nu}(x,y) \delta(A_\nu(y)-a_\nu(y))} \\ & e^{\frac{\alpha}{2}\int d^3x \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho} e^{\frac{-\alpha^2}{2}\int d^3x d^3y b(x) v(|x-y|) b(y)}. \end{aligned} \tag{4.65}$$

Introducing a gauge fixing term $(\partial_\mu a^\mu)^2/(2\beta)$ in order to avoid singularity in the inverse matrix of the quadratic term in a_μ,

$$\begin{aligned} Z(A) &= \int D[\delta a] e^{\frac{1}{2}\int d^3x d^3y \delta(A_\mu(x)-a_\mu(x)) \prod_{\mu\nu}(x,y) \delta(A_\nu(y)-a_\nu(y))} \\ & e^{\frac{\alpha}{2}\int d^3x \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho} e^{\frac{-\alpha^2}{2}\int d^3x d^3y b(x) v(|x-y|) b(y)} \\ & e^{\int d^3x \frac{(\partial_\mu a^\mu)^2}{2\beta}}, \end{aligned} \tag{4.66}$$

matrix elements before integrating over statistical gauge can be written as,

$$mt = \begin{bmatrix} \sqcap_{00} - \frac{\partial_0^2}{\beta} & \sqcap_{01} - \alpha\partial_2 - \frac{\partial_0\partial_1}{\beta} & \sqcap_{02} + \alpha\partial_1 - \frac{\partial_0\partial_2}{\beta} \\ \sqcap_{10} + \alpha\partial_2 - \frac{\partial_0\partial_1}{\beta} & \sqcap_{11} - \frac{\partial_1\partial_1}{\beta} + \alpha^2 v\partial_2\partial_2 & \sqcap_{12} - \alpha\partial_0 - \frac{\partial_1\partial_2}{\beta} - \alpha^2 v\partial_2\partial_1 \\ \sqcap_{20} - \alpha\partial_1 - \frac{\partial_2\partial_0}{\beta} & \sqcap_{21} + \alpha\partial_0 - \frac{\partial_2\partial_1}{\beta} - \alpha^2 v\partial_2\partial_1 & \sqcap_{22} - \frac{\partial_2\partial_2}{\beta} + \alpha^2 v\partial_1\partial_1 \end{bmatrix},$$

where

$$\sqcap_{ij} = \frac{\partial^2 S^{eff}}{\partial a_i \partial a_j}.$$

Fourier transform of above matrix, $e^{-i(qx)+i\omega t}$

$$mt = \begin{bmatrix} ft_{00} + \frac{\omega^2}{\beta} & ft_{01} + i\alpha q_2 - \frac{\omega q_1}{\beta} & ft_{02} - i\alpha q_1 - \frac{\omega q_2}{\beta} \\ ft_{10} - i\alpha q_2 - \frac{\omega q_1}{\beta} & ft_{11} + \frac{q_1^2}{\beta} - \alpha^2 v q_2^2 & ft_{12} - i\alpha\omega + \frac{q_1 q_2}{\beta} + \alpha^2 v q_2 q_1 \\ ft_{20} + i\alpha q_1 - \frac{q_2\omega}{\beta} & ft_{21} + i\alpha\omega + \frac{q_2 q_1}{\beta} + \alpha^2 v q_2 q_1 & ft_{22} + \frac{q_2^2}{\beta} - \alpha^2 v q_1^2 \end{bmatrix}.$$

Consider the polarization tensors in the matrix above under the Fourier transformation (Appendix C). The matrix mt will be written as,

$$\begin{aligned}
mt(0,0) &= q^2 \sqcap_0 + \frac{\omega^2}{\beta}, \\
mt(0,1) &= \omega q_1 \sqcap_0 + i q_2(\sqcap_1 + \alpha) - \frac{\omega q_1}{\beta}, \\
mt(0,2) &= \omega q_2 \sqcap_0 - i q_1 \sqcap_1 - i\alpha q_1 - \frac{\omega q_2}{\beta}, \\
mt(1,0) &= \omega q_1 \sqcap_0 - i q_2 \sqcap_1 - i\alpha q_2 - \frac{\omega q_1}{\beta}, \\
mt(1,1) &= \frac{q_1^2}{\beta} + \omega^2 \sqcap_0 + q_2^2(\sqcap_2 - \alpha^2 v), \\
mt(1,2) &= \frac{q_1 q_2}{\beta} - q_1 q_2(\sqcap_2 - \alpha^2 v) - i\omega(\sqcap_1 + \alpha), \\
mt(2,0) &= \omega q_2 \sqcap_0 + i q_1 \sqcap_1 + i\alpha q_1 - \frac{q_2\omega}{\beta}, \\
mt(2,1) &= \frac{q_2 q_1}{\beta} - q_1 q_2(\sqcap_2 - \alpha^2 v) + i\omega(\sqcap_1 + \alpha), \\
mt(2,2) &= \frac{q_2^2}{\beta} + \omega^2 \sqcap_0 + q_1^2(\sqcap_2 - \alpha^2 v).
\end{aligned}$$

The determinent of the matrix mt is then,

$$det(\mathrm{mt}) = \frac{(\omega^2 - q^2)^2}{\beta}\Sigma(\omega, q), \tag{4.67}$$

where

$$\Sigma(\omega, q) \;\; = \;\; \sqcap_0 \omega^2 - (\sqcap_1 + \alpha)^2 + \sqcap_0(\sqcap_2 - \alpha^2 v) q^2.$$

Inverse of the matrix mt can be shown below,

$$\mathrm{mt}^{-1} = \begin{bmatrix} k_0 q^2 & k_0 \omega q_1 + i k_1 q_2 & k_0 \omega q_2 - i q_1 k_1(\omega, q) \\ \omega q_1 k_0 - i q_2 k_1 & \omega^2 k_0 + (q^2 - q_1^2) k_2 & -i\omega k_1 - q_1 q_2 k_2 \\ \omega q_2 k_0 + i q_1 k_1 & -i\omega k_1 - q_2 q_1 k_2 & \omega^2 k_0 + (q^2 - q_2^2) k_2 \end{bmatrix},$$

where

$$\begin{aligned} k_0(\omega, q) &= -\frac{\alpha^2 \sqcap_0}{\Sigma(\omega, q)}, \\ k_1(\omega, q) &= \alpha + \alpha^2 \frac{\alpha + \sqcap_1}{\Sigma(\omega, q)} + \alpha^3 q^2 v \frac{\sqcap_0}{\Sigma(\omega, q)}, \\ k_2(\omega, q) &= \frac{-\alpha^2 \sqcap_2 + (\omega^2 \sqcap_0^2 - \sqcap_1^2 + q^2 \sqcap_0 \sqcap_2) v}{\Sigma(\omega, q)}. \end{aligned}$$

CHAPTER V
CONCLUSION AND DISSCUSSION

In quantum hall regime, namely at low temperatures and high magnetic fields, very different behaviour is found in ordinary two-dimensional electron gas, ρ_{xy} passes through a series of plateaux, $\rho_{xy} = \nu e^2/h$ where ν is rational number at which ρ_{xx} vanishes. Some insight into this phenomenon can be gained by considering the quantum mechanic of a single electron in a magnetic field. These simple consideration negleted two factors which are crucial to the observation of the QHE, namely the effects of impurities and inter-electron interaction. There are two approaches to explain the FQHE. One is the variational approaches. For example, a laughlin's wave function and Jain wave function. Another approach is the field theory. For example, the Chern-Simons-Landau-Ginzburg theory. The composite particles can be constructed by introducing the Chern-Simons term into the Lagrangian. And the statistical transmutation is connected through the statistical phase factor. In the fermion case, the composite particle is seen the effective magnetic field and can be described in the same way as IQHE. In the bosonic case, the composite can not seen the magnetic field.

In this dissertation, we derive the microscopic theory of the FQHE by using coherent state representation to derive the essential physics in the Laughlin mean field. Besides this, we also derive from quantum fluctuation via saddle point approximation. The quantum fluctuation will be of importance to address the stability of FQHE. This aspect will be investigated in near future.

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APPENDICES

APPENDIX A

In this appendix, we prove some mathematical result in the chapter 2. We first consider the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$. From (2.5) and (2.6)

$$\hat{a} = (\frac{1}{\sqrt{2}})(\hat{q} + i\hat{p}) \tag{A.1}$$

$$\hat{a}^\dagger = (\frac{1}{\sqrt{2}})(\hat{q} - i\hat{p}), \tag{A.2}$$

We obtain the commutation relation,

$$\begin{aligned} [\hat{a}, \hat{a}^\dagger] &= \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} \\ &= (\frac{1}{2})(\hat{q}\hat{q} + i\hat{p}\hat{q} - i\hat{q}\hat{p} + \hat{p}\hat{q}) - (\frac{1}{2})(\hat{q}\hat{q} - i\hat{p}\hat{q} - i\hat{q}\hat{p} + \hat{p}\hat{q}) \\ &= \frac{i}{2}(2\hat{p}\hat{q} - 2\hat{q}\hat{p}) \\ &= 1. \end{aligned} \tag{A.3}$$

We next prove the Baker-Campbell-Hausdorff formula (in the case of [A,[A,B]] = 0). then,

$$\begin{aligned} f(a) &= e^{aA}e^{aB} \\ \frac{df(a)}{da} &= e^{aA}Be^{aB} + Ae^{aA}e^{aB} \\ &= (e^{aA}Be^{-aA} + A)e^{aA}e^{aB} \\ &= (e^{aA}Be^{-aA} + A)f(a). \end{aligned} \tag{A.4}$$

then,

$$g(b) = e^{bA}Be^{-bA}$$

$$\frac{dg(b)}{db} = e^{bA}[A, B]e^{-bA} \tag{A.5}$$

$$\frac{d^2g(b)}{db^2} = e^{bA}[A, [A, B]]e^{-bA} \tag{A.6}$$

$$\frac{d^3g(b)}{db^3} = e^{bA}[A, [A, [A, B]]]e^{-bA} \tag{A.7}$$

We therefore can express g(b) as a Taylor's series

$$\begin{aligned} g(b) &= g(0) + bg^{(1)}(0) + \frac{b^2g^{(2)}(0)}{2} + \dots \\ &= B + b[A, B] + \frac{b^2[A, [A, B]]}{2} + \dots \end{aligned} \tag{A.8}$$

In the case that $[A,[A,B]]=0$, the series terminated, that is $g(b)=B+[A,B]$. Thus,

$$\begin{aligned} \frac{df(a)}{da} &= (e^{aA}Be^{-aA}+A)f(a) \\ &= (B+a[A,B]+A)f(a) \end{aligned}$$

so that,

$$\begin{aligned} \int \frac{df(a)}{f(a)} &= \int (B+a[A,B]+A)da \\ \ln \mid f(a) \mid &= (A+B)a+\frac{a^2[A,B]}{2}+c \end{aligned}$$

where c is an arbtrary constant. Imposing the condition f(0)=1, we get $c=0$ so that $f(a)=e^{(A+B)+\frac{a^2[A,B]}{2}}$.

Setting a=1, we finally obtain,

$$e^Ae^B = e^{(A+B)+\frac{[A,B]}{2}}. \tag{A.9}$$

Next, we prove the completeness relation for the coherent states:

$$1 = \frac{1}{\pi}\int d^2\alpha \mid \alpha >< \alpha \mid$$

Writting,

$$\begin{aligned} \mid \alpha > &= e^{-\frac{|\alpha|^2}{2}}e^{\alpha\hat{a}^\dagger} \mid 0 > \\ &= e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n(\hat{a}^\dagger)^n}{n!} \mid 0 > \\ &= e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}} \mid n > \end{aligned} \tag{A.10}$$

and expressing $d^2\alpha = d(\text{Re})\alpha d(\text{Im})\alpha$, using the polar coordinates, (r,θ), $r=\mid \alpha \mid$ $(\alpha = re^{i\theta})$

$$d^2\alpha = rdrd\theta$$

$$\begin{aligned}
\frac{1}{\pi}\int d^2\alpha \mid \alpha >< \alpha \mid &= \frac{1}{\pi}\sum_{m,n=0}^{\infty}\frac{\mid m >< n \mid}{\sqrt{m!n!}}\int rdrd\theta r^{n+m+1} \\
&\quad e^{-r^2}\int_0^{2pi} d\theta e^{i(m-n)\theta} \\
&= 2\sum_{n}^{\infty}\frac{\mid n >< n \mid}{n!}\int_0^{\infty} drr^{2n+1}e^{-r^2} \\
&= \sum_{n}^{\infty}\frac{\mid n >< n \mid}{n!}\int_0^{\infty} dr^2(r^2)^n e^{-r^2} \qquad \text{(A.11)}
\end{aligned}$$

Gamma functon, $\int_0^\infty e^{-t}t^n dt = n!$, we finally obtain,

$$\begin{aligned}
\frac{1}{\pi}\int d^2\alpha \mid \alpha >< \alpha \mid &= \sum_{n=0}^{\infty} \mid n >< n \mid \\
&= 1. \qquad \text{(A.12)}
\end{aligned}$$

We finally prove the normalization of $\mid \alpha >$ from E.q.(A.10)

$$\begin{aligned}
\mid \alpha > &= e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}} \mid n > \\
< \beta \mid &= e^{-\frac{|\beta|^2}{2}}\sum_{m=0}^{\infty}\frac{< m \mid}{\sqrt{m!}}(\beta^*)^m
\end{aligned} \qquad \text{(A.13)}$$

then,

$$\begin{aligned}
< \beta \mid \alpha > &= e^{-\frac{|\alpha|^2}{2}}e^{-\frac{|\beta|^2}{2}} \\
&\quad \sum_{m=0}^{\infty}\frac{(\beta^*)^m(\alpha)^n}{\sqrt{m!n!}} < m \mid n > \\
&= e^{-\frac{|\alpha|^2}{2}}e^{-\frac{|\beta|^2}{2}}e^{\alpha\beta^*} \\
&= e^{-\frac{|\alpha|^2}{2}}e^{-\frac{|\beta|^2}{2}}\sum_{n=0}^{\infty}\frac{(\alpha\beta^*)}{n!} \qquad \text{(A.14)}
\end{aligned}$$

If α or β is equal to zero, then

$$< 0 \mid \alpha > = e^{-\frac{|\alpha|^2}{2}} \qquad \text{(A.15)}$$

$$< \beta \mid 0 > = e^{-\frac{|\beta|^2}{2}} \qquad \text{(A.16)}$$

APPENDIX B

In this appendix, we derive the so-called Hubbard-Stratonovich transformation following the work by Hubbard [22]. We first recall the formulae for the Gaussian integrals in one dimension,

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}, \tag{B.1}$$

$$\int_{-\infty}^{\infty} e^{-ax^2+bx} dx = \sqrt{\frac{\pi}{a}} e^{-\frac{b^2}{4a}}. \tag{B.2}$$

Consider the function

$$f(a) = e^{a^2}. \tag{B.3}$$

Using the identity

$$\int_{-\infty}^{\infty} e^{-\pi(x-\frac{a}{\sqrt{\pi}})^2} dx = 1 \tag{B.4}$$

which is a consequence of Eq. (B.1), $f(a)$ can be expressed as

$$\begin{aligned} f(a) &= \int_{-\infty}^{\infty} e^{a^2} e^{-\pi(x-\frac{a}{\sqrt{\pi}})^2} dx \\ &= \int_{-\infty}^{\infty} e^{-\pi x^2 - 2\sqrt{\pi} a x} dx. \end{aligned} \tag{B.5}$$

We now generalize the above results to the d-dimensional case [23]. Let v be a d-dimensional vector and B be a real $d \times d$ symmetric and positive-definite matrix. It is well-known that

$$\int d^d v e^{-\frac{1}{2}v^T B v} = (2\pi)^{d/2} e^{-\frac{1}{2}Tr \ln B}, \tag{B.6}$$

$$\int d^d v e^{-\frac{1}{2}v^T B v + \rho^T v} = (2\pi)^{d/2} e^{-\frac{1}{2}Tr \ln B} e^{\frac{1}{2}\rho^T B^{-1} \rho}. \tag{B.7}$$

More generally, if z is a d-dimensional complex vector, then

$$\int dz_1^* dz_1 \ldots dz_d^* dz_d e^{-z^\dagger B z} = (2\pi)^{d/2} e^{-\frac{1}{2}Tr \ln B}. \tag{B.8}$$

For the Grassmann variables $\Theta(x)$, we have the analogous formulae,

$$\int d\Theta(x)_n \ldots d\Theta(x)_1 e^{-\frac{1}{2}\Theta(x)^T A \Theta(x)} = e^{\frac{1}{2}Tr \ln A}, \tag{B.9}$$

$$\int d\Theta(x)_n \ldots d\Theta(x)_1 e^{-\frac{1}{2}\Theta(x)^T A \Theta(x) + \rho^T \Theta} = e^{\frac{1}{2}Tr \ln A} e^{-\frac{1}{2}\rho^T A^{-1} \rho}. \tag{B.10}$$

Consider the function

$$f(w) = e^{\frac{1}{2}w^T B w} \tag{B.11}$$

with w being a d-dimensional vector. Using Eq. (B.7) with $\rho = w$, we get

$$f(w) = N \int d^d v e^{-\frac{1}{2}v^T B v + w^T v} \tag{B.12}$$

where $N = (2\pi)^{-d/2} \exp(\frac{1}{2} Tr \ln B)$. The above result can be generalized to an infinite dimensional case with the result,

$$e^{\frac{1}{2}\int d^2x d^2y d\tau [\phi^*(x)\phi(x) - \bar{\rho}] v [\phi^*(y)\phi(y) - \bar{\rho}]} = N \int D[\lambda] e^{-\frac{1}{2}\int d^2x d^2y d\tau \lambda(x) v^{-1} \lambda(y)} e^{\int d^2x d\tau \lambda(x)[\phi^*(x)\phi(x) - \bar{\rho}]}. \tag{B.13}$$

This is the desired Hubbard-Stratonovich transformation.

APPENDIX C

This appendix follows the papers by Fradkin [28] and Zhang [29] and uses some techniques from a many-body physics textbook [26] in order to obtain the polarization tensors.

Consider

$$
\begin{aligned}
iG(x,y) &= < T[\hat{\phi}(x), \hat{\phi}^\dagger(y)] > \\
&= \theta(x_0 - y_0) < \hat{\phi}(x)\hat{\phi}^\dagger(y) > - < \hat{\phi}^\dagger(y)\hat{\phi}(x) > \theta(x_0 - y_0) \quad \text{(C.1)} \\
&= \theta(x_0 - y_0) \sum_m \int \frac{dk}{2\pi} e^{-i\omega_m(x_0 - y_0)} (\phi_{mk}(x))(\phi^*_{mk}(y)) \\
&\quad - \theta(y_0 - x_0) \sum_m \int \frac{dk}{2\pi} e^{-i\omega_m(x_0 - y_0)} (\phi_{mk}(x)(\phi^*_{mk}(y)) \quad \text{(C.2)}
\end{aligned}
$$

where $\phi_{mk}(x,y) = (\frac{\sqrt{B}}{2^m m! \sqrt{\pi}})^{1/2} e^{ikx} e^{-\frac{1}{2}(\sqrt{B}y - \frac{k}{\sqrt{B}})^2} H_m(y\sqrt{B} - \frac{k}{\sqrt{B}})$ is the eigenfunction of a charged particle moving in the magnetic field in the Landau gauge ($\vec{A} = eB(-y, 0)$). We use the Landau gauge because our system is rectangular. If our system is circular, then it is more convenient to use the symmetric gauge instead. The Fourier transformation of the polarization tensor $\sqcap_{\mu\nu}(x,y)$ is

$$
\begin{aligned}
\sqcap_{\mu\nu}(q,p) &= \int d^3x d^3y e^{-i(xq - x_0 q_0)} e^{-i(yp - y_0 p_0)} \sqcap_{\mu\nu}(x,y) \\
&= \int d^3x d^3y e^{-i(xq - x_0 q_0)} e^{-i(yp - y_0 p_0)} I_1 G(x,y) I_2 G(y,x) \quad \text{(C.3)}
\end{aligned}
$$

where I_1 is the covariant derivative (identity) when μ is non-zero (zero); the same holds for I_2 and ν. Then

$$
\begin{aligned}
\sqcap_{jk}(x,y) &= \frac{i}{M}\delta^{(3)}(x-y)\delta_{jk}G(x,y) - \frac{i}{4M^2}(D^x_j G(x,y)][D^y_k G(y,x)] \\
&\quad - \frac{i}{4M^2}(D^{x^\dagger}_j G(x,y)][D^{y^\dagger}_k G(y,x)] + \frac{i}{4M^2}G(x,y)(D^{x^\dagger}_j D^{y^\dagger}_k] \\
&\quad + \frac{i}{4M^2}(D^{x^\dagger}_j D^{y^\dagger}_k]G(x,y) \quad \text{(C.4)}
\end{aligned}
$$

where $D_\mu = \partial_\mu + ie(A_\mu - a_\mu)$ is the covariant derivative. If the step function is written in the integral form,

$$
\theta(t - t_0) = \frac{i}{2\pi}\int_\infty^\infty \frac{e^{-i\omega(t-t_0)}}{\omega + i\eta} d\omega, \quad \text{(C.5)}
$$

then $\sqcap_{\mu\nu}(x,y)$ can be rewritten as

$$\sqcap_{\mu\nu}(q,p) = \sum_m \int d^3x d^3y d\omega \frac{i}{2\pi} e^{-i\omega(x_0-y_0)} e^{-i(xq-x_0q_0)} e^{-i(yp-y_0p_0)} \left[\frac{I_1(\phi_{mk}(x)\phi^*_{mk}(y))I_2(\phi_{m'k'}(x)\phi^*_{m'k'}(y))}{\omega-(\omega_m-\omega'_m)+i\eta} - \frac{I_1(\phi_{m'k'}(x)\phi^*_{m'k'}(y))I_2(\phi_{mk}(x)\phi^*_{mk}(y))}{\omega+(\omega_m-\omega'_m)-i\eta}\right]. \quad (C.6)$$

We now derive the Lehmann representation. Consider $\sqcap_{01}(q,p)$ for which $I_1 = 1$ and $I_2 = D_1^y = \partial_1^{(y)} - iBy_2$. Then

$$\begin{aligned}\sqcap_{01}(q,p) &= \frac{i}{2\pi}\frac{1}{2M}\sum_m \int d^3x d^3y d\omega e^{-i\omega(x_0-y_0)} e^{-i(xq-x_0q_0)} e^{-i(yp-y_0p_0)} \\ &\quad \left[\frac{(\phi_{mk}(x)\phi^*_{mk}(y))D_1^y(\phi_{m'k'}(y)\phi^*_{m'k'}(x))}{\omega-(\omega_m-\omega'_m)+i\eta} - \frac{(\phi_{m'k'}(x)\phi^*_{m'k'}(y))D_1^y(\phi_{mk}(y)\phi^*_{mk}(x))}{\omega+(\omega_m-\omega'_m)-i\eta}\right] \\ &= (2\pi)^3 \frac{B}{4M(2\pi)} e^{\bar{q}^2} \\ &\quad \sum_m \left[\frac{q_1 L_{m'}^{m-m'}(\bar{q}^2)[\bar{q}^2-(m-m')] + iq_2 F_{m,m'}(\bar{q}^2)]}{[q_0-(\omega_m-\omega'_m)+i\eta]} - \frac{q_1 L_{m'}^{m-m'}(\bar{q}^2)[\bar{q}^2+(m-m')] + iq_2 F_{m,m'}(\bar{q}^2)]}{[q_0+(\omega_m-\omega'_m)-i\eta]}\right] \quad (C.7)\end{aligned}$$

where $F_{m,m'}(\bar{q}^2) = \bar{q}^2[L_{m'}^{m-m'}(\bar{q}^2) + 2L_{m'-1}^{m-m'+1}(\bar{q}^2)(1-\delta_{m',0}) - (m-m')L_{m'}^{m-m'}(\bar{q}^2)]$ and the polarization tensors are now

$$\sqcap_{00}(q) = q^2 \sqcap_0(q) \quad (C.8)$$

$$\sqcap_{0j}(q) = q_0 q_j \sqcap_0(q) + i\epsilon^{jk} q_k \sqcap_1(q) \quad (C.9)$$

$$\sqcap_{j0}(q) = q_0 q_j \sqcap_0(q) - i\epsilon^{jk} q_k \sqcap_1(q) \quad (C.10)$$

$$\sqcap_{jk}(q) = q_0^2 \delta_{jk} \sqcap_0(q) - iq_0\epsilon^{jk} \sqcap_1(q)(q^2\delta_{jk} - q_j q_k) \sqcap_2(q) + \delta_{jk} \sqcap_3(q) \quad (C.11)$$

where

$$\sqcap_0(q) = -\frac{B}{2\pi M}\sum_m e^{-\bar{q}^2} \frac{m-m'}{q_0^2-(\omega_m-\omega'_m)} \frac{m'!}{m!} \bar{q}^{2(m-m'-1)}$$

$$
\begin{aligned}
& \quad \times (L_{m'}^{m-m'}(\bar{q}^2))^2 \\
\sqcap_1(q) &= \frac{B}{2\pi M^2} e^{-\bar{q}^2} \sum_m \frac{m-m'}{q_0^2 - (\omega_m - \omega'_m)} \frac{m'!}{m!} \bar{q}^{2(m-m'-1)} L_{m'}^{m-m'}(\bar{q}^2) \\
& \quad \times (\bar{q}^2 (L_{m'}^{m-m'}(\bar{q}^2) + 2L_{m'-1}^{m-m'+1}(\bar{q}^2)(1-\delta_{m',0})) \\
& \quad - (m-m') L_{m'}^{m-m'}(\bar{q}^2)) \qquad \text{(C.12)} \\
\sqcap_2(q) &= -\frac{B}{2\pi 2M^2} e^{-\bar{q}^2} \sum_m \frac{m-m'}{q_0^2 - (\omega_m - \omega'_m)} \frac{m'!}{m!} \bar{q}^{2(m-m'-1)} \\
& \quad \times (L_{m'}^{m-m'}(\bar{q}^2) + 2L_{m'-1}^{m-m'+1}(\bar{q}^2)(1-\delta_{m',0})) \\
& \quad \times (\bar{q}^2 (L_{m'}^{m-m'}(\bar{q}^2) + 2L_{m'-1}^{m-m'+1}(\bar{q}^2)(1-\delta_{m',0})) \\
& \quad - 2(m-m') L_{m'}^{m-m'}(\bar{q}^2)) \qquad \text{(C.13)} \\
\sqcap_3(q) &= \frac{B}{2\pi M^2} e^{-\bar{q}^2} \sum_m \frac{m'!}{m!} \bar{q}^{2(m-m'-1)} \\
& \quad \times (L_{m'}^{m-m'}(\bar{q}^2))^2 (m-m') - p\frac{B}{2\pi M}. \qquad \text{(C.14)}
\end{aligned}
$$

CURRICULUM VITAE

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