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ลิขสิทธิ์ของจุฬาลงกรณ์มหาวิทยาลัย

REGULAR ELEMENTS OF SOME ORDER-PRESERVING
TRANSFORMATION SEMIGROUPS AND GENERALIZED
ORDER-PRESERVING TRANSFORMATION SEMIGROUPS

Miss Winita Mora

ศูนย์วิทยทรัพยากร
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สำหรับเซตอันดับบางส่วน X และ Y ใดๆ ให้ $OT(X, Y)$, $OP(X, Y)$ และ $OI(X, Y)$ แทนเซตของการแปลงที่รักษาอันดับของ X ไปยัง Y ทั้งหมด เซตของการแปลงบางส่วนที่รักษาอันดับของ X ไปยัง Y ทั้งหมด และเซตของการแปลงบางส่วนหนึ่งต่อหนึ่งที่รักษาอันดับของ X ไปยัง Y ทั้งหมด ตามลำดับ ให้ $OT(X)$, $OP(X)$ และ $OI(X)$ แทน $OT(X, X)$, $OP(X, X)$ และ $OI(X, X)$ ตามลำดับ ดังนั้น $OT(X)$, $OP(X)$ และ $OI(X)$ เป็นกึ่งกรุปภายใต้การประกอบ ถ้า Y เป็นเซตย่อยไม่ว่างของ X แล้ว $OT(X, Y)$, $OP(X, Y)$ และ $OI(X, Y)$ เป็นกึ่งกรุปย่อยของ $OT(X)$, $OP(X)$ และ $OI(X)$ ตามลำดับ สำหรับเซตย่อยไม่ว่าง Y ของเซตอันดับบางส่วน X เราให้ $\overline{OT}(X,Y) = \{\alpha \in OT(X) \mid Y\alpha \subseteq Y\}$, $\overline{OP}(X,Y) = \{\alpha \in OP(X) \mid (\text{dom } \alpha \cap Y)\alpha \subseteq Y\}$ และ $\overline{OI}(X,Y) = \{\alpha \in OI(X) \mid (\text{dom } \alpha \cap Y)\alpha \subseteq Y\}$ เราได้ว่า $\overline{OT}(X,Y)$, $\overline{OP}(X,Y)$ และ $\overline{OI}(X,Y)$ เป็นกึ่งกรุปย่อยของ $OT(X)$, $OP(X)$ และ $OI(X)$ ซึ่งบรรจุ $OT(X, Y)$, $OP(X, Y)$ และ $OI(X, Y)$ ตามลำดับ เราสามารถมองได้ว่ากึ่งกรุป $OT(X,Y)$ และ $\overline{OT}(X,Y)$ เป็นการวางนัยทั่วไปของ $OT(X)$ และในทำนองเดียวกันกับ $OP(X, Y)$, $\overline{OP}(X,Y)$, $OI(X, Y)$ และ $\overline{OI}(X,Y)$

สำหรับเซตอันดับบางส่วน X, Y และ $\theta \in OT(Y, X)$ ให้ $(OT(X, Y), \theta)$ แทนกึ่งกรุป $(OT(X, Y), *)$ โดยที่ $\alpha * \beta = \alpha\theta\beta$ สำหรับทุก $\alpha, \beta \in OT(X, Y)$ เรานิยามกึ่งกรุป $(OP(X, Y), \theta)$ โดยที่ $\theta \in OP(Y, X)$ และกึ่งกรุป $(OI(X, Y), \theta)$ โดยที่ $\theta \in OI(Y, X)$ ในทำนองเดียวกัน เราจะเห็นได้ว่า $(OT(X, Y), \theta)$, $(OP(X, Y), \theta)$ และ $(OI(X, Y), \theta)$ เป็นการวางนัยทั่วไปของ $OT(X)$, $OP(X)$ และ $OI(X)$ ตามลำดับ

เป็นที่รู้กันแล้วว่า สำหรับเซตอันดับทุกส่วน X ใดๆ $OP(X)$ และ $OI(X)$ เป็นกึ่งกรุปปกติ และได้มีการให้ลักษณะของสมาชิกปกติของ $OT(X)$ ยิ่งไปกว่านั้นในกรณีที่ X เป็นเซตอันดับทุกส่วนจำกัด ได้มีการนับจำนวนสมาชิกของ $OT(X)$, $OP(X)$ และ $OI(X)$ จุดมุ่งหมายของงานวิจัยนี้ เพื่อบอกลักษณะของสมาชิกปกติของกึ่งกรุป $OT(X, Y)$, $OP(X, Y)$, $OI(X, Y)$, $\overline{OT}(X,Y)$, $\overline{OP}(X,Y)$ และ $\overline{OI}(X,Y)$ โดยที่ X เป็นเซตอันดับทุกส่วน และ $\varnothing \neq Y \subseteq X$ และ $(OT(X, Y), \theta)$, $(OP(X, Y), \theta)$ และ $(OI(X, Y), \theta)$ โดยที่ X และ Y เป็นเซตอันดับทุกส่วนใดๆ นอกจากนั้นเรายังนับจำนวนสมาชิกปกติของกึ่งกรุป $OT(X, Y)$, $OP(X, Y)$, $OI(X, Y)$, $\overline{OT}(X,Y)$, $\overline{OP}(X,Y)$ และ $\overline{OI}(X,Y)$ ด้วย เมื่อ $X = \{1, 2, \ldots, n\}$ และ $Y = \{1, 2, \ldots, m\}$ โดยที่ $m \leq n$

ภาควิชา.......คณิตศาสตร์...... ลายมือชื่อนิสิต วินิตา โมรา

สาขาวิชา.....คณิตศาสตร์...... ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์หลัก ยุพาภรณ์ เข็มประสิทธิ์

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WINITA MORA : REGULAR ELEMENTS OF SOME ORDER-PRESERVING TRANSFORMATION SEMIGROUPS AND GENERALIZED ORDER-PRESERVING TRANSFORMATION SEMIGROUPS. THESIS ADVISOR : PROF. YUPAPORN KEMPRASIT, Ph.D., 69 pp.

For any posets X and Y, let $OT(X,Y)$, $OP(X,Y)$ and $OI(X,Y)$ denote respectively the set of all order-preserving transformations, the set of all order-preserving partial transformations and the set of all order-preserving 1-1 partial transformations of X into Y. Let $OT(X), OP(X)$ and $OI(X)$ stand for $OT(X,X), OP(X,X)$ and $OI(X,X)$, respectively. Then $OT(X)$, $OP(X)$ and $OI(X)$ are semigroups under composition. If Y is a nonempty subset of X, then $OT(X,Y)$, $OP(X,Y)$ and $OI(X,Y)$ are subsemigroups of $OT(X)$, $OP(X)$ and $OI(X)$, respectively. For a nonempty subset Y of a poset X, we let $\overline{OT}(X,Y) = \{\alpha \in OT(X) \mid Y\alpha \subseteq Y\}$, $\overline{OP}(X,Y) = \{\alpha \in OP(X) \mid (\operatorname{dom}\alpha \cap Y)\alpha \subseteq Y\}$ and $\overline{OI}(X,Y) = \{\alpha \in OI(X) \mid (\operatorname{dom}\alpha \cap Y)\alpha \subseteq Y\}$. We have that $\overline{OT}(X,Y), \overline{OP}(X,Y)$ and $\overline{OI}(X,Y)$ are subsemigroups of $OT(X), OP(X)$ and $OI(X)$ containing $OT(X,Y)$, $OP(X,Y)$ and $OI(X,Y)$, respectively. The semigroups $OT(X,Y)$ and $\overline{OT}(X,Y)$ can be considered as generalizations of $OT(X)$ and likewise for $OP(X,Y)$, $\overline{OP}(X,Y)$, $OI(X,Y)$ and $\overline{OI}(X,Y)$.

For posets X, Y and $\theta \in OT(Y,X)$, let $(OT(X,Y),\theta)$ be the semigroup $(OT(X,Y),*)$ where $\alpha * \beta = \alpha\theta\beta$ for all α, $\beta \in OT(X,Y)$. The semigroups $(OP(X,Y),\theta)$ where $\theta \in OP(Y,X)$ and $(OI(X,Y),\theta)$ where $\theta \in OI(Y,X)$ are defined similarly. We can see that $(OT(X,Y),\theta)$, $(OP(X,Y),\theta)$ and $(OI(X,Y),\theta)$ are generalizations of $OT(X), OP(X)$ and $OI(X)$, respectively.

It is known that for any chain X, $OP(X)$ and $OI(X)$ are regular semigroups and the regular elements of $OT(X)$ have been characterized. Moreover, if X is a finite chain, the cardinalities of $OT(X), OP(X)$ and $OI(X)$ have been determined. The purpose of this research is to characterize the regular elements of $OT(X,Y)$, $OP(X,Y)$, $OI(X,Y)$, $\overline{OT}(X,Y)$, $\overline{OP}(X,Y)$ and $\overline{OI}(X,Y)$ where X is a chain and $\varnothing \neq Y \subseteq X$, and $(OT(X,Y),\theta)$, $(OP(X,Y),\theta)$ and $(OI(X,Y),\theta)$ where X and Y are any chains. In addition, the regular elements of $OT(X,Y)$, $OP(X,Y)$, $OI(X,Y)$, $\overline{OT}(X,Y)$, $\overline{OP}(X,Y)$ and $\overline{OI}(X,Y)$ are counted when $X = \{1,2,\ldots,n\}$ and $Y = \{1,2,\ldots,m\}$ where $m \leq n$.

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INTRODUCTION

Transformation semigroups play an important role in Semigroup Theory. It is well-known that every semigroup can be embedded in a full transformation semigroup ([8], p. 3 or [11], p. 7). As we know, regularity is a crucial notion in Semigroup Theory. All standard transformation semigroups are regular semigroups. In addition, the full linear transformation semigroup on a vector space and the full $n \times n$ matrix semigroup over a division ring are both regular. Semigroups of order-preserving tranformations have been widely studied. Combinatorial results for them have been of interest in this subject. See [5], [6], [7], [9], [10], [12], [13], [14], [15], [16], [17], [18], [22], [23], [24], [26] and [27] for example. Order-preserving transformation semigroups need not be regular in general. In this research, the regular elements of certain order-preserving transformation semigroups on chains are of our interest to characterize. Combinatorial results on the regular elements of some of these semigroups are also considered.

For a nonempty set X, let $T(X), P(X)$ and $I(X)$ denote respectively the full transformation semigroup on X, the partial transformation semigroup on X and the 1-1 partial transformation semigroup on X, respectively. It is well-known that all the semigroups $T(X), P(X)$ and $I(X)$ are regular ([8], p. 4 or [11], p. 63 and 149). For nonempty sets X and Y, let $T(X,Y)$, $P(X,Y)$ and $I(X,Y)$ be the set of all transformations, the set of all partial transformations and the set of all 1-1 partial transformations of X into Y, respectively. If Y is a nonempty subset of X, then $T(X,Y)$, $P(X,Y)$ and $I(X,Y)$ are clearly subsemigroups of $T(X), P(X)$ and $I(X)$, respectively. For $\varnothing \neq Y \subseteq X$, let $\overline{T}(X,Y) = \{\alpha \in T(X) \mid Y\alpha \subseteq Y\}$, $\overline{P}(X,Y) = \{\alpha \in P(X) \mid (\operatorname{dom}\alpha \cap Y)\alpha \subseteq Y\}$ and $\overline{I}(X,Y) = \{\alpha \in I(X) \mid (\operatorname{dom}\alpha \cap Y)\alpha \subseteq Y\}$. Then $\overline{T}(X,Y)$, $\overline{P}(X,Y)$ and $\overline{I}(X,Y)$ are subsemigroups of $T(X), P(X)$ and $I(X)$ containing $T(X,Y), P(X,Y)$

and $I(X, Y)$, respectively. We may consider $T(X, Y)$ and $\overline{T}(X, Y)$ as generalizations of $T(X)$. The semigroups $P(X, Y)$ and $\overline{P}(X, Y)$ generalize $P(X)$ as well as $I(X, Y)$ and $\overline{I}(X, Y)$ generalize $I(X)$. The semigroup $T(X, Y)$ was introduced and studied by Symons [29] in 1975 while Magill [19] introduced and studied the semigroup $\overline{T}(X, Y)$ in 1966. In [25], the authors characterized the regular elements of the transformation semigroups $T(X, Y)$ and $\overline{T}(X, Y)$. In addition, the number of regular elements of these two sets when X is finite was given in terms of $|X|, |Y|$, and their Stirling numbers of second kind.

Let X and Y be nonempty sets. For $\theta \in T(Y, X)$, let $(T(X, Y), \theta)$ denote the semigroup $(T(X, Y), *)$ where $\alpha * \beta = \alpha\theta\beta$ for all $\alpha, \beta \in T(X, Y)$. The semigroups $(P(X, Y), \theta)$ where $\theta \in P(Y, X)$ and $(I(X, Y), \theta)$ where $\theta \in I(Y, X)$ are defined similarly. These semigroups can be also considered as generalizations of $T(X), P(X)$ and $I(X)$, respectively. They are special cases of generalized partial transformation semigroups introduced by Sullivan [28] in 1975. In 1975, Magill and Subbiah [20] characterized the regular elements of the semigroups $(T(X, Y), \theta)$ and $(P(X, Y), \theta)$. Recently, Chinram [3] considered when the semigroup $(P(X, Y), \theta)$ is regular and gave a characterization of its regular elements in a different version. A characterization of the regularity of the semigroup $(T(X, Y), \theta)$ was given in [2]. The regularity and the regular elements of the semigroups $(I(X, Y), \theta)$ were introduced in [4].

For a partially ordered set X, let $OT(X), OP(X)$ and $OI(X)$ denote the order-preserving full transformation semigroup on X, the order-preserving partial transformation semigroup on X and the order-preserving 1-1 partial transformation semigroup on X, respectively. It is known that $OT(X)$ is a regular semigroup if X is a finite chain ([8], p. 203). Kemprasit and Changphas [14] extended this result by showing that $OT(X)$ is regular for any chain which is order-isomorphic to a subset of $\mathbb{Z}$, the set of integers under the natural order. It was also shown in [14] that for any chain X, $OP(X)$ and $OI(X)$ are regular semigroups. In fact, Kim and Kozhukhov [16] characterized a countable chain X for which $OT(X)$ is a regular semigroup. It was also proved in [14] that if X is an interval in $\mathbb{R}$, the set of real numbers under the usual order, then $OT(X)$ is a regular semigroup if and only if X is closed and bounded. Rungrattrakoon and Kemprasit [26] extended this fact by showing that for a nontrivial interval X in a subfield F of $\mathbb{R}$, $OT(X)$ is regular if and only if $F = \mathbb{R}$ and X is closed and bounded. Then it follows as a direct consequence that for any nontrivial interval X in $\mathbb{Q}$, the set of rational numbers under the usual order, $OT(X)$ is not a regular semigroup. In fact, the result in [26] mentioned above is a consequence of the main theorem in [13]. In [23], the regularity of the semigroup $OT(X)$ was investigated for a certain dictionary chain X and it was studied in [24] for X being an other dictionary chain. In general, $OT(X)$ need not be regular. Then we gave in [22] a characterization determining when an element of $OT(X)$ is regular where X is any chain. In the case of a finite chain X, Howie [10] gave the cardinality of $OT(X)$ and in [7], Howie and Gomes provided the cardinality of $OP(X)$. See also the papers [17] and [18] of Laradji and Umar and the paper [9] of Higgins. The cardinality of $OI(X)$ was first presented by Garba in [6]. It was also given in [5].

Let X and Y be partially ordered sets. Denote by $OT(X,Y), OP(X,Y)$ and $OI(X,Y)$ the set of all order-preserving transformations, the set of all order-preserving partial transformations and the set of all order-preserving 1-1 partial transformations of X into Y, respectively. If Y is a nonempty subset of X, then $OT(X,Y), OP(X,Y)$ and $OI(X,Y)$ are subsemigroups of $OT(X), OP(X)$ and $OI(X)$, respectively. For $\varnothing \neq Y \subseteq X$, let $\overline{OT}(X,Y) = \{\alpha \in OT(X) \mid Y\alpha \subseteq Y\}$, $\overline{OP}(X,Y) = \{\alpha \in OP(X) \mid (\operatorname{dom}\alpha \cap Y)\alpha \subseteq Y\}$ and $\overline{OI}(X,Y) = \{\alpha \in OI(X) \mid (\operatorname{dom}\alpha \cap Y)\alpha \subseteq Y\}$. Then $\overline{OT}(X,Y), \overline{OP}(X,Y)$ and $\overline{OI}(X,Y)$ are subsemigroups of $OT(X)$, $OP(X)$ and $OI(X)$ containing $OT(X,Y), OP(X,Y)$ and $OI(X,Y)$, respectively. Also, we have that $OT(X,Y)$ and $\overline{OT}(X,Y)$ generalize $OT(X)$ and likewise for $OP(X,Y)$, $\overline{OP}(X,Y)$, $OI(X,Y)$ and $\overline{OI}(X,Y)$. The regularity of the semigroups $OT(X,Y), OP(X,Y)$ and $OI(X,Y)$ was studied in [27] where X is a chain.

For any partially ordered sets X, Y and $\theta \in OT(Y,X)$, let $(OT(X,Y),\theta)$ be the semigroup $(OT(X,Y),*)$ where $\alpha * \beta = \alpha\theta\beta$ for all $\alpha, \beta \in OT(X,Y)$. The semigroups $(OP(X,Y),\theta)$ where $\theta \in OP(Y,X)$ and $(OI(X,Y),\theta)$ where $\theta \in OI(Y,X)$ are defined analogously. We also have that $(OT(X,Y),\theta)$, $(OP(X,Y),\theta)$ and $(OI(X,Y),\theta)$ generalize $OT(X)$, $OP(X)$ and $OI(X)$, respectively. In [15], the authors considered when the semigroup $(OT(X,Y),\theta)$ is regular where X and Y are any chains. Also, the regularity of the semigroups $(OP(X,Y),\theta)$ and $(OI(X,Y),\theta)$ was determined in [12] where X and Y are chains.

In this research, we extend above results for order-preserving transformation semigroups. The regular elements of following semigroups are characterized: $OT(X,Y), OP(X,Y)$, $OI(X,Y)$, $\overline{OT}(X,Y), \overline{OP}(X,Y)$ and $\overline{OI}(X,Y)$ where X is a chain and $\varnothing \neq Y \subseteq X$ and $(OT(X,Y),\theta)$, $(OP(X,Y),\theta)$ and $(OI(X,Y),\theta)$ where X and Y are any chains and θ belongs to $OT(Y,X), OP(Y,X)$ and $OI(Y,X)$, respectively. In addition, if $X = \{1,2,\ldots,n\}$ and $Y = \{1,2,\ldots,m\}$ where $m \leq n$, the number of regular elements of the semigroups $OT(X,Y), OP(X,Y)$, $OI(X,Y)$, $\overline{OT}(X,Y), \overline{OP}(X,Y)$ and $\overline{OI}(X,Y)$ is investigated.

This research is organized as follows:

Chapter I contains the basic definitions, notations and quoted results which will be used for this research.

In Chapter II, we give necessary and sufficient conditions for the elements of the semigroups $OT(X,Y), OP(X,Y)$ and $OI(X,Y)$ to be regular when X, Y are chains and $\varnothing \neq Y \subseteq X$. Then these characterizations are applied to prove the above known results concerning the regularity of $OT(X,Y), OP(X,Y)$ and $OI(X,Y)$. In addition, the regular elements of $OT(X,Y), OP(X,Y)$ and $OI(X,Y)$ are counted when $X = \{1,2,\ldots,n\}$ and $Y = \{1,2,\ldots,m\}$ where $m \leq n$.

In Chapter III, necessary and sufficient conditions for the elements of the semigroups $\overline{OT}(X,Y)$, $\overline{OP}(X,Y)$ and $\overline{OI}(X,Y)$ to be regular are provided when X, Y are chains and $\varnothing \neq Y \subseteq X$. These conditions are then applied to determine the regularity of $\overline{OT}(X,Y)$, $\overline{OP}(X,Y)$ and $\overline{OI}(X,Y)$. Moreover, we also provide the number of regular elements in each of the semigroups $\overline{OT}(X,Y)$, $\overline{OP}(X,Y)$ and $\overline{OI}(X,Y)$ when $X = \{1,2,\ldots,n\}$ and $Y = \{1,2,\ldots,m\}$ where $m \leq n$.

Chapter IV contains characterizations of the regular elements of the generalized order-preserving transformation semigroups $(OT(X,Y),\theta)$, $(OP(X,Y),\theta)$ and $(OI(X,Y),\theta)$ where X and Y are any chains. In addition, the regularity of $(OT(X,Y),\theta)$, $(OP(X,Y),\theta)$ and $(OI(X,Y),\theta)$ is determined by making use of our characterizations.

Note that a condition of the regularity of an element in some semigroups of our interest is given in terms of the regularity of an elements in $OT(X)$ where X is a chain. Recall that the regular elements of $OT(X)$ were characterized in [22].

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CHAPTER I
PRELIMINARIES

For a set X, let $|X|$ denote the cardinality of X. The notation $\dot{\cup}$ stands for a disjoint union.

An element a of a semigroup S is said to be *regular* if $a = axa$ for some $x \in S$, and S is called a *regular semigroup* if every element of S is regular. The set of all regular elements of a semigroup S will be denoted by $\mathrm{Reg}(S)$, that is,

$$\mathrm{Reg}(S) = \{a \in S \mid a = axa \text{ for some } x \in S\}.$$

The domain and the range of a mapping α will be denoted by $\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha$, respectively. For an element x in the domain of a mapping α, the image of x under α is written as $x\alpha$. Notice that $\operatorname{dom}\alpha = \dot{\bigcup}_{x \in \operatorname{ran}\alpha} x\alpha^{-1}$. For $A \subseteq \operatorname{dom}\alpha$, denote by $\alpha_{|_A}$ the restriction of α to A. The identity mapping on a nonempty set A is denoted by 1_A. For any mappings α and β, the *composition* $\alpha\beta$ of α and β is defined as follows: $\alpha\beta = 0$ if $\operatorname{ran}\alpha \cap \operatorname{dom}\beta = \varnothing$, otherwise, $\alpha\beta$ is the usual composition of the mappings $\alpha_{|_{(\operatorname{ran}\alpha \cap \operatorname{dom}\beta)\alpha^{-1}}}$ and $\beta_{|_{(\operatorname{ran}\alpha \cap \operatorname{dom}\beta)}}$ where 0 is the empty transformation, that is, the mapping with empty domain. Then for any mappings α, β and γ, we have

$$\begin{aligned} \operatorname{dom}(\alpha\beta) &= (\operatorname{ran}\alpha \cap \operatorname{dom}\beta)\alpha^{-1} \subseteq \operatorname{dom}\alpha, \\ \operatorname{ran}(\alpha\beta) &= (\operatorname{ran}\alpha \cap \operatorname{dom}\beta)\beta \subseteq \operatorname{ran}\beta, \\ \text{for } x \in X,\ \ x \in \operatorname{dom}(\alpha\beta) &\Leftrightarrow x \in \operatorname{dom}\alpha \text{ and } x\alpha \in \operatorname{dom}\beta, \\ (\alpha\beta)\gamma &= \alpha(\beta\gamma). \end{aligned}$$

Let X be a nonempty set. We call a mapping α from X into itself a *transformation* of X. By a *partial transformation* of X we mean a mapping from a subset of X into X. Then the empty transformation 0 is a partial transformation of X.

Let $T(X)$, $P(X)$ and $I(X)$ denote the set of all transformations of X, the set of all partial transformations of X and the set of all 1-1 partial transformations of X, respectively, that is,

$$T(X) = \{\alpha \mid \alpha : X \to X\},$$
$$P(X) = \{\alpha : A \to X \mid A \subseteq X\},$$
$$I(X) = \{\alpha \in P(X) \mid \alpha \text{ is 1-1}\}.$$

We can see that all of $T(X), P(X)$ and $I(X)$ contain 1_X, 0 is contained in $P(X)$ and $I(X)$ but not in $T(X)$ and $T(X)$, and $I(X)$ are subsets of $P(X)$. Therefore, under the composition of mappings, $P(X)$ is a semigroup having $T(X)$ and $I(X)$ as its subsemigroups. The semigroups $T(X), P(X)$ and $I(X)$ are called the *full transformation semigroup* on X, the *partial transformation semigroup* on X and the *1-1 partial transformation semigroup* or the *symmetric inverse semigroup* on X, respectively. By a *transformation semigroup* on X we mean a subsemigroup of $P(X)$. It is well-known that all the semigroups $P(X), T(X)$ and $I(X)$ are regular for every set X ([8], p. 4 or [11], p. 63 and 149).

For convenience, we sometimes write a mapping by using a bracket notation. For example,

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ stands for the mapping α with $\operatorname{dom}\alpha = \{a, b\}$, $\operatorname{ran}\alpha = \{c, d\}$, $a\alpha = c$ and $b\alpha = d$,

$\begin{pmatrix} A & x \\ a & x' \end{pmatrix}_{x \in X \setminus A}$ stands for the mapping β with $\operatorname{dom}\beta = X$,

$$\operatorname{ran}\beta = \{a\} \cup \{x' \mid x \in X \setminus A\} \text{ and } x\alpha = \begin{cases} a & \text{if } x \in A, \\ x' & \text{if } x \in X \setminus A. \end{cases}$$

By the above notations, a mapping α can be written as $\alpha = \begin{pmatrix} x\alpha^{-1} \\ x \end{pmatrix}_{x \in \operatorname{ran}\alpha}$.

For nonempty sets X and Y, let

$$T(X,Y) = \{\alpha \mid \alpha : X \to Y\},$$
$$P(X,Y) = \{\alpha : A \to Y \mid A \subseteq X\},$$
$$I(X,Y) = \{\alpha \in P(X,Y) \mid \alpha \text{ is } 1-1\}.$$

Notice that $T(X,X) = T(X), P(X,X) = P(X)$ and $I(X,X) = I(X)$. If Y is a nonempty subset of X, then

$$T(X,Y) = \{\alpha \in T(X) \mid \operatorname{ran}\alpha \subseteq Y\},$$
$$P(X,Y) = \{\alpha \in P(X) \mid \operatorname{ran}\alpha \subseteq Y\},$$
$$I(X,Y) = \{\alpha \in I(X) \mid \operatorname{ran}\alpha \subseteq Y\}$$

which are clearly subsemigroups of $T(X), P(X)$ and $I(X)$, respectively.

For $\varnothing \neq Y \subseteq X$, let

$$\overline{T}(X,Y) = \{\alpha \in T(X) \mid Y\alpha \subseteq Y\},$$
$$\overline{P}(X,Y) = \{\alpha \in P(X) \mid (\operatorname{dom}\alpha \cap Y)\alpha \subseteq Y\},$$
$$\overline{I}(X,Y) = \{\alpha \in I(X) \mid (\operatorname{dom}\alpha \cap Y)\alpha \subseteq Y\}.$$

Then $T(X,Y) \subseteq \overline{T}(X,Y), P(X,Y) \subseteq \overline{P}(X,Y)$ and $I(X,Y) \subseteq \overline{I}(X,Y)$. Also, $\overline{T}(X,Y)$, $\overline{P}(X,Y)$ and $\overline{I}(X,Y)$ are subsemigroups of $T(X)$, $P(X)$ and $I(X)$, respectively. Notice that $1_X \in \overline{T}(X,Y)$ but $1_X \notin T(X,Y)$ if $Y \subsetneq X$. The semigroups $\overline{T}(X,Y)$ and $T(X,Y)$ were introduced and studied by Magill [19] in 1966 and Symons [29] in 1975, respectively. We observe that $T(X,X) = \overline{T}(X,X) = T(X)$, $P(X,X) = \overline{P}(X,X) = P(X)$ and $I(X,X) = \overline{I}(X,X) = I(X)$.

The characterizations of the regular elements in $T(X,Y)$ and $\overline{T}(X,Y)$ are respectively given as follows:

Theorem 1.1. ([25]) *Let X be a nonempty set, $\varnothing \neq Y \subseteq X$ and $\alpha \in T(X,Y)$. Then $\alpha \in \operatorname{Reg}(T(X,Y))$ if and only if $\operatorname{ran}\alpha = Y\alpha$.*

Theorem 1.2. ([25]) *Let X be a nonempty set, $\varnothing \neq Y \subseteq X$ and $\alpha \in \overline{T}(X,Y)$. Then $\alpha \in \operatorname{Reg}(\overline{T}(X,Y))$ if and only if $\operatorname{ran}\alpha \cap Y = Y\alpha$.*

Next, let X and Y be any nonempty sets. Let $S(X,Y)$ be $T(X,Y), P(X,Y)$ or $I(X,Y)$. For $\theta \in S(Y,X)$, we define a *sandwich* operation $*$ on $S(X,Y)$ by

$$\alpha * \beta \ = \ \alpha\theta\beta \quad \text{for all } \alpha, \beta \in S(X,Y).$$

Then $(S(X,Y),*)$ is a semigroup which we denote by $(S(X,Y),\theta)$. The semigroups $(T(X,Y),\theta)$, $(P(X,Y),\theta)$ and $(I(X,Y),\theta)$ are called the *generalized full transformation semigroup*, the *generalized partial transformation semigroup* and the *generalized 1-1 partial transformation semigroup* of X into Y induced by θ, respectively. Generalized partial transformation semigroups introduced by Sullivan [28] in 1975 have these semigroups as special cases. In particular, $(T(X,X),1_X)$, $(P(X,X),1_X)$ and $(I(X,X),1_X)$ are respectively the semigroups $T(X), P(X)$ and $I(X)$.

Example 1.3. ([12]) Let X and Y be nonempty sets and $a \in X$. Then $\big(T(X,Y), \binom{Y}{a}\big)$ is the semigroup $T(X,Y)$ with the operation $*$ defined by

$$\alpha * \beta = \alpha \begin{pmatrix} Y \\ a \end{pmatrix} \beta = \begin{pmatrix} X \\ a\beta \end{pmatrix} \quad \text{for all } \alpha, \beta \in T(X,Y).$$

Also, $\big(P(X,Y), \binom{Y}{a}\big)$ is the semigroup $P(X,Y)$ with the operation $\circ$ defined by

$$\alpha \circ \beta = \alpha \begin{pmatrix} Y \\ a \end{pmatrix} \beta = \begin{cases} \begin{pmatrix} \operatorname{dom}\alpha \\ a\beta \end{pmatrix} & \text{if } \alpha \neq 0 \text{ and } a \in \operatorname{dom}\beta, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, for $b \in Y$, the semigroup $\big(I(X,Y), \binom{b}{a}\big)$ is the semigroup $(I(X,Y), \bullet)$ where

$$\alpha \bullet \beta = \alpha \begin{pmatrix} b \\ a \end{pmatrix} \beta = \begin{cases} \begin{pmatrix} b\alpha^{-1} \\ a\beta \end{pmatrix} & \text{if } b \in \operatorname{ran}\alpha \text{ and } a \in \operatorname{dom}\beta, \\ 0 & \text{otherwise.} \end{cases}$$

For a nonempty subset A of a partially ordered set (poset) X, we let $\max(A)$ and $\min(A)$ denote the maximum and the minimum of A, respectively if they exist. Also, for nonempty subsets A and B of X, let $A < B$ mean that $a < b$

for all $a \in A$ and $b \in B$. For $x \in X$, let $x < A$ stand for $\{x\} < A$. We define $A > B, A \leq B, A \geq B, x > A, x \leq A$ and $x \geq A$ analogously. The set of all upper bounds of A in X and the set of all lower bounds of A in X are denoted by $\mathrm{ub}(A)$ and $\mathrm{lb}(A)$, respectively. Notice that $x \in \mathrm{ub}(A)$ if and only if $x \geq A$, and $x \in \mathrm{lb}(A)$ if and only if $x \leq A$.

Let X and Y be partially ordered sets. For $\alpha \in P(X, Y)$, α is said to be *order-preserving* if

$$\text{for any } x_1, x_2 \in \operatorname{dom}\alpha, \quad x_1 \leq x_2 \text{ in } X \;\Rightarrow\; x_1\alpha \leq x_2\alpha \text{ in } Y.$$

A bijection $\varphi : X \to Y$ is called an *order-isomorphism* if φ and φ^{-1} are order-preserving. It is clear that if both X and Y are chains and $\varphi : X \to Y$ is an order-preserving bijection, then φ is an order-isomorphism from X onto Y. We say that X and Y are *order-isomorphic* if there is an order-isomorphism from X onto Y.

A transformation semigroup on a partially ordered set X is said to be an *order-preserving transformation semigroup* on X if all of its elements are order-preserving. Let

$$OT(X) = \{\alpha \in T(X) \mid \alpha \text{ is order-preserving}\},$$

$$OP(X) = \{\alpha \in P(X) \mid \alpha \text{ is order-preserving}\},$$

$$OI(X) = \{\alpha \in I(X) \mid \alpha \text{ is order-preserving}\}.$$

Then $OT(X), OP(X)$ and $OI(X)$ are respectively subsemigroups of $T(X), P(X)$ and $I(X)$. Observe that 0 and 1_X belong to $OP(X)$ and $OI(X)$ and $1_X \in OT(X)$. The semigroups $OT(X), OP(X)$ and $OI(X)$ are called the *order-preserving full transformation semigroup* on X, the *order-preserving partial transformation semigroup* on X and the *order-preserving 1-1 partial transformation semigroup* on X, respectively.

The following results for the semigroups $OT(X)$, $OP(X)$ and $OI(X)$ are known.

Theorem 1.4. ([8], p.203) *If X is a finite chain, then $OT(X)$ is a regular semigroup.*

Theorem 1.5. ([14]) *For any chain X, the semigroups $OP(X)$ and $OI(X)$ are regular.*

A characterization determining when an element of $OT(X)$ is regular where X is a chain was given in [22] as follows:

Theorem 1.6. ([22]) *Let X be a chain and $\alpha \in OT(X)$. Then $\alpha \in \mathrm{Reg}(OT(X))$ if and only if the following three conditions hold.*

(i) *If* $\mathrm{ub}(\mathrm{ran}\,\alpha) \neq \varnothing$, *then* $\max(\mathrm{ran}\,\alpha)$ *exists.*

(ii) *If* $\mathrm{lb}(\mathrm{ran}\,\alpha) \neq \varnothing$, *then* $\min(\mathrm{ran}\,\alpha)$ *exists.*

(iii) *If* $x \in X \smallsetminus \big(\mathrm{ran}\,\alpha \cup \mathrm{ub}(\mathrm{ran}\,\alpha) \cup \mathrm{lb}(\mathrm{ran}\,\alpha)\big)$, *then* $\max(\{t \in \mathrm{ran}\,\alpha \mid t < x\})$ *exists or* $\min(\{t \in \mathrm{ran}\,\alpha \mid t > x\})$ *exists.*

The following corollary is a direct consequence of Theorem 1.6.

Corollary 1.7. *Let $\alpha \in OT(X)$. If $\mathrm{ran}\,\alpha$ is finite, then $\alpha \in \mathrm{Reg}(OT(X))$.*

Notice that Corollary 1.7 is a generalization of Theorem 1.4.

In the case that X is a finite chain, the cardinalities of $OT(X), OP(X)$ and $OI(X)$ were given as follows:

Theorem 1.8. ([9], [10], [18]) *If X is a finite chain of n elements, then*

$$|\,OT(X)| = \binom{2n-1}{n-1} = \binom{2n-1}{n}.$$

Theorem 1.9. ([7], [17]) *If X is a finite chain of n elements, then*

$$|\,OP(X)| = \sum_{r=0}^{n} \binom{n}{r}\binom{n+r-1}{r}.$$

Theorem 1.10. ([5], [6]) *If X is a finite chain of n elements, then*

$$|\,OI(X)| = \sum_{r=0}^{n} \binom{n}{r}^2 = \binom{2n}{n}.$$

For partially ordered sets X and Y, let

$$OT(X,Y) = \{\alpha \in T(X,Y) \mid \alpha \text{ is order-preserving}\},$$
$$OP(X,Y) = \{\alpha \in P(X,Y) \mid \alpha \text{ is order-preserving}\},$$
$$OI(X,Y) = \{\alpha \in I(X,Y) \mid \alpha \text{ is order-preserving}\}.$$

Proposition 1.11. *Let X and Y be chains. If $\alpha \in OP(X,Y)$ and $a, b \in \operatorname{ran}\alpha$ are such that $a < b$ in Y, then $a\alpha^{-1} < b\alpha^{-1}$ in X.*

Proof. Let $x \in a\alpha^{-1}$ and $y \in b\alpha^{-1}$. Then $x\alpha = a$ and $y\alpha = b$. Since X is a chain, $x < y$ or $x \geq y$. If $x \geq y$, then $x\alpha \geq y\alpha$ since α is order-preserving. This implies that $a \geq b$, a contradiction. Hence $x < y$. $\square$

If α and β are mappings with disjoint domains, we define the mapping $\alpha \cup \beta$ as follows: $\operatorname{dom}(\alpha \cup \beta) = \operatorname{dom}\alpha \cup \operatorname{dom}\beta$, $(\alpha \cup \beta)_{|\operatorname{dom}\alpha} = \alpha$ and $(\alpha \cup \beta)_{|\operatorname{dom}\beta} = \beta$.

The following facts are clearly seen.

Proposition 1.12. *Let X and Y be partially ordered sets and $\alpha \in OP(X,Y)$. If $\operatorname{dom}\alpha = A \dot{\cup} B$, then $\alpha_{|A} \in OT(A,Y), \alpha_{|B} \in OT(B,Y)$ and $\alpha = \alpha_{|A} \cup \alpha_{|B}$.*

Proposition 1.13. *Let X and Y be chains. If $\alpha, \beta \in OP(X,Y)$ are such that $\operatorname{dom}\alpha < \operatorname{dom}\beta$ and $\operatorname{ran}\alpha \leq \operatorname{ran}\beta$, then $\alpha \cup \beta \in OP(X,Y)$.*

If Y is a nonempty subset of a partially ordered set X, then

$$OT(X,Y) = \{\alpha \in OT(X) \mid \operatorname{ran}\alpha \subseteq Y\},$$
$$OP(X,Y) = \{\alpha \in OP(X) \mid \operatorname{ran}\alpha \subseteq Y\},$$
$$OI(X,Y) = \{\alpha \in OI(X) \mid \operatorname{ran}\alpha \subseteq Y\}.$$

It is easy to see that $OT(X,Y), OP(X,Y)$ and $OI(X,Y)$ are subsemigroups of $OT(X), OP(X)$ and $OI(X)$, respectively.

Due to the semigroup $\overline{T}(X,Y)$ introduced by Magill [19] and those $\overline{P}(X,Y)$ and $\overline{I}(X,Y)$ mentioned previously for a set X and $\varnothing \neq Y \subseteq X$, the following order-preserving transformation semigroups are defined for a partially ordered set

X and $\varnothing \neq Y \subseteq X$ analogously as follows:

$$\overline{OT}(X,Y) = \{\alpha \in OT(X) \mid Y\alpha \subseteq Y\},$$
$$\overline{OP}(X,Y) = \{\alpha \in OP(X) \mid (\operatorname{dom}\alpha \cap Y)\alpha \subseteq Y\},$$
$$\overline{OI}(X,Y) = \{\alpha \in OI(X) \mid (\operatorname{dom}\alpha \cap Y)\alpha \subseteq Y\}.$$

Then

$$OT(X,Y) \subseteq \overline{OT}(X,Y) \subseteq OT(X), \quad OP(X,Y) \subseteq \overline{OP}(X,Y) \subseteq OP(X),$$
$$OI(X,Y) \subseteq \overline{OI}(X,Y) \subseteq OI(X), \quad OT(X,X) = \overline{OT}(X,X) = OT(X),$$
$$OP(X,X) = \overline{OP}(X,X) = OP(X), OI(X,X) = \overline{OI}(X,X) = OI(X)$$

and 0 belongs to all of the semigroups $OP(X,Y)$, $\overline{OP}(X,Y)$, $OI(X,Y)$ and $\overline{OI}(X,Y)$.

The regularity of $OT(X,Y), OP(X,Y)$ and $OI(X,Y)$ where X is a chain and $\varnothing \neq Y \subseteq X$ was studied in [27].

Theorem 1.14. ([27]) *Let X be a chain and $\varnothing \neq Y \subseteq X$. Then $OT(X,Y)$ is a regular semigroup if and only if one of the following statements holds.*

(i) *$Y = X$ and $OT(X)$ is a regular semigroup.*

(ii) $|Y| = 1$.

(iii) $|Y| = 2$, *$\min(X)$ and $\max(X)$ exist, and $Y = \{\min(X), \max(X)\}$.*

Theorem 1.15. ([27]) *Let X be a chain and $\varnothing \neq Y \subseteq X$.*

(i) *$OP(X,Y)$ is a regular semigroup if and only if $Y = X$.*

(ii) *$OI(X,Y)$ is a regular semigroup if and only if $Y = X$.*

Next, let X and Y be any partially ordered sets. For $\theta \in OT(Y,X)$, let $(OT(X,Y),\theta)$ denote the semigroup $OT(X,Y)$ under the sandwich operation determined by θ and likewise for $(OP(X,Y),\theta)$ with $\theta \in OP(Y,X)$ and $(OI(X,Y),\theta)$ with $\theta \in OI(Y,X)$. We call the semigroups $(OT(X,Y),\theta)$, $(OP(X,Y),\theta)$ and $(OI(X,Y),\theta)$ the *generalized order-preserving full transformation semigroup*, the *generalized order-preserving partial transformation semigroup* and the *generalized order-preserving 1-1 partial transformation semigroup* of X into Y induced by θ, respectively. As before, $(OT(X,X),1_X)$, $(OP(X,X),1_X)$ and $(OI(X,X),1_X)$ are respectively the semigroups $OT(X)$, $OP(X)$ and $OI(X)$.

The following theorem provided in [15] can be considered as a generalization of Theorem 1.14.

Theorem 1.16. ([15]) *Let X,Y be any chains and $\theta \in OT(Y,X)$. Then the semigroup $(OT(X,Y),\theta)$ is regular if and only if one of the following statements holds.*

(i) *The semigroup $OT(X)$ is regular and θ is an order-isomorphism from Y onto X.*

(ii) $|X| = 1$.

(iii) $|Y| = 1$.

(iv) $|Y| = 2$, $\min(X)$ *and* $\max(X)$ *exist, and* $\operatorname{ran}\theta = \{\min(X), \max(X)\}$.

The following two theorems given in [12] can be also considered as generalizations of Theorem 1.15(i) and Theorem 1.15(ii), respectively.

Theorem 1.17. ([12]) *Let X and Y be chains. For $\theta \in OP(Y,X)$, the semigroup $(OP(X,Y),\theta)$ is regular if and only if*

(i) *θ is an order-isomorphism from Y onto X or*

(ii) $\operatorname{dom}\theta = Y$, $\operatorname{ran}\theta = X$ *and* $|X| = 1$.

Theorem 1.18. ([12]) *Let X and Y be chains. For $\theta \in OI(Y,X)$, the semigroup $(OI(X,Y),\theta)$ is regular if and only if θ is an order-isomorphism from Y onto X.*

Recall that for nonnegative integers n and r, $\binom{n}{r} = 0$ if $r > n$.

To count the regular elements of the semigroups $OT(X,Y)$, $OP(X,Y)$, $OI(X,Y)$, $\overline{OT}(X,Y)$, $\overline{OP}(X,Y)$ and $\overline{OI}(X,Y)$ when $X = \{1,2,\ldots,n\}$ and $Y = \{1,2,\ldots,m\}$ where $m \le n$, the following proposition will be used. It is obtained from some combinatorial ideas given in [7].

Proposition 1.19. ([7]) *Let X and Y be finite chains. If $|X| = n$ and $|Y| = r$, then the number of all order-preserving transformations from X onto Y is $\binom{n-1}{r-1}$.*

Moreover, the following standard combinatorial results are also used for our counting.

Result 1.20. ([9]) *For all natural numbers m and n with $n \le m$,*

$$\sum_{k=0}^{n} \binom{n}{k}\binom{m}{k} = \binom{n+m}{m}.$$

Result 1.21. ([17]) *For all natural numbers n and r,*

$$\sum_{k=1}^{n} \binom{k+r-2}{k-1} = \binom{n+r-1}{n-1} = \binom{n+r-1}{r}.$$

Result 1.22. ([18]) *For all natural numbers n and r,*

$$\sum_{k=r}^{n} \binom{k-1}{r-1} = \binom{n}{r}.$$

Result 1.23. ([21], p.68) *For every natural number n,*

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n.$$

Result 1.24. ([21], p.53) *For all natural numbers m and n with $n \le m$,*

$$\sum_{r=1}^{n} \binom{m}{r}\binom{n-1}{r-1} = \binom{m+n-1}{n}.$$

Result 1.25. ([1], p.42) *For all natural numbers m, p and q,*

$$\sum_{k=0}^{m} \binom{p}{k}\binom{q}{m-k} = \binom{p+q}{m}.$$

Result 1.25 yields the following result.

Result 1.26. *For all natural numbers m and n,*

$$\sum_{r=1}^{m} \binom{m}{r}\binom{n-1}{r-1} = \binom{m+n-1}{n}.$$

Proof. For all natural numbers m and n, we have

$$\begin{aligned}
\sum_{r=1}^{m}\binom{m}{r}\binom{n-1}{r-1} &= \sum_{k=0}^{m-1}\binom{m}{k+1}\binom{n-1}{k} \\
&= \sum_{k=0}^{m-1}\binom{m}{m-(k+1)}\binom{n-1}{k} \\
&= \sum_{k=0}^{m-1}\binom{n-1}{k}\binom{m}{(m-1)-k} \\
&= \binom{m+n-1}{m-1} \qquad \text{by Result 1.25} \\
&= \binom{m+n-1}{n}.
\end{aligned}$$

□

The following result is a direct consequence of Result 1.24 and Result 1.26.

Result 1.27. *For all natural numbers m and n,*

$$\sum_{r=1}^{\min\{m,n\}}\binom{m}{r}\binom{n-1}{r-1} = \binom{m+n-1}{n}.$$

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CHAPTER II
SEMIGROUPS OF ORDER-PRESERVING TRANSFORMATIONS WITH RESTRICTED RANGE

The purpose of this chapter is to characterize the regular elements of the semigroups $OT(X,Y), OP(X,Y)$ and $OI(X,Y)$ where X is a chain and $\varnothing \neq Y \subseteq X$. These results are then applied to prove Theorem 1.14 and Theorem 1.15, respectively. In addition, the number of regular elements in each of the semigroups $OT(X,Y), OP(X,Y)$ and $OI(X,Y)$ is provided when $X = \{1,2,\ldots,n\}$ and $Y = \{1,2,\ldots,m\}$ where $m \leq n$. First, we recall that

$$OT(X,Y) = \{\alpha \in OT(X) \mid \operatorname{ran}\alpha \subseteq Y\},$$
$$OP(X,Y) = \{\alpha \in OP(X) \mid \operatorname{ran}\alpha \subseteq Y\},$$
$$OI(X,Y) = \{\alpha \in OI(X) \mid \operatorname{ran}\alpha \subseteq Y\}.$$

2.1 Characterizations of Regular Elements

Throughout this section, X denotes a chain and $\varnothing \neq Y \subseteq X$.

We begin this section by characterizing the regular elements of the semigroup $OT(X,Y)$. Recall that the regular elements of the semigroups $T(X,Y)$ and $OT(X)$ are introduced in Theorem 1.1 and Theorem 1.6, respectively.

Theorem 2.1.1. *For* $\alpha \in OT(X,Y)$, $\alpha \in \operatorname{Reg}(OT(X,Y))$ *if and only if* $\alpha \in \operatorname{Reg}(T(X,Y))$ *and* $\alpha \in \operatorname{Reg}(OT(X))$. *Consequently,*

$$\operatorname{Reg}(OT(X,Y)) = \operatorname{Reg}(T(X,Y)) \cap \operatorname{Reg}(OT(X)).$$

Proof. Assume that $\alpha \in \operatorname{Reg}(OT(X,Y))$. Since $OT(X,Y)$ is a subsemigroup of $T(X,Y)$ and $OT(X)$, it follows that α is regular in $T(X,Y)$ and $OT(X)$, i.e.,

$\alpha \in \operatorname{Reg}(T(X,Y))$ and $\alpha \in \operatorname{Reg}(OT(X))$.

For the converse, assume that $\alpha \in \operatorname{Reg}(T(X,Y))$ and $\alpha \in \operatorname{Reg}(OT(X))$. By Theorem 1.1, $\operatorname{ran}\alpha = Y\alpha$ or equivalently, $x\alpha^{-1} \cap Y \neq \varnothing$ for all $x \in \operatorname{ran}\alpha$. For each $x \in \operatorname{ran}\alpha$, choose $y_x \in x\alpha^{-1} \cap Y$. Then $y_x\alpha = x$ for all $x \in \operatorname{ran}\alpha$. Let $\beta \in OT(X)$ be such that $\alpha = \alpha\beta\alpha$. Then $X\alpha = X\alpha\beta\alpha \subseteq X\beta\alpha \subseteq X\alpha = \operatorname{ran}\alpha$. It follows that $\operatorname{ran}\alpha = \operatorname{ran}(\beta\alpha)$. Thus $X = \dot{\bigcup\limits_{x \in \operatorname{ran}(\beta\alpha)}} x(\beta\alpha)^{-1} = \dot{\bigcup\limits_{x \in \operatorname{ran}\alpha}} x(\beta\alpha)^{-1}$. Define $\beta' : X \to Y$ by a bracket notation as follows:

$$\beta' = \begin{pmatrix} x(\beta\alpha)^{-1} \\ y_x \end{pmatrix}_{x \in \operatorname{ran}\alpha}.$$

If $x \in X$, then $x\alpha = (x\alpha)\beta\alpha$, so $x\alpha \in (x\alpha)(\beta\alpha)^{-1}$ which implies that $x\alpha\beta'\alpha = y_{x\alpha}\alpha = x\alpha$. Hence $\alpha = \alpha\beta'\alpha$. To show that β' is order-preserving, let $x_1, x_2 \in X$ be such that $x_1 < x_2$. Then $x_1\beta\alpha \leq x_2\beta\alpha$. If $x_1\beta\alpha = x_2\beta\alpha$, then $x_1, x_2 \in (x_1\beta\alpha)(\beta\alpha)^{-1}$, so $x_1\beta' = y_{x_1\beta\alpha} = x_2\beta'$. If $x_1\beta\alpha < x_2\beta\alpha$, then by Proposition 1.11, $(x_1\beta\alpha)\alpha^{-1} < (x_2\beta\alpha)\alpha^{-1}$. It follows that $y_{x_1\beta\alpha} < y_{x_2\beta\alpha}$. Since $((x_1\beta\alpha)(\beta\alpha)^{-1})\beta' = \{y_{x_1\beta\alpha}\}$ and $((x_2\beta\alpha)(\beta\alpha)^{-1})\beta' = \{y_{x_2\beta\alpha}\}$, we have that $x_1\beta' = y_{x_1\beta\alpha} < y_{x_2\beta\alpha} = x_2\beta'$.

The proof is thereby complete. □

The following theorem is another version of Theorem 2.1.1. It follows directly from Theorem 1.1, Theorem 1.6 and Theorem 2.1.1.

Theorem 2.1.2. *For* $\alpha \in OT(X,Y)$, $\alpha \in \operatorname{Reg}(OT(X,Y))$ *if and only if the following four conditions hold.*

(i) $\operatorname{ran}\alpha = Y\alpha$.

(ii) *If* $\operatorname{ub}(\operatorname{ran}\alpha) \neq \varnothing$, *then* $\max(\operatorname{ran}\alpha)$ *exists.*

(iii) *If* $\operatorname{lb}(\operatorname{ran}\alpha) \neq \varnothing$, *then* $\min(\operatorname{ran}\alpha)$ *exists.*

(iv) *If* $x \in X \smallsetminus \big(\operatorname{ran}\alpha \cup \operatorname{ub}(\operatorname{ran}\alpha) \cup \operatorname{lb}(\operatorname{ran}\alpha)\big)$, *then* $\max(\{t \in \operatorname{ran}\alpha \mid t < x\})$ *exists or* $\min(\{t \in \operatorname{ran}\alpha \mid t > x\})$ *exists.*

The next result follows directly from Theorem 1.1, Corollary 1.7 and Theorem 2.1.1.

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Corollary 2.1.3. *Let* $\alpha \in OT(X,Y)$*. If* $\operatorname{ran}\alpha$ *is finite, then* $\alpha \in \operatorname{Reg}(OT(X,Y))$ *if and only if* $\operatorname{ran}\alpha = Y\alpha$*.*

Example 2.1.4. (1) Let $X = \mathbb{R}$ and $Y = (-2,2)$. Define $\alpha : X \to Y$ by

$$x\alpha = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

Then $\alpha \in OT(X,Y), \operatorname{ran}\alpha = \{0,1\}$ and $Y\alpha = \{0,1\}$. By Corollary 2.1.3, $\alpha \in \operatorname{Reg}(OT(X,Y))$. Let $Y' = [0,2)$. Then $\alpha \in OT(X,Y')$ and $Y'\alpha = \{1\}$, so $\alpha \notin \operatorname{Reg}(OT(X,Y'))$ by Corollary 2.1.3.

(2) Let $X = \mathbb{R}$ and $Y = [0,\infty)$. Define $\beta : X \to Y$ by

$$x\beta = \begin{cases} \dfrac{x}{x+1} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Then $\beta \in OT(X,Y)$ and $\operatorname{ran}\beta = [0,1)$. Since $\operatorname{ran}\beta$ has an upper bound in X but $\operatorname{ran}\beta$ has no maximum, by Theorem 2.1.2, $\beta \notin \operatorname{Reg}(OT(X,Y))$.

(3) Let $X = Y = [0,1) \cup (1,2]$. Define $\lambda : X \to Y$ by

$$x\lambda = \begin{cases} \dfrac{x}{3} & \text{if } x \in [0,1), \\ x & \text{if } x \in (1,2]. \end{cases}$$

Then $\lambda \in OT(X,Y)$ and $\operatorname{ran}\lambda = [0,\frac{1}{3}) \cup (1,2]$. Since $\frac{2}{3} \in X \smallsetminus \big(\operatorname{ran}\lambda \cup \operatorname{ub}(\operatorname{ran}\lambda) \cup \operatorname{lb}(\operatorname{ran}\lambda)\big)$,

$$\{x \in \operatorname{ran}\lambda \mid x < \frac{2}{3}\} = [0,\frac{1}{3})$$

and

$$\{x \in \operatorname{ran}\lambda \mid x > \frac{2}{3}\} = (1,2],$$

it follows that $\{x \in \operatorname{ran}\lambda \mid x < \frac{2}{3}\}$ has no maximum and $\{x \in \operatorname{ran}\lambda \mid x > \frac{2}{3}\}$ has no minimum. By Theorem 2.1.2, $\lambda \notin \operatorname{Reg}(OT(X,Y))$.

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Next, we shall apply Theorem 2.1.2 to prove Theorem 1.14 given in [27]. The following series of lemmas is needed.

Lemma 2.1.5. *Let* $|Y| \geq 2$. *If there is an element* $a \in X$ *such that* $a > Y$ *or* $a < Y$, *then the semigroup* $OT(X, Y)$ *is not regular.*

Proof. Let $e, f \in Y$ be such that $e < f$. Define $\alpha : X \to Y$ by

$$\alpha = \begin{pmatrix} u & v \\ e & f \end{pmatrix}_{\substack{u < a \\ v \geq a}} \quad \text{if } a > Y \quad \text{and} \quad \alpha = \begin{pmatrix} u & v \\ e & f \end{pmatrix}_{\substack{u \leq a \\ v > a}} \quad \text{if } a < Y.$$

Then $\alpha \in OT(X, Y), \operatorname{ran}\alpha = \{e, f\}, Y\alpha = \{e\}$ for $a > Y$ and $Y\alpha = \{f\}$ for $a < Y$. By Corollary 2.1.3, $\alpha \notin \operatorname{Reg}(OT(X, Y))$. Hence $OT(X, Y)$ is not a regular semigroup. □

Lemma 2.1.6. *If* $Y \subsetneq X$ *and* $|Y| \geq 3$, *then* $OT(X, Y)$ *is not a regular semigroup.*

Proof. Let $e, f, g \in Y$ be such that $e < f < g$ and let $a \in X \smallsetminus Y$. If $a > Y$ or $a < Y$, then by Lemma 2.1.5, $OT(X, Y)$ is not regular. Assume that $a \not> Y$ and $a \not< Y$. Then $\{t \in Y \mid t < a\}$ and $\{t \in Y \mid t > a\}$ are nonempty. Define $\alpha : X \to Y$ by

$$\alpha = \begin{pmatrix} u & a & v \\ e & f & g \end{pmatrix}_{\substack{u < a \\ v > a}}.$$

Then $\alpha \in OT(X, Y)$, $\operatorname{ran}\alpha = \{e, f, g\}$ and $Y\alpha = \{e, g\}$. It follows from Corollary 2.1.3 that $\alpha \notin \operatorname{Reg}(OT(X, Y))$ and we conclude that $OT(X, Y)$ is not a regular semigroup. □

Lemma 2.1.7. *Let* $|Y| = 2$. *Then* $OT(X, Y)$ *is a regular semigroup if and only if* $\min(X)$ *and* $\max(X)$ *exist, and* $Y = \{\min(X), \max(X)\}$.

Proof. Let $Y = \{e, f\}$ be such that $e < f$. Assume that $OT(X, Y)$ is regular. Then by Lemma 2.1.5, for every $a \in X$, $a \not> Y$ and $a \not< Y$. Thus $e \leq a \leq f$ for all $a \in X$. This implies that $e = \min(X)$ and $f = \max(X)$.

Conversely, assume that $\min(X)$ and $\max(X)$ exist, $e = \min(X)$ and $f =$

$\max(X)$. Let $\alpha \in OT(X,Y)$. If $|\operatorname{ran}\alpha| = 1$, then $\alpha^2 = \alpha$, so $\alpha \in \operatorname{Reg}(OT(X,Y))$. If $\operatorname{ran}\alpha = \{e, f\}$, then $e\alpha = e$ and $f\alpha = f$ since α is order-preserving. Thus $\operatorname{ran}\alpha = Y\alpha$, so Corollary 2.1.3 implies that $\alpha \in \operatorname{Reg}(OT(X,Y))$. □

Theorem 2.1.8. *The semigroup $OT(X,Y)$ is regular if and only if one of the following statements holds.*

(i) $Y = X$ *and* $OT(X)$ *is a regular semigroup.*

(ii) $|Y| = 1$.

(iii) $|Y| = 2$, $\min(X)$ *and* $\max(X)$ *exist, and* $Y = \{\min(X), \max(X)\}$.

Proof. Assume that $OT(X,Y)$ is regular and suppose that (i) and (ii) are false. Then ($Y \subsetneq X$ or $OT(X)$ is not regular) and $|Y| \geq 2$, so there are two cases to be considered.

Case 1: $Y \subsetneq X$ and $|Y| \geq 2$. Then the regularity of $OT(X,Y)$ and Lemma 2.1.6 yield $|Y| = 2$. Hence (iii) holds by Lemma 2.1.7.

Case 2: $OT(X)$ is not regular and $|Y| \geq 2$. Since $OT(X,Y)$ is regular, it follows that $Y \subsetneq X$, so by Lemma 2.1.6, $|Y| = 2$. Thus (iii) holds by Lemma 2.1.7.

Conversely, $OT(X,Y)$ is obviously regular if (i) or (ii) holds. We have by Lemma 2.1.7 that $OT(X,Y)$ is regular if (iii) holds.

Therefore the theorem is proved. □

Next, we give characterizations of the regular elements in $OP(X,Y)$ and $OI(X,Y)$, respectively.

Lemma 2.1.9. *Let A be a nonempty set and $\varnothing \neq B \subseteq A$. For $\alpha \in P(A,B)$, $\alpha \in \operatorname{Reg}(P(A,B))$ if and only if $\operatorname{ran}\alpha = (\operatorname{dom}\alpha \cap B)\alpha$.*

Proof. Assume that $\alpha \in \operatorname{Reg}(P(A,B))$. Let $\beta \in P(A,B)$ be such that $\alpha = \alpha\beta\alpha$. Then $\operatorname{ran}(\alpha\beta) \subseteq B$, so

$$\operatorname{ran}\alpha = \operatorname{ran}(\alpha\beta\alpha) = (\operatorname{ran}(\alpha\beta) \cap \operatorname{dom}\alpha)\alpha \subseteq (B \cap \operatorname{dom}\alpha)\alpha \subseteq \operatorname{ran}\alpha,$$

which implies that $\operatorname{ran}\alpha = (\operatorname{dom}\alpha \cap B)\alpha$.

Conversely, assume that $\operatorname{ran}\alpha = (\operatorname{dom}\alpha \cap B)\alpha$. Then $x\alpha^{-1} \cap B \neq \varnothing$ for all $x \in \operatorname{ran}\alpha$. For each $x \in \operatorname{ran}\alpha$, choose $d_x \in x\alpha^{-1} \cap B$. Then $d_x\alpha = x$ for all $x \in \operatorname{ran}\alpha$. Define $\beta : \operatorname{ran}\alpha \to B$ by

$$\beta = \begin{pmatrix} x \\ d_x \end{pmatrix}_{x \in \operatorname{ran}\alpha}.$$

Then $\beta \in P(A,B)$. Since for $x \in \operatorname{dom}\alpha, x\alpha \in \operatorname{dom}\beta$ and $x\alpha\beta \in \operatorname{dom}\alpha$, it follows that $\operatorname{dom}(\alpha\beta\alpha) = \operatorname{dom}\alpha$. If $x \in \operatorname{dom}\alpha$, then $x\alpha\beta\alpha = (x\alpha)\beta\alpha = d_{x\alpha}\alpha = x\alpha$. Therefore $\alpha = \alpha\beta\alpha$, so $\alpha \in \operatorname{Reg}(P(A,B))$, as desired. □

Theorem 2.1.10. *For* $\alpha \in OP(X,Y)$, $\alpha \in \operatorname{Reg}(OP(X,Y))$ *if and only if* $\operatorname{ran}\alpha = (\operatorname{dom}\alpha \cap Y)\alpha$. *Consequently,*

$$\operatorname{Reg}(OP(X,Y)) = \operatorname{Reg}(P(X,Y)) \cap OP(X).$$

Proof. If $\alpha \in \operatorname{Reg}(OP(X,Y))$, then $\alpha \in \operatorname{Reg}(P(X,Y))$ since $OP(X,Y)$ is a subsemigroup of $P(X,Y)$, so $\operatorname{ran}\alpha = (\operatorname{dom}\alpha \cap Y)\alpha$ by Lemma 2.1.9.

For the converse, assume that $\operatorname{ran}\alpha = (\operatorname{dom}\alpha \cap Y)\alpha$. Define $\beta : \operatorname{ran}\alpha \to Y$ as in the proof of Lemma 2.1.9. Then $\beta \in P(X,Y)$ and $\alpha = \alpha\beta\alpha$. Since α is order-preserving, it follows from Proposition 1.11 that β is order-preserving. Thus $\beta \in OP(X,Y)$, and so $\alpha \in \operatorname{Reg}(OP(X,Y))$, as desired. □

Lemma 2.1.11. *Let A be a nonempty set and $\varnothing \neq B \subseteq A$. For $\alpha \in I(A,B)$, $\alpha \in \operatorname{Reg}(I(A,B))$ if and only if* $\operatorname{dom}\alpha \subseteq B$.

Proof. Assume that $\alpha \in \operatorname{Reg}(I(A,B))$. Since $I(A,B)$ is a subsemigroup of $P(A,B)$, it follows that $\alpha \in \operatorname{Reg}(P(A,B))$. By Lemma 2.1.9, $\operatorname{ran}\alpha = (\operatorname{dom}\alpha \cap B)\alpha$. Then $(\operatorname{dom}\alpha)\alpha = (\operatorname{dom}\alpha \cap B)\alpha$, so $\operatorname{dom}\alpha = \operatorname{dom}\alpha \cap B$ since α is 1-1. Hence $\operatorname{dom}\alpha \subseteq B$.

Conversely, assume that $\operatorname{dom}\alpha \subseteq B$. Then $\alpha \in I(B)$. Since $I(B)$ is a regular subsemigroup of $I(A,B)$, it follows that $\alpha \in \operatorname{Reg}(I(A,B))$. □

Theorem 2.1.12. *For* $\alpha \in OI(X,Y)$, $\alpha \in \operatorname{Reg}(OI(X,Y))$ *if and only if* $\operatorname{dom}\alpha \subseteq Y$. *Consequently,* $\operatorname{Reg}(OI(X,Y)) = OI(Y)$.

Proof. If $\alpha \in \operatorname{Reg}(OI(X,Y))$, then $\alpha \in \operatorname{Reg}(I(X,Y))$ since $OI(X,Y)$ is a subsemigroup of $I(X,Y)$. So $\operatorname{dom}\alpha \subseteq Y$ by Lemm 2.1.11.

Conversely, assume that $\operatorname{dom}\alpha \subseteq Y$. Then $\alpha \in OI(Y)$, so $\alpha \in \operatorname{Reg}(OI(Y))$ by Theorem 1.5, and hence $\alpha \in \operatorname{Reg}(OI(X,Y))$ since $OI(Y)$ is a subsemigroup of $OI(X,Y)$. $\square$

We close this section with the proof of Theorem 1.15 by using Theorem 2.1.10 and Theorem 2.1.12.

Theorem 2.1.13. *Let $OS(X,Y)$ be $OP(X,Y)$ or $OI(X,Y)$. Then $OS(X,Y)$ is a regular semigroup if and only if $Y = X$.*

Proof. Suppose that $Y \subsetneq X$. Let $a \in X \smallsetminus Y$ and $b \in Y$. Then $\alpha = \binom{a}{b} \in OI(X,Y) \subseteq OP(X,Y)$. But $\operatorname{dom}\alpha \cap Y = \varnothing, \operatorname{ran}\alpha = \{b\}$ and $\operatorname{dom}\alpha = \{a\} \nsubseteq Y$, so by Theorem 2.1.10 and Theorem 2.1.12, $\alpha \notin \operatorname{Reg}(OS(X,Y))$. If $Y = X$, then $OP(X,Y) = OP(X)$, $OI(X,Y) = OI(X)$, and both $OP(X)$ and $OI(X)$ are regular semigroups by Theorem 1.5, completing the proof. $\square$

2.2 Combinatorial Results on Regular Elements

We begin this section by determining $|OT(X,Y)|$, $|OP(X,Y)|$ and $|OI(X,Y)|$ where X and Y are any finite chains. Then for $X = \{1, 2, \ldots, n\}$ and $Y = \{1, 2, \ldots, m\}$ where $m \leq n$, $|\operatorname{Reg}(OT(X,Y))|$, $|\operatorname{Reg}(OP(X,Y))|$ and $|\operatorname{Reg}(OI(X,Y))|$ are provided. In this case, the nonregular elements in $OT(X,Y)$, $OP(X,Y)$ and $OI(X,Y)$ can be counted.

The following two lemmas are needed to obtain the first purpose.

Lemma 2.2.1. *Let X and Y be finite chains, $|X| = n$ and $|Y| = m$. Then for $1 \leq r \leq n$ and $1 \leq s \leq m$,*

$$\left|\{\alpha \in OP(X,Y) \mid |\operatorname{dom}\alpha| = r \text{ and } |\operatorname{ran}\alpha| = s\}\right| = \binom{n}{r}\binom{m}{s}\binom{r-1}{s-1}.$$

Proof. Let $\varnothing \neq X' \subseteq X$ and $\varnothing \neq Y' \subseteq Y$ be such that $|X'| = r$ and $|Y'| = s$. Then by Proposition 1.19, the number of order-preserving transformations from

X' onto Y' is $\binom{r-1}{s-1}$. It follows that

$$\left|\{\alpha \in OP(X,Y) \mid \operatorname{dom}\alpha = X' \text{ and } \operatorname{ran}\alpha = Y'\}\right| = \binom{r-1}{s-1}.$$

This implies that for $1 \leq r \leq n$ and $1 \leq s \leq m$,

$$\left|\{\alpha \in OP(X,Y) \mid |\operatorname{dom}\alpha| = r \text{ and } |\operatorname{ran}\alpha| = s\}\right| = \binom{n}{r}\binom{m}{s}\binom{r-1}{s-1}.$$

□

Lemma 2.2.2. *Let X and Y be finite chains, $|X| = n$ and $|Y| = m$. Then for $1 \leq r \leq n$,*

$$\left|\{\alpha \in OP(X,Y) \mid |\operatorname{dom}\alpha| = r\}\right| = \binom{n}{r}\binom{m+r-1}{r}.$$

Proof. Note that for all $\alpha \in OP(X,Y) \smallsetminus \{0\}$, $1 \leq |\operatorname{ran}\alpha| \leq \min\{|\operatorname{dom}\alpha|, |Y|\}$. Then

$$\begin{aligned}
&\left|\{\alpha \in OP(X,Y) \mid |\operatorname{dom}\alpha| = r\}\right| \\
&= \left|\bigcup_{s=1}^{\min\{r,m\}} \{\alpha \in OP(X,Y) \mid |\operatorname{dom}\alpha| = r \text{ and } |\operatorname{ran}\alpha| = s\}\right| \\
&= \sum_{s=1}^{\min\{r,m\}} \left|\{\alpha \in OP(X,Y) \mid |\operatorname{dom}\alpha| = r \text{ and } |\operatorname{ran}\alpha| = s\}\right| \\
&= \sum_{s=1}^{\min\{r,m\}} \binom{n}{r}\binom{m}{s}\binom{r-1}{s-1} \qquad \text{by Lemma 2.2.1} \\
&= \binom{n}{r}\sum_{s=1}^{\min\{r,m\}} \binom{m}{s}\binom{r-1}{s-1} \\
&= \binom{n}{r}\binom{m+r-1}{r} \qquad \text{by Result 1.27.}
\end{aligned}$$

□

Theorem 2.2.3. *Let X and Y be finite chains. If $|X| = n$ and $|Y| = m$, then*

(i) $|\,OT(X,Y)| \;=\; \binom{m+n-1}{n}.$

(ii) $|\,OP(X,Y)| \;=\; \sum_{r=0}^{n}\binom{n}{r}\binom{m+r-1}{r}.$

(iii) $|\,OI(X,Y)| \;=\; \binom{n+m}{m}.$

Proof. (i) We have that

$$\begin{aligned}
|\,OT(X,Y)| &= \Big|\bigcup_{s=1}^{\min\{n,m\}}\{\alpha\in OT(X,Y)\mid |\operatorname{ran}\alpha| = s\}\Big| \\
&= \sum_{s=1}^{\min\{n,m\}}\big|\{\alpha\in OT(X,Y)\mid |\operatorname{ran}\alpha| = s\}\big| \\
&= \sum_{s=1}^{\min\{n,m\}}\big|\{\alpha\in OP(X,Y)\mid |\operatorname{dom}\alpha| = n \text{ and } |\operatorname{ran}\alpha| = s\}\big| \\
&= \sum_{s=1}^{\min\{n,m\}}\binom{n}{n}\binom{m}{s}\binom{n-1}{s-1} \qquad \text{by Lemma 2.2.1} \\
&= \sum_{s=1}^{\min\{n,m\}}\binom{m}{s}\binom{n-1}{s-1} \\
&= \binom{m+n-1}{n} \qquad \text{by Result 1.27.}
\end{aligned}$$

(ii) We see that

$$\begin{aligned}
|\,OP(X,Y)| &= \Big|\{0\}\cup\bigcup_{r=1}^{n}\{\alpha\in OP(X,Y)\mid |\operatorname{dom}\alpha| = r\}\Big| \\
&= 1+\sum_{r=1}^{n}\big|\{\alpha\in OP(X,Y)\mid |\operatorname{dom}\alpha| = r\}\big| \\
&= 1+\sum_{r=1}^{n}\binom{n}{r}\binom{m+r-1}{r} \qquad \text{by Lemma 2.2.2} \\
&= \sum_{r=0}^{n}\binom{n}{r}\binom{m+r-1}{r}.
\end{aligned}$$

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(iii) The following equalities hold.

$$\begin{aligned}
|\,OI(X,Y)| &= \left|\{0\} \cup \bigcup_{r=1}^{n} \{\alpha \in OI(X,Y) \mid |\operatorname{dom}\alpha| = r\}\right| \\
&= 1 + \sum_{r=1}^{n} \left|\{\alpha \in OI(X,Y) \mid |\operatorname{dom}\alpha| = r\}\right| \\
&= 1 + \sum_{r=1}^{n} \left|\{\alpha \in OP(X,Y) \mid |\operatorname{dom}\alpha| = |\operatorname{ran}\alpha| = r\}\right| \\
&= 1 + \sum_{r=1}^{n} \binom{n}{r}\binom{m}{r}\binom{r-1}{r-1} \qquad \text{by Lemma 2.2.1} \\
&= \begin{cases} \displaystyle\sum_{r=0}^{n} \binom{n}{r}\binom{m}{r} & \text{if } n \le m, \\ \displaystyle\sum_{r=0}^{m} \binom{n}{r}\binom{m}{r} & \text{if } n > m, \end{cases} \\
&= \binom{n+m}{m} \qquad \text{by Result 1.20.}
\end{aligned}$$

□

Notice that Theorem 1.8, Theorem 1.9 and Theorem 1.10 are special cases of Theorem 2.2.3 when $Y = X$.

The following lemma is needed to determine $|\operatorname{Reg}(OT(X,Y))|$ when $X = \{1, 2, \ldots, n\}$, $Y = \{1, 2, \ldots, m\}$ and $m \le n$.

Lemma 2.2.4. *Let $X = \{1, 2, \ldots, n\}$ and $Y = \{1, 2, \ldots, m\}$ where $m < n$. Then for $\alpha \in OT(X,Y)$, $\operatorname{ran}\alpha = Y\alpha$ if and only if $(X \smallsetminus Y)\alpha = \{m\alpha\}$.*

Proof. Suppose that $\operatorname{ran}\alpha = Y\alpha$. Let $x \in X \smallsetminus Y$ be arbitrary. Then $x > m$. Since $x\alpha \in \operatorname{ran}\alpha = Y\alpha$, we can choose $y \in Y$ such that $x\alpha = y\alpha$. Since α is order-preserving, it follows that $m\alpha \le x\alpha = y\alpha \le m\alpha$, so $x\alpha = m\alpha$. This proves that $(X \smallsetminus Y)\alpha = \{m\alpha\}$. If $(X \smallsetminus Y)\alpha = \{m\alpha\}$, then $(X \smallsetminus Y)\alpha \subseteq Y\alpha$, this implies that $\operatorname{ran}\alpha = Y\alpha$, and the proof is complete. □

Theorem 2.2.5. *Let $X = \{1, 2, \ldots, n\}$ and $Y = \{1, 2, \ldots, m\}$ where $m \le n$. Then*

$$|\operatorname{Reg}(OT(X,Y))| = \binom{2m-1}{m}.$$

Proof. If $m = n$, then $Y = X$, so $OT(X,Y) = OT(Y)$. This together with Theorem 1.4 yields $\operatorname{Reg}(OT(X,Y)) = OT(Y)$. Then the result for $m = n$ follows from Theorem 1.8. Assume that $m < n$. Let $\varnothing \neq Y' \subseteq Y$ and $|Y'| = s$. By Proposition 1.19, the number of all order-preserving transformations from Y onto Y' is $\binom{m-1}{s-1}$. Then

$$\big|\{\alpha \in OT(X,Y) \mid Y\alpha = Y' \text{ and } (X \smallsetminus Y)\alpha = \{m\alpha\}\}\big| = \binom{m-1}{s-1}.$$

It follows from Lemma 2.2.4 that

$$\begin{aligned}\{\alpha \in OT(X,Y) \mid Y\alpha = Y' \text{ and } (X \smallsetminus Y)\alpha = \{m\alpha\}\}\\ = \{\alpha \in OT(X,Y) \mid \operatorname{ran}\alpha = Y\alpha = Y'\}.\end{aligned}$$

Hence

$$|\{\alpha \in OT(X,Y) \mid \operatorname{ran}\alpha = Y\alpha = Y'\}| = \binom{m-1}{s-1}.$$

But we have from Corollary 2.1.3 that

$$\{\alpha \in OT(X,Y) \mid \operatorname{ran}\alpha = Y\alpha = Y'\} = \{\alpha \in \operatorname{Reg}(OT(X,Y)) \mid \operatorname{ran}\alpha = Y'\},$$

so

$$|\{\alpha \in \operatorname{Reg}(OT(X,Y)) | \operatorname{ran}\alpha = Y'\}| = \binom{m-1}{s-1}.$$

This implies that for $1 \leq s \leq m$,

$$|\{\alpha \in \operatorname{Reg}(OT(X,Y)) \mid |\operatorname{ran}\alpha| = s\}| = \binom{m}{s}\binom{m-1}{s-1}.$$

Therefore, it follows that

$$|\operatorname{Reg}(OT(X,Y))| = \sum_{s=1}^{m}\binom{m}{s}\binom{m-1}{s-1}.$$

We obtain from Result 1.26 that

$$|\operatorname{Reg}(OT(X,Y))| = \binom{2m-1}{m}.$$

□

Next, we count the regular elements of $OP(X,Y)$ when $X = \{1, 2, \dots, n\}$ and $Y = \{1, 2, \dots, m\}$ where $m \leq n$. Before proceeding, we require the following lemmas.

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Lemma 2.2.6. *Let* $X = \{1, 2, \ldots, n\}$ *and* $Y = \{1, 2, \ldots, m\}$ *where* $m < n$. *For* $\alpha \in OP(X, Y)$, $\operatorname{ran}\alpha = (\operatorname{dom}\alpha \cap Y)\alpha$ *if and only if either*

(i) $\operatorname{dom}\alpha \subseteq Y$ *or*

(ii) $\operatorname{dom}\alpha \cap Y \neq \varnothing$, $\operatorname{dom}\alpha \cap (X \smallsetminus Y) \neq \varnothing$ *and* $(\operatorname{dom}\alpha \cap (X \smallsetminus Y))\alpha = \{\max((\operatorname{dom}\alpha \cap Y)\alpha)\}$.

Proof. Assume that $\operatorname{ran}\alpha = (\operatorname{dom}\alpha \cap Y)\alpha$ and suppose that (i) is false, i.e., $\operatorname{dom}\alpha \nsubseteq Y$. Then $\operatorname{dom}\alpha \cap Y \neq \varnothing$ and $\operatorname{dom}\alpha \cap (X \smallsetminus Y) \neq \varnothing$. To show that $(\operatorname{dom}\alpha \cap (X \smallsetminus Y))\alpha = \{\max((\operatorname{dom}\alpha \cap Y)\alpha)\}$, let $x \in \operatorname{dom}\alpha \cap (X \smallsetminus Y)$. Then $x > m \geq \max(\operatorname{dom}\alpha \cap Y)$ and $x\alpha \in (\operatorname{dom}\alpha \cap Y)\alpha$ by assumption. Since α is order-preserving, we obtain that

$$\max((\operatorname{dom}\alpha \cap Y)\alpha) = (\max(\operatorname{dom}\alpha \cap Y))\alpha \leq x\alpha \leq \max((\operatorname{dom}\alpha \cap Y)\alpha),$$

and we deduce that $x\alpha = \max((\operatorname{dom}\alpha \cap Y)\alpha)$. Hence (ii) holds.

Conversely, if (i) holds, then $(\operatorname{dom}\alpha \cap Y)\alpha = (\operatorname{dom}\alpha)\alpha = \operatorname{ran}\alpha$. Next, assume that (ii) holds. Then $(\operatorname{dom}\alpha \cap (X \smallsetminus Y))\alpha = \{\max((\operatorname{dom}\alpha \cap Y)\alpha)\} \subseteq (\operatorname{dom}\alpha \cap Y)\alpha$. This implies that $\operatorname{ran}\alpha = (\operatorname{dom}\alpha \cap Y)\alpha \cup (\operatorname{dom}\alpha \cap (X \smallsetminus Y))\alpha = (\operatorname{dom}\alpha \cap Y)\alpha$.

Hence the proof is complete. □

Lemma 2.2.7. *Let* $X = \{1, 2, \ldots, n\}$ *and* $Y = \{1, 2, \ldots, m\}$ *where* $m < n$. *Then for* $1 \leq s \leq m$ *and* $1 \leq t \leq n - m$,

$$\Big|\{\alpha \in OP(X, Y) \mid |\operatorname{dom}\alpha \cap Y| = s, |\operatorname{dom}\alpha \cap (X \smallsetminus Y)| = t \text{ and } (\operatorname{dom}\alpha \cap (X \smallsetminus Y))\alpha = \{\max((\operatorname{dom}\alpha \cap Y)\alpha)\}\}\Big|$$

$$= \binom{m}{s}\binom{n-m}{t}\binom{m+s-1}{s}.$$

Proof. Let $\varnothing \neq Y' \subseteq Y$ and $\varnothing \neq Z' \subseteq X \smallsetminus Y$ be such that $|Y'| = s$ and $|Z'| = t$. Then by Theorem 2.2.3(i), the number of order-preserving transformations from Y' into Y is $\binom{m+s-1}{s}$. Therefore it follows that the number of order-preserving transformations $\alpha : Y' \cup Z' \to Y$ such that $Z'\alpha = \max(Y'\alpha)$ is also $\binom{m+s-1}{s}$.

Consequently,

$$\Big|\big\{\alpha \in OP(X,Y) \;\big|\; \operatorname{dom}\alpha \cap Y = Y', \operatorname{dom}\alpha \cap (X \smallsetminus Y) = Z' \text{ and } (\operatorname{dom}\alpha \cap (X \smallsetminus Y))\alpha = \{\max((\operatorname{dom}\alpha \cap Y)\alpha)\}\big\}\Big|$$

$$= \binom{m+s-1}{s}.$$

This implies that for $1 \leq s \leq m$ and $1 \leq t \leq n-m$,

$$\Big|\big\{\alpha \in OP(X,Y) \;\big|\; |\operatorname{dom}\alpha \cap Y| = s, |\operatorname{dom}\alpha \cap (X \smallsetminus Y)| = t \text{ and } (\operatorname{dom}\alpha \cap (X \smallsetminus Y))\alpha = \{\max((\operatorname{dom}\alpha \cap Y)\alpha)\}\big\}\Big|$$

$$= \binom{m}{s}\binom{n-m}{t}\binom{m+s-1}{s}.$$

□

Theorem 2.2.8. *Let $X = \{1, 2, \ldots, n\}$ and $Y = \{1, 2, \ldots, m\}$ where $m \leq n$. Then*

$$|\operatorname{Reg}(OP(X,Y))| = 1 + 2^{n-m}\sum_{s=1}^{m}\binom{m}{s}\binom{m+r-1}{s}.$$

Proof. If $m = n$, then $OP(X,Y) = OP(Y)$, so $\operatorname{Reg}(OP(X,Y)) = OP(Y)$ by Theorem 1.5 and then the result for $m = n$ follows from Theorem 1.9. Next, assume that $m < n$. Then by Theorem 2.1.10 and Lemma 2.2.6, we have

$$\begin{aligned}
\operatorname{Reg}(OP(X,Y)) &= \{\alpha \in OP(X,Y) \mid \operatorname{ran}\alpha = (\operatorname{dom}\alpha \cap Y)\alpha\} \\
&= \{\alpha \in OP(X,Y) \mid \operatorname{dom}\alpha \subseteq Y\} \cup \\
&\quad \big\{\alpha \in OP(X,Y) \mid \operatorname{dom}\alpha \cap Y \neq \varnothing, \operatorname{dom}\alpha \cap (X \smallsetminus Y) \neq \varnothing \text{ and } \\
&\qquad (\operatorname{dom}\alpha \cap (X \smallsetminus Y))\alpha = \{\max((\operatorname{dom}\alpha \cap Y)\alpha)\}\big\} \\
&= OP(Y) \cup \big\{\alpha \in OP(X,Y) \mid \operatorname{dom}\alpha \cap Y \neq \varnothing, \operatorname{dom}\alpha \cap (X \smallsetminus Y) \neq \varnothing \text{ and } \\
&\qquad (\operatorname{dom}\alpha \cap (X \smallsetminus Y))\alpha = \{\max((\operatorname{dom}\alpha \cap Y)\alpha)\}\big\}.
\end{aligned} \tag{1}$$

We know from Theorem 1.9 that

$$|\,OP(Y)| = \sum_{s=0}^{m} \binom{m}{s}\binom{m+s-1}{s}. \tag{2}$$

Also, by Lemma 2.2.7, we have

$$\begin{aligned}
&\Big|\{\alpha \in OP(X,Y) \mid \operatorname{dom}\alpha \cap Y \neq \varnothing, \operatorname{dom}\alpha \cap (X \smallsetminus Y) \neq \varnothing \text{ and} \\
&\qquad\qquad (\operatorname{dom}\alpha \cap (X \smallsetminus Y))\alpha = \{\max((\operatorname{dom}\alpha \cap Y)\alpha)\}\}\Big| \\
&= \sum_{s=1}^{m}\sum_{t=1}^{n-m} \Big|\{\alpha \in OP(X,Y) \mid |\operatorname{dom}\alpha \cap Y| = s, |\operatorname{dom}\alpha \cap (X \smallsetminus Y)| = t \text{ and} \\
&\qquad\qquad (\operatorname{dom}\alpha \cap (X \smallsetminus Y))\alpha = \{\max((\operatorname{dom}\alpha \cap Y)\alpha)\}\}\Big| \\
&= \sum_{s=1}^{m}\sum_{t=1}^{n-m} \binom{m}{s}\binom{n-m}{t}\binom{m+s-1}{s}. \qquad (3)
\end{aligned}$$

From (1), (2) and (3), we have that

$$\begin{aligned}
|\operatorname{Reg}(OT(X,Y))| &= \sum_{s=0}^{m} \binom{m}{s}\binom{m+s-1}{s} + \sum_{s=1}^{m}\sum_{t=1}^{n-m} \binom{m}{s}\binom{n-m}{t}\binom{m+s-1}{s} \\
&= 1 + \sum_{s=1}^{m} \binom{m}{s}\binom{m+s-1}{s} + \sum_{t=1}^{n-m}\binom{n-m}{t}\sum_{s=1}^{m}\binom{m}{s}\binom{m+s-1}{s} \\
&= 1 + \sum_{t=0}^{n-m}\binom{n-m}{t}\sum_{s=1}^{m}\binom{m}{s}\binom{m+s-1}{s} \\
&= 1 + 2^{n-m}\sum_{s=1}^{m}\binom{m}{s}\binom{m+s-1}{s} \qquad \text{by Result 1.23.}
\end{aligned}$$

□

The last result of this section follows directly from Theorem 2.1.12 and Theorem 1.10.

Theorem 2.2.9. *If X is a finite chain and $\varnothing \neq Y \subseteq X$, then*

$$|\operatorname{Reg}(OI(X,Y))| = \binom{2|Y|}{|Y|}.$$

CHAPTER III

SEMIGROUPS OF ORDER-PRESERVING TRANSFORMATIONS SENDING A FIXED SET INTO ITSELF

In this chapter, we consider the semigroups $\overline{OT}(X,Y)$, $\overline{OP}(X,Y)$ and $\overline{OI}(X,Y)$ where Y is a nonempty subset of a chain X. The main purpose of this chapter is to characterize the regular elements of $\overline{OT}(X,Y)$, $\overline{OP}(X,Y)$ and $\overline{OI}(X,Y)$. We also give necessary and sufficient conditions in terms of Y for $\overline{OT}(X,Y)$, $\overline{OP}(X,Y)$ and $\overline{OI}(X,Y)$ to be regular semigroups. Moreover, the cardinalities of $\mathrm{Reg}(\overline{OT}(X,Y))$, $\mathrm{Reg}(\overline{OP}(X,Y))$ and $\mathrm{Reg}(\overline{OI}(X,Y))$ are provided when $X = \{1,2,\ldots,n\}$ and $Y = \{1,2,\ldots,m\}$ where $m \leq n$.

Recall that the semigroups $\overline{OT}(X,Y)$, $\overline{OP}(X,Y)$ and $\overline{OI}(X,Y)$, where Y is a nonempty subset of a chain X, are defined as follows:

$$\overline{OT}(X,Y) = \{\alpha \in OT(X) \mid Y\alpha \subseteq Y\},$$

$$\overline{OP}(X,Y) = \{\alpha \in OP(X) \mid (\operatorname{dom}\alpha \cap Y)\alpha \subseteq Y\},$$

$$\overline{OI}(X,Y) = \{\alpha \in OI(X) \mid (\operatorname{dom}\alpha \cap Y)\alpha \subseteq Y\}.$$

3.1 Characterizations of Regular Elements

Throughout this section, let X be any chain and $\varnothing \neq Y \subseteq X$.

We first give a necessary and sufficient condition for an element of $\overline{OT}(X,Y)$ to be regular. The following two lemmas give necessary conditions for the regular elements of $\overline{OT}(X,Y)$.

Lemma 3.1.1. *Let $\alpha \in \mathrm{Reg}(\overline{OT}(X,Y))$. Then the following statements hold.*

(i) *If $\mathrm{ub}(\operatorname{ran}\alpha) \cap Y \neq \varnothing$, then $\max(\operatorname{ran}\alpha)$ exists and belongs to Y.*

(ii) *If $\mathrm{lb}(\operatorname{ran}\alpha) \cap Y \neq \varnothing$, then $\min(\operatorname{ran}\alpha)$ exists and belongs to Y.*

Proof. Assume that $\mathrm{ub}(\mathrm{ran}\,\alpha) \cap Y \neq \varnothing$. Let $u \in \mathrm{ub}(\mathrm{ran}\,\alpha) \cap Y$ and let $\beta \in \overline{OT}(X,Y)$ be such that $\alpha = \alpha\beta\alpha$. Then $\mathrm{ran}\,\alpha \leq u$, and thus

$$\mathrm{ran}\,\alpha = X\alpha = X\alpha\beta\alpha = (\mathrm{ran}\,\alpha)\beta\alpha \leq u\beta\alpha \in \mathrm{ran}\,\alpha.$$

This implies that $\max(\mathrm{ran}\,\alpha) = u\beta\alpha$ and $u\beta\alpha \in Y\beta\alpha \subseteq Y$. This proves (i), and (ii) follows in the same way. □

Lemma 3.1.2. *Let* $\alpha \in \mathrm{Reg}(\overline{OT}(X,Y))$ *and* $x \in X \smallsetminus \big(\mathrm{ran}\,\alpha \cup \mathrm{ub}(\mathrm{ran}\,\alpha) \cup \mathrm{lb}(\mathrm{ran}\,\alpha)\big)$. *If* $x \in Y$, *then*

(i) $\max(\{t \in \mathrm{ran}\,\alpha \mid t < x\})$ *exists and belongs to* Y *or*

(ii) $\min(\{t \in \mathrm{ran}\,\alpha \mid t > x\})$ *exists and belongs to* Y.

Proof. Let $\beta \in \overline{OT}(X,Y)$ be such that $\alpha = \alpha\beta\alpha$. Since $x \in X \smallsetminus \big(\mathrm{ran}\,\alpha \cup \mathrm{ub}(\mathrm{ran}\,\alpha) \cup \mathrm{lb}(\mathrm{ran}\,\alpha)\big)$, it follows that

$$\{t \in \mathrm{ran}\,\alpha \mid t < x\} \neq \varnothing,\ \{t \in \mathrm{ran}\,\alpha \mid t > x\} \neq \varnothing,$$
$$\mathrm{ran}\,\alpha = \{t \in \mathrm{ran}\,\alpha \mid t < x\} \,\dot\cup\, \{t \in \mathrm{ran}\,\alpha \mid t > x\}.$$

Since $x\beta\alpha \in \mathrm{ran}\,\alpha$, it follows that $x\beta\alpha < x$ or $x\beta\alpha > x$. For $s \in X$, if $s\alpha < x$, then $s\alpha = (s\alpha)\beta\alpha \leq x\beta\alpha$. If $s\alpha > x$, then $s\alpha = (s\alpha)\beta\alpha \geq x\beta\alpha$.
This shows that

$$x\beta\alpha = \begin{cases} \max(\{t \in \mathrm{ran}\,\alpha \mid t < x\}) & \text{if } x\beta\alpha < x, \\ \min(\{t \in \mathrm{ran}\,\alpha \mid t > x\}) & \text{if } x\beta\alpha > x. \end{cases}$$

Since $x \in Y$, we have $x\beta\alpha \in Y$, so the result follows. □

Now we give a necessary and sufficient condition for an element of $\overline{OT}(X,Y)$ to be regular.

Theorem 3.1.3. *Let* $\alpha \in \overline{OT}(X,Y)$. *Then* $\alpha \in \mathrm{Reg}(\overline{OT}(X,Y))$ *if and only if the following four conditions hold.*

(i) $\mathrm{ran}\,\alpha \cap Y = Y\alpha$.

(ii) *If* $\mathrm{ub}(\mathrm{ran}\,\alpha) \neq \varnothing$, *then* $\max(\mathrm{ran}\,\alpha)$ *exists.*

If $\mathrm{ub}(\mathrm{ran}\,\alpha) \cap Y \neq \varnothing$, *then* $\max(\mathrm{ran}\,\alpha) \in Y$.

(iii) *If* $\mathrm{lb}(\mathrm{ran}\,\alpha) \neq \varnothing$, *then* $\min(\mathrm{ran}\,\alpha)$ *exists.*

If $\mathrm{lb}(\mathrm{ran}\,\alpha) \cap Y \neq \varnothing$, *then* $\min(\mathrm{ran}\,\alpha) \in Y$.

(iv) *If* $x \in X \smallsetminus \big(\mathrm{ran}\,\alpha \cup \mathrm{ub}(\mathrm{ran}\,\alpha) \cup \mathrm{lb}(\mathrm{ran}\,\alpha)\big)$, *then* $\max(\{t \in \mathrm{ran}\,\alpha \mid t < x\})$ *exists or* $\min(\{t \in \mathrm{ran}\,\alpha \mid t > x\})$ *exists.*

If x is also in Y, then $\max(\{t \in \mathrm{ran}\,\alpha \mid t < x\})$ *exists and belongs to Y or* $\min(\{t \in \mathrm{ran}\,\alpha \mid t > x\})$ *exists and belongs to Y.*

Proof. Assume that $\alpha \in \mathrm{Reg}(\overline{OT}(X,Y))$. Since $\overline{OT}(X,Y)$ is a subsemigroup of $\overline{T}(X,Y)$ and $OT(X)$, it follows that $\alpha \in \mathrm{Reg}\big(\overline{T}(X,Y)\big)$ and $\alpha \in \mathrm{Reg}(OT(X))$. By Theorem 1.2, $\mathrm{ran}\,\alpha \cap Y = Y\alpha$, so (i) holds. Also, by Theorem 1.6, the first parts of (ii), (iii) and (iv) are true. For the second parts of (ii), (iii) and (iv), we obtain from Lemma 3.1.1(i), Lemma 3.1.1(ii) and Lemma 3.1.2, respectively.

For the converse, assume that (i), (ii), (iii) and (iv) hold. If $\mathrm{ub}(\mathrm{ran}\,\alpha) \neq \varnothing$, let $u = \max(\mathrm{ran}\,\alpha)$ and so $u \in Y$ if $\mathrm{ub}(\mathrm{ran}\,\alpha) \cap Y \neq \varnothing$. If $\mathrm{lb}(\mathrm{ran}\,\alpha) \neq \varnothing$, let $l = \min(\mathrm{ran}\,\alpha)$ and so $l \in Y$ if $\mathrm{lb}(\mathrm{ran}\,\alpha) \cap Y \neq \varnothing$. For $x \in \big(X \smallsetminus (\mathrm{ran}\,\alpha \cup \mathrm{ub}(\mathrm{ran}\,\alpha) \cup \mathrm{lb}(\mathrm{ran}\,\alpha))\big)$, if $x \in Y$, let

$$m_x = \begin{cases} \max(\{t \in \mathrm{ran}\,\alpha \mid t < x\}) & \text{if } \max(\{t \in \mathrm{ran}\,\alpha \mid t < x\}) \text{ exists} \\ & \qquad\text{and belongs to } Y, \\ \min(\{t \in \mathrm{ran}\,\alpha \mid t > x\}) & \text{otherwise,} \end{cases}$$

and if $x \notin Y$, let

$$n_x = \begin{cases} \max(\{t \in \mathrm{ran}\,\alpha \mid t < x\}) & \text{if } \max(\{t \in \mathrm{ran}\,\alpha \mid t < x\}) \text{ exists,} \\ \min(\{t \in \mathrm{ran}\,\alpha \mid t > x\}) & \text{otherwise.} \end{cases}$$

By (iv), $m_x \in Y$ for all $x \in \big(X \smallsetminus (\mathrm{ran}\,\alpha \cup \mathrm{ub}(\mathrm{ran}\,\alpha) \cup \mathrm{lb}(\mathrm{ran}\,\alpha))\big) \cap Y$.
For each $y \in \big(X \smallsetminus (\mathrm{ran}\,\alpha \cup \mathrm{ub}(\mathrm{ran}\,\alpha) \cup \mathrm{lb}(\mathrm{ran}\,\alpha))\big) \cap Y$, let

$$A_y = \big\{x \in X \mid \{t \in \mathrm{ran}\,\alpha \mid t < x\} = \{t \in \mathrm{ran}\,\alpha \mid t < y\} \text{ and } \{t \in \mathrm{ran}\,\alpha \mid t > x\} = \{t \in \mathrm{ran}\,\alpha \mid t > y\}\big\}.$$

Notice that $y \in A_y$ for all $y \in \big(X \smallsetminus (\operatorname{ran}\alpha \cup \operatorname{ub}(\operatorname{ran}\alpha) \cup \operatorname{lb}(\operatorname{ran}\alpha))\big) \cap Y$ and for $y_1, y_2 \in \big(X \smallsetminus (\operatorname{ran}\alpha \cup \operatorname{ub}(\operatorname{ran}\alpha) \cup \operatorname{lb}(\operatorname{ran}\alpha))\big) \cap Y$, either $A_{y_1} \cap A_{y_2} = \varnothing$ or $A_{y_1} = A_{y_2}$. It follows that if $x \notin A_y$ for all $y \in \big(X \smallsetminus (\operatorname{ran}\alpha \cup \operatorname{ub}(\operatorname{ran}\alpha) \cup \operatorname{lb}(\operatorname{ran}\alpha))\big) \cap Y$, then $x \notin Y$. Since $\operatorname{ran}\alpha \cap Y = Y\alpha$ by (i), this implies that $x\alpha^{-1} \cap Y \neq \varnothing$ for all $x \in \operatorname{ran}\alpha \cap Y$. For each $x \in \operatorname{ran}\alpha$, choose an element

$$x' \in \begin{cases} x\alpha^{-1} \cap Y & \text{if } x \in Y, \\ x\alpha^{-1} & \text{if } x \notin Y. \end{cases}$$

Then $x' \in Y$ for all $x \in \operatorname{ran}\alpha \cap Y$ and $x'\alpha = x$ for all $x \in \operatorname{ran}\alpha$. Also, we have from Proposition 1.11 that

$$\text{for } x_1, x_2 \in \operatorname{ran}\alpha, \;\; x_1 < x_2 \;\; \text{implies} \;\; x_1' < x_2'.$$

Define $\beta : X \to X$ by

$$x\beta = \begin{cases} x' & \text{if } x \in \operatorname{ran}\alpha, \\ u' & \text{if } x > \operatorname{ran}\alpha, \\ l' & \text{if } x < \operatorname{ran}\alpha, \\ {m_y}' & \text{if } x \in X \smallsetminus (\operatorname{ran}\alpha \cup \operatorname{ub}(\operatorname{ran}\alpha) \cup \operatorname{lb}(\operatorname{ran}\alpha)) \text{ and } x \in A_y \\ & \quad \text{for some } y \in \big(X \smallsetminus (\operatorname{ran}\alpha \cup \operatorname{ub}(\operatorname{ran}\alpha) \cup \operatorname{lb}(\operatorname{ran}\alpha))\big) \cap Y, \\ {n_x}' & \text{if } x \in X \smallsetminus (\operatorname{ran}\alpha \cup \operatorname{ub}(\operatorname{ran}\alpha) \cup \operatorname{lb}(\operatorname{ran}\alpha)) \text{ and } x \notin A_y \\ & \quad \text{for all } y \in \big(X \smallsetminus (\operatorname{ran}\alpha \cup \operatorname{ub}(\operatorname{ran}\alpha) \cup \operatorname{lb}(\operatorname{ran}\alpha))\big) \cap Y. \end{cases}$$

We see that $Y\beta \subseteq Y$ and for $x \in X, x\alpha \in \operatorname{ran}\alpha$, and thus

$$x\alpha\beta\alpha = (x\alpha)\beta\alpha = (x\alpha)'\alpha = x\alpha.$$

Hence $\beta \in \overline{T}(X,Y)$ and $\alpha = \alpha\beta\alpha$. It remains to show that β is order-preserving. Let $x_1, x_2 \in X$ be such that $x_1 < x_2$. We can see that $u' = \max(\operatorname{ran}\beta)$ if $\operatorname{ub}(\operatorname{ran}\alpha) \neq \varnothing$ and $l' = \min(\operatorname{ran}\beta)$ if $\operatorname{lb}(\operatorname{ran}\alpha) \neq \varnothing$. It follows that if $x_2 \in \operatorname{ub}(\operatorname{ran}\alpha)$ or $x_1 \in \operatorname{lb}(\operatorname{ran}\alpha)$, then $x_1\beta \leq x_2\beta$. Also, we have that if $x_1, x_2 \in \operatorname{ran}\alpha$, then $x_1\beta = {x_1}' < {x_2}' = x_2\beta$. Therefore there are six cases to clarify as follows:

Case 1: $x_1 \in \operatorname{ran}\alpha$ and $x_2 \in X \setminus \big(\operatorname{ran}\alpha \cup \operatorname{ub}(\operatorname{ran}\alpha) \cup \operatorname{lb}(\operatorname{ran}\alpha)\big)$.

Subcase 1.1: $x_2 \in A_y$ for some $y \in \big(X \setminus (\operatorname{ran}\alpha \cup \operatorname{ub}(\operatorname{ran}\alpha) \cup \operatorname{lb}(\operatorname{ran}\alpha))\big) \cap Y$. Then $\{t \in \operatorname{ran}\alpha \mid t < x_2\} = \{t \in \operatorname{ran}\alpha \mid t < y\}$ and $\{t \in \operatorname{ran}\alpha \mid t > x_2\} = \{t \in \operatorname{ran}\alpha \mid t > y\}$. If $m_y = \max(\{t \in \operatorname{ran}\alpha \mid t < y\})$, then $x_1 \leq m_y$ since $x_1 \in \{t \in \operatorname{ran}\alpha \mid t < x_2\} = \{t \in \operatorname{ran}\alpha \mid t < y\}$. Thus $x_1\beta = {x_1}' \leq {m_y}' = x_2\beta$. If $m_y = \min(\{t \in \operatorname{ran}\alpha \mid t > y\})$, then $x_1 < x_2 < m_y$ since $m_y \in \{t \in \operatorname{ran}\alpha \mid t > y\} = \{t \in \operatorname{ran}\alpha \mid t > x_2\}$. So $x_1\beta = {x_1}' < {m_y}' = x_2\beta$.

Subcase 1.2: $x_2 \notin A_y$ for all $y \in \big(X \setminus (\operatorname{ran}\alpha \cup \operatorname{ub}(\operatorname{ran}\alpha) \cup \operatorname{lb}(\operatorname{ran}\alpha))\big) \cap Y$. If $n_{x_2} = \max(\{t \in \operatorname{ran}\alpha \mid t < x_2\})$, then $x_1 \leq n_{x_2}$, and so $x_1\beta = {x_1}' \leq {n_{x_2}}' = x_2\beta$. If $n_{x_2} = \min(\{t \in \operatorname{ran}\alpha \mid t > x_2\})$, then $x_1 < x_2 < n_{x_2}$, and thus $x_1\beta = {x_1}' < {n_{x_2}}' = x_2\beta$.

Case 2: $x_1 \in X \setminus \big(\operatorname{ran}\alpha \cup \operatorname{ub}(\operatorname{ran}\alpha) \cup \operatorname{lb}(\operatorname{ran}\alpha)\big)$ and $x_2 \in \operatorname{ran}\alpha$.

Subcase 2.1: $x_1 \in A_y$ for some $y \in \big(X \setminus (\operatorname{ran}\alpha \cup \operatorname{ub}(\operatorname{ran}\alpha) \cup \operatorname{lb}(\operatorname{ran}\alpha))\big) \cap Y$. Then $\{t \in \operatorname{ran}\alpha \mid t < x_1\} = \{t \in \operatorname{ran}\alpha \mid t < y\}$ and $\{t \in \operatorname{ran}\alpha \mid t > x_1\} = \{t \in \operatorname{ran}\alpha \mid t > y\}$. If $m_y = \max(\{t \in \operatorname{ran}\alpha \mid t < y\})$, then $m_y < x_1 < x_2$ since $m_y \in \{t \in \operatorname{ran}\alpha \mid t < y\} = \{t \in \operatorname{ran}\alpha \mid t < x_1\}$, so $x_1\beta = {m_y}' < {x_2}' = x_2\beta$. If $m_y = \min(\{t \in \operatorname{ran}\alpha \mid t > y\})$, then $m_y \leq x_2$ since $x_2 \in \{t \in \operatorname{ran}\alpha \mid t > x_1\} = \{t \in \operatorname{ran}\alpha \mid t > y\}$. Therefore $x_1\beta = {m_y}' \leq {x_2}' = x_2\beta$.

Subcase 2.2: $x_1 \notin A_y$ for all $y \in \big(X \setminus (\operatorname{ran}\alpha \cup \operatorname{ub}(\operatorname{ran}\alpha) \cup \operatorname{lb}(\operatorname{ran}\alpha))\big) \cap Y$. If $n_{x_1} = \max(\{t \in \operatorname{ran}\alpha \mid t < x_1\})$, then $n_{x_1} < x_1 < x_2$, so $x_1\beta = n'_{x_1} < x'_2 = x_2\beta$. If $n_{x_1} = \min(\{t \in \operatorname{ran}\alpha \mid t > x_1\})$, then $n_{x_2} \leq x_2$, and hence $x_1\beta = {n_{x_1}}' \leq {x_2}' = x_2\beta$.

Case 3: $x_1, x_2 \in X \setminus \big(\operatorname{ran}\alpha \cup \operatorname{ub}(\operatorname{ran}\alpha) \cup \operatorname{lb}(\operatorname{ran}\alpha)\big)$, $x_1 \in A_{y_1}$ and $x_2 \in A_{y_2}$ for some $y_1, y_2 \in \big(X \setminus (\operatorname{ran}\alpha \cup \operatorname{ub}(\operatorname{ran}\alpha) \cup \operatorname{lb}(\operatorname{ran}\alpha))\big) \cap Y$. Then $\{t \in \operatorname{ran}\alpha \mid t < x_1\} = \{t \in \operatorname{ran}\alpha \mid t < y_1\}$, $\{t \in \operatorname{ran}\alpha \mid t > x_1\} = \{t \in \operatorname{ran}\alpha \mid t > y_1\}$, $\{t \in \operatorname{ran}\alpha \mid t < x_2\} = \{t \in \operatorname{ran}\alpha \mid t < y_2\}$ and $\{t \in \operatorname{ran}\alpha \mid t > x_2\} = \{t \in \operatorname{ran}\alpha \mid t > y_2\}$.

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Subcase 3.1: $m_{y_1} = \max(\{t \in \operatorname{ran}\alpha \mid t < y_1\})$ and $m_{y_2} = \max(\{t \in \operatorname{ran}\alpha \mid t < y_2\})$. Since $\{t \in \operatorname{ran}\alpha \mid t < y_1\} = \{t \in \operatorname{ran}\alpha \mid t < x_1\} \subseteq \{t \in \operatorname{ran}\alpha \mid t < x_2\} = \{t \in \operatorname{ran}\alpha \mid t < y_2\}$, it follows that $m_{y_1} \leq m_{y_2}$, and so $x_1\beta = {m_{y_1}}' \leq {m_{y_2}}' = x_2\beta$.

Subcase 3.2: $m_{y_1} = \max(\{t \in \operatorname{ran}\alpha \mid t < y_1\})$ and $m_{y_2} = \min(\{t \in \operatorname{ran}\alpha \mid t > y_2\})$. Then $m_{y_1} \in \{t \in \operatorname{ran}\alpha \mid t < y_1\} = \{t \in \operatorname{ran}\alpha \mid t < x_1\}$ and $m_{y_2} \in \{t \in \operatorname{ran}\alpha \mid t > y_2\} = \{t \in \operatorname{ran}\alpha \mid t > x_2\}$. Hence $m_{y_1} < x_1 < x_2 < m_{y_2}$, so $x_1\beta = {m_{y_1}}' < {m_{y_2}}' = x_2\beta$.

Subcase 3.3: $m_{y_1} = \min(\{t \in \operatorname{ran}\alpha \mid t > y_1\})$ and $m_{y_2} = \max(\{t \in \operatorname{ran}\alpha \mid t < y_2\})$. If $\{t \in \operatorname{ran}\alpha \mid y_1 < t < y_2\} = \varnothing$, then $\{t \in \operatorname{ran}\alpha \mid t < y_1\} = \{t \in \operatorname{ran}\alpha \mid t < y_2\}$ which impossible since $\{t \in \operatorname{ran}\alpha \mid t < y_1\}$ has no maximum or $\max(\{t \in \operatorname{ran}\alpha \mid t < y_1\})$ exists and $\max(\{t \in \operatorname{ran}\alpha \mid t < y_1\}) \notin Y$ but $\max(\{t \in \operatorname{ran}\alpha \mid t < y_2\})$ exists and $\max(\{t \in \operatorname{ran}\alpha \mid t < y_2\}) \in Y$. Then there exists an element $c \in \operatorname{ran}\alpha$ such that $y_1 < c < y_2$. Consequently, $m_{y_1} \leq c \leq m_{y_2}$. Hence $x_1\beta = m'_{y_1} \leq m'_{y_2} = x_2\beta$.

Subcase 3.4: $m_{y_1} = \min(\{t \in \operatorname{ran}\alpha \mid t > y_1\})$ and $m_{y_2} = \min(\{t \in \operatorname{ran}\alpha \mid t > y_2\})$. Since $\{t \in \operatorname{ran}\alpha \mid t > y_1\} = \{t \in \operatorname{ran}\alpha \mid t > x_1\} \supseteq \{t \in \operatorname{ran}\alpha \mid t > x_2\} = \{t \in \operatorname{ran}\alpha \mid t > y_2\}$, it follows that $m_{y_1} \leq m_{y_2}$, and then $x_1\beta = m'_{y_1} \leq m'_{y_2} = x_2\beta$.

Case 4: $x_1, x_2 \in X \setminus \big(\operatorname{ran}\alpha \cup \operatorname{ub}(\operatorname{ran}\alpha) \cup \operatorname{lb}(\operatorname{ran}\alpha)\big)$, $x_1 \in A_y$ for some $y \in \big(X \setminus (\operatorname{ran}\alpha \cup \operatorname{ub}(\operatorname{ran}\alpha) \cup \operatorname{lb}(\operatorname{ran}\alpha))\big) \cap Y$ and $x_2 \notin A_y$ for all $y \in \big(X \setminus (\operatorname{ran}\alpha \cup \operatorname{ub}(\operatorname{ran}\alpha) \cup \operatorname{lb}(\operatorname{ran}\alpha))\big) \cap Y$. Then $\{t \in \operatorname{ran}\alpha \mid t < x_1\} = \{t \in \operatorname{ran}\alpha \mid t < y\}$ and $\{t \in \operatorname{ran}\alpha \mid t > x_1\} = \{t \in \operatorname{ran}\alpha \mid t > y\}$.

Subcase 4.1: $m_y = \max(\{t \in \operatorname{ran}\alpha \mid t < y\})$ and $n_{x_2} = \max(\{t \in \operatorname{ran}\alpha \mid t < x_2\})$. Since $\{t \in \operatorname{ran}\alpha \mid t < y\} = \{t \in \operatorname{ran}\alpha \mid t < x_1\} \subseteq \{t \in \operatorname{ran}\alpha \mid t < x_2\}$, we get $m_y \leq n_{x_2}$, and it follows that $x_1\beta = {m_y}' \leq {n_{x_2}}' = x_2\beta$.

Subcase 4.2: $m_y = \max(\{t \in \operatorname{ran}\alpha \mid t < y\})$ and $n_{x_2} = \min(\{t \in \operatorname{ran}\alpha \mid t > x_2\})$. Then $m_y \in \{t \in \operatorname{ran}\alpha \mid t < y\} = \{t \in \operatorname{ran}\alpha \mid t < x_1\}$

and $n_{x_2} \in \{t \in \operatorname{ran}\alpha \mid t > x_2\}$. Hence $m_y < x_1 < x_2 < n_{x_2}$, and therefore $x_1\beta = m_y{}' < n_{x_2}{}' = x_2\beta$.

Subcase 4.3: $m_y = \min(\{t \in \operatorname{ran}\alpha \mid t > y\})$ and $n_{x_2} = \max(\{t \in \operatorname{ran}\alpha \mid t < x_2\})$. If $\{t \in \operatorname{ran}\alpha \mid y < t < x_2\} = \varnothing$, then $\{t \in \operatorname{ran}\alpha \mid t < x_2\} = \{t \in \operatorname{ran}\alpha \mid t < y\}$ and $\{t \in \operatorname{ran}\alpha \mid t > x_2\} = \{t \in \operatorname{ran}\alpha \mid t > y\}$, so $x_2 \in A_y$, contradicting the assumption. Then there exists an element $c \in \operatorname{ran}\alpha$ such that $y < c < x_2$. This implies that $m_y \leq c \leq n_{x_2}$. Hence $x_1\beta = m_y' \leq n_{x_2}' = x_2\beta$.

Subcase 4.4: $m_y = \min(\{t \in \operatorname{ran}\alpha \mid t > y\})$ and $n_{x_2} = \min(\{t \in \operatorname{ran}\alpha \mid t > x_2\})$. Since $\{t \in \operatorname{ran}\alpha \mid t > y\} = \{t \in \operatorname{ran}\alpha \mid t > x_1\} \supseteq \{t \in \operatorname{ran}\alpha \mid t > x_2\}$, we have $m_y \leq n_{x_2}$, so $x_1\beta = m_y' \leq n_{x_2}' = x_2\beta$.

Case 5: $x_1, x_2 \in X \smallsetminus \big(\operatorname{ran}\alpha \cup \operatorname{ub}(\operatorname{ran}\alpha) \cup \operatorname{lb}(\operatorname{ran}\alpha)\big)$, $x_1 \notin A_y$ for all $y \in \big(X \smallsetminus (\operatorname{ran}\alpha \cup \operatorname{ub}(\operatorname{ran}\alpha) \cup \operatorname{lb}(\operatorname{ran}\alpha))\big) \cap Y$ and $x_2 \in A_y$ for some $y \in \big(X \smallsetminus (\operatorname{ran}\alpha \cup \operatorname{ub}(\operatorname{ran}\alpha) \cup \operatorname{lb}(\operatorname{ran}\alpha))\big) \cap Y$. Then $\{t \in \operatorname{ran}\alpha \mid t < x_2\} = \{t \in \operatorname{ran}\alpha \mid t < y\}$ and $\{t \in \operatorname{ran}\alpha \mid t > x_2\} = \{t \in \operatorname{ran}\alpha \mid t > y\}$.

Subcase 5.1: $n_{x_1} = \max(\{t \in \operatorname{ran}\alpha \mid t < x_1\})$ and $m_y = \max(\{t \in \operatorname{ran}\alpha \mid t < y\})$. Since $\{t \in \operatorname{ran}\alpha \mid t < x_1\} \subseteq \{t \in \operatorname{ran}\alpha \mid t < x_2\} = \{t \in \operatorname{ran}\alpha \mid t < y\}$, we obtain that $n_{x_1} \leq m_y$. Then $x_1\beta = n_{x_1}{}' \leq m_y{}' = x_2\beta$.

Subcase 5.2: $n_{x_1} = \max(\{t \in \operatorname{ran}\alpha \mid t < x_1\})$ and $m_y = \min(\{t \in \operatorname{ran}\alpha \mid t > y\})$. Then $n_{x_1} \in \{t \in \operatorname{ran}\alpha \mid t < x_1\}$ and $m_y \in \{t \in \operatorname{ran}\alpha \mid t > y\} = \{t \in \operatorname{ran}\alpha \mid t > x_2\}$. Thus $n_{x_1} < x_1 < x_2 < m_y$, so $x_1\beta = n_{x_1}{}' < m_y{}' = x_2\beta$.

Subcase 5.3: $n_{x_1} = \min(\{t \in \operatorname{ran}\alpha \mid t > x_1\})$ and $m_y = \max(\{t \in \operatorname{ran}\alpha \mid t < y\})$. If $\{t \in \operatorname{ran}\alpha \mid x_1 < t < y\} = \varnothing$, then $\{t \in \operatorname{ran}\alpha \mid t < x_1\} = \{t \in \operatorname{ran}\alpha \mid t < y\}$ and $\{t \in \operatorname{ran}\alpha \mid t > x_1\} = \{t \in \operatorname{ran}\alpha \mid t > y\}$, so $x_1 \in A_y$, a contradiction. Then there is an element $c \in \operatorname{ran}\alpha$ such that $x_1 < c < y$. This implies that $n_{x_1} \leq c \leq m_y$, and thus $x_1\beta = n_{x_1}{}' \leq m_y{}' = x_2\beta$.

Subcase 5.4: $n_{x_1} = \min(\{t \in \operatorname{ran}\alpha \mid t > x_1\})$ and $m_y = \min(\{t \in \operatorname{ran}\alpha \mid t > y\})$. Since $\{t \in \operatorname{ran}\alpha \mid t > x_1\} \supseteq \{t \in \operatorname{ran}\alpha \mid t > x_2\} = \{t \in \operatorname{ran}\alpha \mid t > y\}$, it follows that $n_{x_1} \leq m_y$. Hence $x_1\beta = n_{x_1}{}' \leq m_y{}' = x_2\beta$.

Case 6: $x_1, x_2 \in X \setminus \big(\operatorname{ran}\alpha \cup \operatorname{ub}(\operatorname{ran}\alpha) \cup \operatorname{lb}(\operatorname{ran}\alpha)\big)$, $x_1 \notin A_y$ for all $y \in \big(X \setminus (\operatorname{ran}\alpha \cup \operatorname{ub}(\operatorname{ran}\alpha) \cup \operatorname{lb}(\operatorname{ran}\alpha))\big) \cap Y$ and $x_2 \notin A_y$ for all $y \in \big(X \setminus (\operatorname{ran}\alpha \cup \operatorname{ub}(\operatorname{ran}\alpha) \cup \operatorname{lb}(\operatorname{ran}\alpha))\big) \cap Y$.

Subcase 6.1: $n_{x_1} = \max(\{t \in \operatorname{ran}\alpha \mid t < x_1\})$ and $n_{x_2} = \max(\{t \in \operatorname{ran}\alpha \mid t < x_2\})$. Since $\{t \in \operatorname{ran}\alpha \mid t < x_1\} \subseteq \{t \in \operatorname{ran}\alpha \mid t < x_2\}$, we have $n_{x_1} \leq n_{x_2}$, so $x_1\beta = {n_{x_1}}' \leq {n_{x_2}}' = x_2\beta$.

Subcase 6.2: $n_{x_1} = \max(\{t \in \operatorname{ran}\alpha \mid t < x_1\})$ and $n_{x_2} = \min(\{t \in \operatorname{ran}\alpha \mid t > x_2\})$. Then $n_{x_1} < x_1 < x_2 < n_{x_2}$, so $x_1\beta = {n_{x_1}}' < {n_{x_2}}' = x_2\beta$.

Subcase 6.3: $n_{x_1} = \min(\{t \in \operatorname{ran}\alpha \mid t > x_1\})$ and $n_{x_2} = \max(\{t \in \operatorname{ran}\alpha \mid t < x_2\})$. Then $\{t \in \operatorname{ran}\alpha \mid t < x_1\}$ has no maximum. It follows that $\{t \in \operatorname{ran}\alpha \mid t < x_1\} \subsetneq \{t \in \operatorname{ran}\alpha \mid t < x_2\}$. Hence $x_1 < c < x_2$ for some $c \in \operatorname{ran}\alpha$. This implies that $n_{x_1} \leq c \leq n_{x_2}$, and thus $x_1\beta = {n_{x_1}}' \leq {n_{x_2}}' = x_2\beta$.

Subcase 6.4: $n_{x_1} = \min(\{t \in \operatorname{ran}\alpha \mid t > x_1\})$ and $n_{x_2} = \min(\{t \in \operatorname{ran}\alpha \mid t > x_2\})$. Since $\{t \in \operatorname{ran}\alpha \mid t > x_1\} \supseteq \{t \in \operatorname{ran}\alpha \mid t > x_2\}$, it follows that $n_{x_1} \leq n_{x_2}$, and hence $x_1\beta = {n_{x_1}}' \leq {n_{x_2}}' = x_2\beta$.

Hence $\beta \in \overline{OT}(X, Y)$, and the theorem is completely proved. □

As an immediate consequence of Theorem 3.1.3, we have

Corollary 3.1.4. *Let* $\alpha \in \overline{OT}(X, Y)$ *be such that* $\operatorname{ran}\alpha$ *is finite. Then* $\alpha \in \operatorname{Reg}(\overline{OT}(X, Y))$ *if and only if the following four conditions hold.*

(i) $\operatorname{ran}\alpha \cap Y = Y\alpha$.

(ii) *If* $\operatorname{ub}(\operatorname{ran}\alpha) \cap Y \neq \varnothing$, *then* $\max(\operatorname{ran}\alpha) \in Y$.

(iii) *If* $\operatorname{lb}(\operatorname{ran}\alpha) \cap Y \neq \varnothing$, *then* $\min(\operatorname{ran}\alpha) \in Y$.

(iv) *If* $x \in \big(X \setminus (\operatorname{ran}\alpha \cup \operatorname{ub}(\operatorname{ran}\alpha) \cup \operatorname{lb}(\operatorname{ran}\alpha))\big) \cap Y$, *then* $\max(\{t \in \operatorname{ran}\alpha \mid t < x\}) \in Y$ *or* $\min(\{t \in \operatorname{ran}\alpha \mid t > x\}) \in Y$.

The following result which is obtained from Theorem 2.1.2 and Theorem 3.1.3 shows that any nonregular element of $OT(X, Y)$ cannot be a regular element of $\overline{OT}(X, Y)$.

Corollary 3.1.5. $\operatorname{Reg}(\overline{OT}(X,Y) \subseteq \operatorname{Reg}(OT(X,Y)) \cup (\overline{OT}(X,Y) \smallsetminus OT(X,Y))$, *or equivalently,*

$$OT(X,Y) \smallsetminus \operatorname{Reg}(OT(X,Y)) \subseteq \overline{OT}(X,Y) \smallsetminus \operatorname{Reg}(\overline{OT}(X,Y)).$$

Proof. Let $\alpha \in \operatorname{Reg}(\overline{OT}(X,Y))$ and assume that $\alpha \in OT(X,Y)$. Then $\operatorname{ran}\alpha \cap Y = Y\alpha$ by Theorem 3.1.3 and $\operatorname{ran}\alpha \subseteq Y$. Combining these two facts, we have that $\operatorname{ran}\alpha = Y\alpha$, i.e., α satisfies (i) of Theorem 2.1.2. Also, by Theorem 3.1.3, α satisfies (ii), (iii) and (iv) of Theorem 2.1.2. Hence $\alpha \in \operatorname{Reg}(OT(X,Y))$ by Theorem 2.1.2. □

From the second inclusion of Corollary 3.1.5, we directly obtain the following fact.

Corollary 3.1.6. *If $\overline{OT}(X,Y)$ is a regular semigroup, then $OT(X,Y)$ is also regular.*

Next, we characterize when $\overline{OT}(X,Y)$ is a regular semigroup. For our required result, the following lemmas are needed.

Lemma 3.1.7. *If $|Y| = 1$ and $\overline{OT}(X,Y) \subseteq \operatorname{Reg}(OT(X))$, then $\overline{OT}(X,Y)$ is a regular semigroup.*

Proof. Assume that $|Y| = 1$ and $\overline{OT}(X,Y) \subseteq \operatorname{Reg}(OT(X))$. Let $Y = \{c\}$. To show that $\overline{OT}(X,Y)$ is regular, let $\alpha \in \overline{OT}(X,Y)$. Then $\alpha \in \operatorname{Reg}(OT(X))$ and $c\alpha = c$. Thus $Y = \{c\} \subseteq \operatorname{ran}\alpha$, so $\operatorname{ran}\alpha \cap Y = Y = Y\alpha$. Hence α satisfies (i) of Theorem 3.1.3. Since $\alpha \in \operatorname{Reg}(OT(X))$, it follows from Theorem 1.6 that α satisfies the first part of (ii), (iii) and (iv) in Theorem 3.1.3. If $\operatorname{ub}(\operatorname{ran}\alpha) \cap Y \neq \varnothing$, then $c \in \operatorname{ub}(\operatorname{ran}\alpha)$, so $\max(\operatorname{ran}\alpha) = c \in Y$ since $c \in \operatorname{ran}\alpha$. This shows that α satisfies the second part of (ii) in Theorem 3.1.3. Similarly, if $\operatorname{lb}(\operatorname{ran}\alpha) \cap Y \neq \varnothing$, then $\min(\operatorname{ran}\alpha) = c \in Y$, so α satisfies the second part of (iii) in Theorem 3.1.3. Since $\big(X \smallsetminus (\operatorname{ran}\alpha \cup \operatorname{ub}(\operatorname{ran}\alpha) \cup \operatorname{lb}(\operatorname{ran}\alpha))\big) \cap Y = \varnothing$, we immediately obtain the second part of (iv) in Theorem 3.1.3. Hence by Theorem 3.1.3, $\alpha \in \operatorname{Reg}(\overline{OT}(X,Y))$. □

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Lemma 3.1.8. *Let* $|Y| = 2$*. If* $\overline{OT}(X,Y) \subseteq \mathrm{Reg}(OT(X))$*,* $\min(X)$ *and* $\max(X)$ *exist and* $Y = \{\min(X), \max(X)\}$*, then* $\overline{OT}(X,Y)$ *is a regular semigroup.*

Proof. Assume that $\overline{OT}(X,Y) \subseteq \mathrm{Reg}(OT(X))$, $\min(X)$ and $\max(X)$ exist and $Y = \{\min(X), \max(X)\}$. Let $\alpha \in \overline{OT}(X,Y)$. Then $|Y\alpha| = 1$ or $|Y\alpha| = 2$ because $|Y| = 2$.

Case 1: $|Y\alpha| = 1$. Then $Y\alpha = \{\min(X)\}$ or $Y\alpha = \{\max(X)\}$. If $Y\alpha = \{\min(X)\}$, then $\min(X)\alpha = \max(X)\alpha = \min(X)$. Since α is order-preserving,

$$\min(X) = \min(X)\alpha \leq x\alpha \leq \max(X)\alpha = \min(X) \text{ for all } x \in X,$$

and we deduce that $x\alpha = \min(X)$ for all $x \in X$. Hence $\alpha^2 = \alpha$, so $\alpha \in \mathrm{Reg}(\overline{OT}(X,Y))$. Likewise, if $Y\alpha = \{\max(X)\}$, then $x\alpha = \max(X)$ for all $x \in X$ and it follows that $\alpha \in \mathrm{Reg}(\overline{OT}(X,Y))$.

Case 2: $|Y\alpha| = 2$. Then $Y\alpha = Y$. Since α is order-preserving, we have $\min(X)\alpha = \min(X)$ and $\max(X)\alpha = \max(X)$. It follows that $\mathrm{ran}\,\alpha \cap Y = Y = Y\alpha$, $\min(\mathrm{ran}\,\alpha) = \min(X) \in Y$ and $\max(\mathrm{ran}\,\alpha) = \max(X) \in Y$. This implies that α satisfies (i), (ii) and (iii) of Theorem 3.1.3. We have $\alpha \in \mathrm{Reg}(OT(X))$ by assumption. Then Theorem 1.6 together with the fact that $\big(X \setminus (\mathrm{ran}\,\alpha \cup \mathrm{ub}(\mathrm{ran}\,\alpha) \cup \mathrm{lb}(\mathrm{ran}\,\alpha))\big) \cap Y = \varnothing$ implies that α also satisfies (iv) of Theorem 3.1.3. Hence $\alpha \in \mathrm{Reg}(\overline{OT}(X,Y))$ by Theorem 3.1.3.

This shows that $\overline{OT}(X,Y)$ is a regular semigroup, so the proof is complete. □

Theorem 3.1.9. *The semigroup* $\overline{OT}(X,Y)$ *is a regular semigroup if and only if* $\overline{OT}(X,Y) \subseteq \mathrm{Reg}(OT(X))$ *and one of the following conditions holds.*

(i) $Y = X$.

(ii) $|Y| = 1$.

(iii) $|Y| = 2$, $\min(X)$ *and* $\max(X)$ *exist, and* $Y = \{\min(X), \max(X)\}$.

Proof. Assume that $\overline{OT}(X,Y)$ is regular. Then $\mathrm{Reg}(\overline{OT}(X,Y)) = \overline{OT}(X,Y)$, and by Corollary 3.1.6, $OT(X,Y)$ is regular. Since $\overline{OT}(X,Y)$ is a subsemigroup of $OT(X)$, it follows that $\overline{OT}(X,Y) = \mathrm{Reg}(\overline{OT}(X,Y)) \subseteq \mathrm{Reg}(OT(X))$. Suppose that (i) and (ii) are false. Then $Y \subsetneq X$ and $|Y| \geq 2$. Then the regularity of $OT(X,Y)$ and Lemma 2.1.6 imply that $|Y| = 2$. Let $Y = \{e, f\}$ be such that $e < f$. Since $OT(X,Y)$ is regular, by Lemma 2.1.5, we have for every $a \in X$, $a \not> Y$ and $a \not< Y$. Thus $e \leq a \leq f$ for all $a \in X$. This implies that $e = \min(X)$ and $f = \max(X)$.

Conversely, $\overline{OT}(X,Y)$ is obviously regular if $\overline{OT}(X,Y) \subseteq \operatorname{Reg}(OT(X))$ and $Y = X$. If $\overline{OT}(X,Y) \subseteq \operatorname{Reg}(OT(X))$ and $|Y| = 1$, then by Lemma 3.1.7, $\overline{OT}(X,Y)$ is regular. Also, by Lemma 3.1.8, $\overline{OT}(X,Y)$ is regular if $\overline{OT}(X,Y) \subseteq \operatorname{Reg}(OT(X))$ and (iii) holds.

Hence the theorem is proved. □

Next, the regular elements of the semigroups $\overline{OP}(X,Y)$ and $\overline{OI}(X,Y)$ are characterized.

Theorem 3.1.10. *For* $\alpha \in \overline{OP}(X,Y)$, $\alpha \in \operatorname{Reg}(\overline{OP}(X,Y))$ *if and only if* $\operatorname{ran}\alpha \cap Y = (\operatorname{dom}\alpha \cap Y)\alpha$.

Proof. Assume that $\alpha \in \operatorname{Reg}(\overline{OP}(X,Y))$. Since $(\operatorname{dom}\alpha \cap Y)\alpha \subseteq Y$, we have that $(\operatorname{dom}\alpha \cap Y)\alpha \subseteq \operatorname{ran}\alpha \cap Y$. To show that $\operatorname{ran}\alpha \cap Y \subseteq (\operatorname{dom}\alpha \cap Y)\alpha$, let $\beta \in \overline{OP}(X,Y)$ be such that $\alpha = \alpha\beta\alpha$ and let $x \in \operatorname{ran}\alpha \cap Y$. Then $x = a\alpha$ for some $a \in \operatorname{dom}\alpha$. Thus $x = a\alpha = a\alpha\beta\alpha = x\beta\alpha$ which implies that $x \in \operatorname{dom}\beta$ and $x\beta \in \operatorname{dom}\alpha$. It follows that $x \in \operatorname{dom}\beta \cap Y$ and hence $x\beta \in (\operatorname{dom}\beta \cap Y)\beta \subseteq Y$. We then deduce that $x\beta \in \operatorname{dom}\alpha \cap Y$. Consequently, $x = x\beta\alpha \in (\operatorname{dom}\alpha \cap Y)\alpha$. This proves that $\operatorname{ran}\alpha \cap Y = (\operatorname{dom}\alpha \cap Y)\alpha$.

For the converse, assume that $\operatorname{ran}\alpha \cap Y = (\operatorname{dom}\alpha \cap Y)\alpha$. Then $x\alpha^{-1} \cap Y \neq \varnothing$ for all $x \in \operatorname{ran}\alpha \cap Y$. For each $x \in \operatorname{ran}\alpha \cap Y$, choose $d_x \in x\alpha^{-1} \cap Y$ and for each $x \in \operatorname{ran}\alpha \smallsetminus Y$, choose $e_x \in x\alpha^{-1}$. Then $d_x\alpha = x$ for all $x \in \operatorname{ran}\alpha \cap Y$ and $e_x\alpha = x$ for all $x \in \operatorname{ran}\alpha \smallsetminus Y$. Define $\beta : \operatorname{ran}\alpha \to \operatorname{dom}\alpha$ by

$$\beta = \begin{pmatrix} x & u \\ d_x & e_u \end{pmatrix}_{\substack{x \in \operatorname{ran}\alpha \cap Y \\ u \in \operatorname{ran}\alpha \smallsetminus Y}}.$$

Then $(\operatorname{dom}\beta \cap Y)\beta = (\operatorname{ran}\alpha \cap Y)\beta = \{d_x \mid x \in \operatorname{ran}\alpha \cap Y\} \subseteq Y$. Since $\alpha \in OP(X)$, it follows from Proposition 1.11 that β is order-preserving. Hence $\beta \in \overline{OP}(X,Y)$.

Since for $x \in \operatorname{dom}\alpha, x\alpha \in \operatorname{dom}\beta$ and $x\alpha\beta \in \operatorname{dom}\alpha$, we deduce that $\operatorname{dom}\alpha = \operatorname{dom}(\alpha\beta\alpha)$. If $x \in \operatorname{dom}\alpha$, then

$$x\alpha\beta\alpha = \begin{cases} d_{x\alpha}\alpha = x\alpha & \text{if } \ x\alpha \in Y, \\ e_{x\alpha}\alpha = x\alpha & \text{if } \ x\alpha \notin Y, \end{cases}$$

so $\alpha = \alpha\beta\alpha$. Thus $\alpha \in \operatorname{Reg}(\overline{OP}(X,Y))$, as desired. □

It can be seen that β constructed in the proof of Theorem 3.1.10 is 1-1. Then $\beta \in \overline{OI}(X,Y)$.

Theorem 3.1.11. *For* $\alpha \in \overline{OI}(X,Y)$, $\alpha \in \operatorname{Reg}(\overline{OI}(X,Y))$ *if and only if* $(\operatorname{ran}\alpha \cap Y)\alpha^{-1} \subseteq Y$.

Proof. Assume that $\alpha \in \operatorname{Reg}(\overline{OI}(X,Y))$. Since $\overline{OI}(X,Y)$ is a subsemigroup of $\overline{OP}(X,Y)$, we have that $\alpha \in \operatorname{Reg}(\overline{OP}(X,Y))$. By Theorem 3.1.10, $\operatorname{ran}\alpha \cap Y = (\operatorname{dom}\alpha \cap Y)\alpha$. Then $(\operatorname{ran}\alpha \cap Y)\alpha^{-1} = (\operatorname{dom}\alpha \cap Y)\alpha\alpha^{-1}$. Since $\alpha\alpha^{-1}$ is the identity mapping on $\operatorname{dom}\alpha$, it follows that $(\operatorname{dom}\alpha \cap Y)\alpha\alpha^{-1} = \operatorname{dom}\alpha \cap Y$. Hence $(\operatorname{ran}\alpha \cap Y)\alpha^{-1} = \operatorname{dom}\alpha \cap Y \subseteq Y$.

Conversely, assume that $(\operatorname{ran}\alpha \cap Y)\alpha^{-1} \subseteq Y$. But $(\operatorname{ran}\alpha \cap Y)\alpha^{-1} \subseteq \operatorname{dom}\alpha$, so $(\operatorname{ran}\alpha \cap Y)\alpha^{-1} \subseteq \operatorname{dom}\alpha \cap Y$. Thus $(\operatorname{ran}\alpha \cap Y)\alpha^{-1}\alpha \subseteq (\operatorname{dom}\alpha \cap Y)\alpha \subseteq \operatorname{ran}\alpha \cap Y$. Since $\alpha^{-1}\alpha$ is the identity mapping on $\operatorname{ran}\alpha$, we have that $(\operatorname{ran}\alpha \cap Y)\alpha^{-1}\alpha = \operatorname{ran}\alpha \cap Y$. Therefore $(\operatorname{dom}\alpha \cap Y)\alpha = \operatorname{ran}\alpha \cap Y$. From the proof of Theorem 3.1.10, $\alpha = \alpha\beta\alpha$ for some $\beta \in \overline{OI}(X,Y)$. Hence $\alpha \in \operatorname{Reg}(\overline{OI}(X,Y))$, as desired. □

We provide a different version in determining $\operatorname{Reg}(\overline{OI}(X,Y))$ as follows:

Theorem 3.1.12. *For* $\alpha \in \overline{OI}(X,Y)$, $\alpha \in \operatorname{Reg}(\overline{OI}(X,Y))$ *if and only if* $\big(\operatorname{dom}\alpha \cap (X \setminus Y)\big)\alpha \subseteq X \setminus Y$.

Proof. It suffices to show that $(\operatorname{ran}\alpha \cap Y)\alpha^{-1} \subseteq Y$ if and only if $\big(\operatorname{dom}\alpha \cap (X \setminus Y)\big)\alpha \subseteq X \setminus Y$. Suppose first that $(\operatorname{ran}\alpha \cap Y)\alpha^{-1} \subseteq Y$. Let $x \in \operatorname{dom}\alpha \cap (X \setminus Y)$. Then $x\alpha \in \operatorname{ran}\alpha$. If $x\alpha \in Y$, then $x \in (\operatorname{ran}\alpha \cap Y)\alpha^{-1} \subseteq Y$, a contradiction. Hence $x\alpha \in X \setminus Y$, proving that $\big(\operatorname{dom}\alpha \cap (X \setminus Y)\big)\alpha \subseteq X \setminus Y$. Now suppose that $\big(\operatorname{dom}\alpha \cap (X \setminus Y)\big)\alpha \subseteq X \setminus Y$. Let $x \in (\operatorname{ran}\alpha \cap Y)\alpha^{-1}$. Then $x \in \operatorname{dom}\alpha$ and $x\alpha \in Y$. If $x \in X \setminus Y$, then $x\alpha \in \big(\operatorname{dom}\alpha \cap (X \setminus Y)\big)\alpha \subseteq X \setminus Y$, a contradiction. Thus $x \in Y$. This proves that $(\operatorname{ran}\alpha \cap Y)\alpha^{-1} \subseteq Y$. □

As a consequence of Theorem 3.1.10 and Theorem 3.1.11, a necessary and sufficient condition for the $\overline{OP}(X,Y)$ and $\overline{OI}(X,Y)$ to be regular semigroups can be given as follows:

Corollary 3.1.13. *Let $\overline{OS}(X,Y)$ be $\overline{OP}(X,Y)$ or $\overline{OI}(X,Y)$. Then $\overline{OS}(X,Y)$ is a regular semigroup if and only if $Y = X$.*

Proof. Suppose that $Y \subsetneq X$. Let $a \in X \setminus Y$ and $b \in Y$. Then $\alpha = \binom{a}{b} \in \overline{OS}(X,Y)$. Since $\operatorname{dom}\alpha \cap Y = \varnothing, \operatorname{ran}\alpha \cap Y = \{b\}$ and $b\alpha^{-1} = a \notin Y$, by Theorem 3.1.10 and Theorem 3.1.11, $\alpha \notin \operatorname{Reg}(\overline{OS}(X,Y))$. If $Y = X$, then $\overline{OP}(X,Y) = OP(X)$, $\overline{OI}(X,Y) = OI(X)$, and both $OP(X)$ and $OI(X)$ are regular semigroups by Theorem 1.5. Hence the result follows. □

3.2 Combinatorial Results on Regular Elements

Throughout this section, let $X = \{1,2,\ldots,n\}$ and $Y = \{1,2,\ldots,m\}$ where $m \le n$.

First of all, we determine the cardinalities of $\overline{OT}(X,Y)$, $\overline{OP}(X,Y)$ and $\overline{OI}(X,Y)$ and then we investigate the numbers of the regular elements of $\overline{OT}(X,Y)$, $\overline{OP}(X,Y)$ and $\overline{OI}(X,Y)$. Hence the numbers of the nonregular elements in $\overline{OT}(X,Y)$, $\overline{OP}(X,Y)$ and $\overline{OI}(X,Y)$ are directly obtained.

The following two lemmas given in [17] are needed to obtain our required results.

Lemma 3.2.1. ([17]) *For $r, s, k \in Y$,*

$$|\{\alpha \in OP(Y) \mid |\operatorname{dom}\alpha| = r, |\operatorname{ran}\alpha| = s \text{ and } \max(\operatorname{ran}\alpha) = k\}|$$

$$= \binom{m}{r}\binom{k-1}{s-1}\binom{r-1}{s-1}.$$

Lemma 3.2.2. ([17]) *For* $r, k \in Y$,

$$|\{\alpha \in OP(Y) \mid |\operatorname{dom}\alpha| = r \textit{ and } \max(\operatorname{ran}\alpha) = k\}| = \binom{m}{r}\binom{k+r-2}{k-1}.$$

Theorem 3.2.3. $|\overline{OT}(X,Y)| = \displaystyle\sum_{k=1}^{m}\binom{k+m-2}{k-1}\binom{2n-k-m}{n-m}$.

Proof. We see that

$$\begin{aligned}\overline{OT}(X,Y) &= \bigcup_{k=1}^{m}\{\alpha \in \overline{OT}(X,Y) \mid \max(Y\alpha) = k\} \\ &= \bigcup_{k=1}^{m}\big\{\alpha \in OT(X) \mid Y\alpha \subseteq Y, \max(Y\alpha) = k \text{ and } \\ &\qquad\qquad (X \smallsetminus Y)\alpha \subseteq \{k,\ldots,n\}\big\}.\end{aligned}$$

It follows from Proposition 1.12 and Proposition 1.13 that for $1 \le k \le m$,

$$\begin{aligned}&\big\{\alpha \in OT(X) \mid Y\alpha \subseteq Y, \max(Y\alpha) = k \text{ and } (X \smallsetminus Y)\alpha \subseteq \{k,\ldots,n\}\big\} \\ &= \big\{\alpha_1 \cup \alpha_2 \mid \alpha_1 \in OT(Y), \max(\operatorname{ran}\alpha_1) = k \text{ and } \alpha_2 \in OT(X \smallsetminus Y, \{k,\ldots,n\})\big\}.\end{aligned}$$

Then we get

$$\begin{aligned}\overline{OT}(X,Y) &= \bigcup_{k=1}^{m}\big\{\alpha_1 \cup \alpha_2 \mid \alpha_1 \in OT(Y), \max(\operatorname{ran}\alpha_1) = k \text{ and } \\ &\qquad\qquad \alpha_2 \in OT(X \smallsetminus Y, \{k,\ldots,n\})\big\}.\end{aligned}$$

For $1 \le k \le m$, we have

$$\begin{aligned}&\big|\big\{\alpha_1 \cup \alpha_2 \mid \alpha_1 \in OT(Y), \max(\operatorname{ran}\alpha_1) = k \text{ and } \alpha_2 \in OT(X \smallsetminus Y, \{k,\ldots,n\})\big\}\big| \\ &= \big|\{\alpha \in OT(Y) \mid \max(\operatorname{ran}\alpha) = k\}\big|\,\big|OT(X \smallsetminus Y, \{k,\ldots,n\})\big| \\ &= \big|\{\alpha \in OP(Y) \mid |\operatorname{dom}\alpha| = m \text{ and } \max(\operatorname{ran}\alpha) = k\}\big|\,\big|OT(X \smallsetminus Y, \{k,\ldots,n\})\big| \\ &= \binom{m}{m}\binom{k+m-2}{k-1}\binom{(n-k+1)+(n-m)-1}{n-m} \quad \text{by Lemma 3.2.2 and Theorem 2.2.3(i)} \\ &= \binom{k+m-2}{k-1}\binom{2n-m-k}{n-m}.\end{aligned}$$

Hence

$$|\overline{OT}(X,Y)| \;=\; \sum_{k=1}^{m} \binom{k+m-2}{k-1}\binom{2n-m-k}{n-m}.$$

□

Theorem 3.2.4.

$$|\overline{OP}(X,Y)| \;=\; \sum_{r=0}^{n-m} \binom{n-m}{r}\binom{n+r-1}{r} +$$
$$\sum_{s=1}^{m}\sum_{k=1}^{m} \binom{m}{s}\binom{k+s-2}{k-1} \sum_{t=0}^{n-m} \binom{n-m}{t}\binom{n-k+t}{t}.$$

Proof. We have

$$\begin{aligned}
\overline{OP}(X,Y) \;&=\; \{\alpha \in \overline{OP}(X,Y) \mid \operatorname{dom}\alpha \subseteq X \smallsetminus Y\} \cup \\
&\quad\; \{\alpha \in \overline{OP}(X,Y) \mid \varnothing \neq \operatorname{dom}\alpha \subseteq Y\} \cup \\
&\quad\; \{\alpha \in \overline{OP}(X,Y) \mid \operatorname{dom}\alpha \cap Y \neq \varnothing \text{ and } \operatorname{dom}\alpha \cap (X \smallsetminus Y) \neq \varnothing\} \\
&=\; OP(X \smallsetminus Y, X) \cup OP(Y) \smallsetminus \{0\} \cup \\
&\quad\; \{\alpha \in \overline{OP}(X,Y) \mid \operatorname{dom}\alpha \cap Y \neq \varnothing \text{ and } \operatorname{dom}\alpha \cap (X \smallsetminus Y) \neq \varnothing\}.
\end{aligned} \tag{1}$$

We know from Theorem 2.2.3(ii) that

$$|OP(X \smallsetminus Y, X)| = \sum_{r=0}^{n-m} \binom{n-m}{r}\binom{n+r-1}{r} \tag{2}$$

and

$$|OP(Y) \smallsetminus \{0\}| \;=\; \sum_{s=1}^{m} \binom{m}{s}\binom{m+s-1}{s}. \tag{3}$$

To obtain the cardinality of $\overline{OP}(X,Y)$, it remains to find $\big|\{\alpha \in \overline{OP}(X,Y) \mid \operatorname{dom}\alpha \cap Y \neq \varnothing \text{ and } \operatorname{dom}\alpha \cap (X \smallsetminus Y) \neq \varnothing\}\big|$. We see that

$$\begin{aligned}
&\{\alpha \in \overline{OP}(X,Y) \mid \operatorname{dom}\alpha \cap Y \neq \varnothing \text{ and } \operatorname{dom}\alpha \cap (X \smallsetminus Y) \neq \varnothing\} \\
&\quad = \bigcup_{s=1}^{m} \bigcup_{t=1}^{n-m} \bigcup_{k=1}^{m} \big\{\alpha \in \overline{OP}(X,Y) \mid |\operatorname{dom}\alpha \cap Y| = s, |\operatorname{dom}\alpha \cap (X \smallsetminus Y)| = t \\
&\qquad\qquad\qquad \text{and } \max((\operatorname{dom}\alpha \cap Y)\alpha) = k\big\}
\end{aligned}$$

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$$= \bigcup_{s=1}^{m} \bigcup_{t=1}^{n-m} \bigcup_{k=1}^{m} \big\{\alpha \in OP(X) \mid |\operatorname{dom}\alpha \cap Y| = s, |\operatorname{dom}\alpha \cap (X \smallsetminus Y)| = t, (\operatorname{dom}\alpha \cap Y)\alpha \subseteq Y, \max((\operatorname{dom}\alpha \cap Y)\alpha) = k \text{ and } (\operatorname{dom}\alpha \cap (X \smallsetminus Y))\alpha \subseteq \{k, \ldots, n\}\big\}.$$

For $1 \leq s \leq m, 1 \leq t \leq n-m$ and $1 \leq k \leq m$, we have from Proposition 1.12 and Proposition 1.13 that

$$\big\{\alpha \in OP(X) \mid |\operatorname{dom}\alpha \cap Y| = s, |\operatorname{dom}\alpha \cap (X \smallsetminus Y)| = t, (\operatorname{dom}\alpha \cap Y)\alpha \subseteq Y, \max((\operatorname{dom}\alpha \cap Y)\alpha) = k \text{ and } (\operatorname{dom}\alpha \cap (X \smallsetminus Y))\alpha \subseteq \{k, \ldots, n\}\big\}$$

$$= \big\{\alpha_1 \cup \alpha_2 \mid \alpha_1 \in OP(Y), |\operatorname{dom}\alpha_1| = s, \max(\operatorname{ran}\alpha_1) = k, \alpha_2 \in OP(X \smallsetminus Y, \{k, \ldots, n\}) \text{ and } |\operatorname{dom}\alpha_2| = t\big\}.$$

From this, we get

$$\Big|\big\{\alpha \in OP(X) \mid |\operatorname{dom}\alpha \cap Y| = s, |\operatorname{dom}\alpha \cap (X \smallsetminus Y)| = t, (\operatorname{dom}\alpha \cap Y)\alpha \subseteq Y, \max((\operatorname{dom}\alpha \cap Y)\alpha) = k \text{ and } (\operatorname{dom}\alpha \cap (X \smallsetminus Y))\alpha \subseteq \{k, \ldots, n\}\big\}\Big|$$

$$= \Big|\big\{\alpha \in OP(Y) \mid |\operatorname{dom}\alpha| = s \text{ and } \max(\operatorname{ran}\alpha) = k\big\}\Big| \cdot \Big|\big\{\alpha \in OP(X \smallsetminus Y, \{k, \ldots, n\}) \mid |\operatorname{dom}\alpha| = t\big\}\Big|$$

$$= \binom{m}{s}\binom{k+s-2}{k-1}\binom{n-m}{t}\binom{(n-k+1)+t-1}{t} \quad \text{by Lemma 3.2.2 and Lemma 2.2.2}$$

$$= \binom{m}{s}\binom{k+s-2}{k-1}\binom{n-m}{t}\binom{n-k+t}{t}.$$

This shows that for $1 \leq s \leq m, 1 \leq t \leq n-m$ and $1 \leq k \leq m$,

$$\Big|\big\{\alpha \in OP(X) \mid |\operatorname{dom}\alpha \cap Y| = s, |\operatorname{dom}\alpha \cap (X \smallsetminus Y)| = t, (\operatorname{dom}\alpha \cap Y)\alpha \subseteq Y, \max((\operatorname{dom}\alpha \cap Y)\alpha) = k \text{ and } (\operatorname{dom}\alpha \cap (X \smallsetminus Y))\alpha \subseteq \{k, \ldots, n\}\big\}\Big|$$

$$= \binom{m}{s}\binom{k+s-2}{k-1}\binom{n-m}{t}\binom{n-k+t}{t}.$$

Consequently,

$$\begin{aligned}
&\left|\{\alpha \in \overline{OP}(X,Y) \mid \operatorname{dom}\alpha \cap Y \neq \varnothing \text{ and } \operatorname{dom}\alpha \cap (X \smallsetminus Y) \neq \varnothing\}\right| \\
&= \sum_{s=1}^{m}\sum_{t=1}^{n-m}\sum_{k=1}^{m}\binom{m}{s}\binom{k+s-2}{k-1}\binom{n-m}{t}\binom{n-k+t}{t} \\
&= \sum_{s=1}^{m}\sum_{k=1}^{m}\binom{m}{s}\binom{k+s-2}{k-1}\sum_{t=1}^{n-m}\binom{n-m}{t}\binom{n-k+t}{t}. \qquad (4)
\end{aligned}$$

From (1), (2), (3) and (4) and using Result 1.21, we obtain

$$\begin{aligned}
|\overline{OP}(X,Y)| &= \sum_{r=0}^{n-m}\binom{n-m}{r}\binom{n+r-1}{r} + \sum_{s=1}^{m}\binom{m}{s}\binom{m+s-1}{s} + \\
&\quad \sum_{s=1}^{m}\sum_{k=1}^{m}\binom{m}{s}\binom{k+s-2}{k-1}\sum_{t=1}^{n-m}\binom{n-m}{t}\binom{n-k+t}{t} \\
&= \sum_{r=0}^{n-m}\binom{n-m}{r}\binom{n+r-1}{r} + \sum_{s=1}^{m}\binom{m}{s}\sum_{k=1}^{m}\binom{k+s-2}{k-1} + \\
&\quad \sum_{s=1}^{m}\sum_{k=1}^{m}\binom{m}{s}\binom{k+s-2}{k-1}\sum_{t=1}^{n-m}\binom{n-m}{t}\binom{n-k+t}{t} \\
&= \sum_{r=0}^{n-m}\binom{n-m}{r}\binom{n+r-1}{r} + \\
&\quad \sum_{s=1}^{m}\sum_{k=1}^{m}\binom{m}{s}\binom{k+s-2}{k-1}\sum_{t=0}^{n-m}\binom{n-m}{t}\binom{n-k+t}{t}.
\end{aligned}$$

Hence the result follows. □

Theorem 3.2.5. $|\overline{OI}(X,Y)| = \binom{2n-m}{n} + \sum_{s=1}^{m}\sum_{k=s}^{m}\binom{m}{s}\binom{k-1}{s-1}\binom{2n-m-k}{n-k}$.

Proof. We see that

$$\begin{aligned}
\overline{OI}(X,Y) &= \{\alpha \in \overline{OI}(X,Y) \mid \operatorname{dom}\alpha \subseteq X \smallsetminus Y\} \cup \\
&\quad \{\alpha \in \overline{OI}(X,Y) \mid \varnothing \neq \operatorname{dom}\alpha \subseteq Y\} \cup \\
&\quad \{\alpha \in \overline{OI}(X,Y) \mid \operatorname{dom}\alpha \cap Y \neq \varnothing \text{ and } \operatorname{dom}\alpha \cap (X \smallsetminus Y) \neq \varnothing\}
\end{aligned}$$

$$= OI(X \setminus Y, X) \cup OI(Y) \setminus \{0\} \cup$$
$$\{\alpha \in \overline{OI}(X,Y) \mid \operatorname{dom}\alpha \cap Y \neq \varnothing \text{ and } \operatorname{dom}\alpha \cap (X \setminus Y) \neq \varnothing\}. \qquad (1)$$

It follows from Theorem 2.2.3(iii) that

$$|\,OI(X \setminus Y, X)| = \binom{2n-m}{n} \qquad (2)$$

and by Theorem 1.10 and Result 1.22, we have

$$|\,OI(Y) \setminus \{0\}| = \sum_{s=1}^{m} \binom{m}{s}\binom{m}{s} = \sum_{s=1}^{m} \binom{m}{s} \sum_{k=s}^{m} \binom{k-1}{s-1}$$
$$= \sum_{s=1}^{m} \sum_{k=s}^{m} \binom{m}{s}\binom{k-1}{s-1}. \qquad (3)$$

Next, we will find $\big|\{\alpha \in \overline{OI}(X,Y) \mid \operatorname{dom}\alpha \cap Y \neq \varnothing \text{ and } \operatorname{dom}\alpha \cap (X \setminus Y) \neq \varnothing\}\big|$. We see that if $\alpha \in \overline{OI}(X,Y)$ is such that $|\operatorname{dom}\alpha \cap Y| = s$ and $\max((\operatorname{dom}\alpha \cap Y)\alpha) = k$, then $|(\operatorname{dom}\alpha \cap Y)\alpha| = |\operatorname{dom}\alpha \cap Y| = s$. This together with $(\operatorname{dom}\alpha \cap Y)\alpha \subseteq \{1, 2, \ldots, m\}$ implies that $k \geq s$. Then we have

$$\{\alpha \in \overline{OI}(X,Y) \mid \operatorname{dom}\alpha \cap Y \neq \varnothing \text{ and } \operatorname{dom}\alpha \cap (X \setminus Y) \neq \varnothing\}$$
$$= \bigcup_{s=1}^{m} \bigcup_{t=1}^{n-m} \bigcup_{k=s}^{m} \{\alpha \in \overline{OI}(X,Y) \mid |\operatorname{dom}\alpha \cap Y| = s, |\operatorname{dom}\alpha \cap (X \setminus Y)| = t$$
$$\text{and } \max((\operatorname{dom}\alpha \cap Y)\alpha) = k\big\}$$
$$= \bigcup_{s=1}^{m} \bigcup_{t=1}^{n-m} \bigcup_{k=s}^{m} \big\{\alpha \in OI(X) \mid |\operatorname{dom}\alpha \cap Y| = s, |\operatorname{dom}\alpha \cap (X \setminus Y)| = t$$
$$(\operatorname{dom} \cap Y)\alpha \subseteq Y, \max((\operatorname{dom}\alpha \cap Y)\alpha) = k \text{ and}$$
$$(\operatorname{dom}\alpha \cap (X \setminus Y))\alpha \subseteq \{k+1, \ldots, n\}\big\}.$$

For $1 \leq s \leq k \leq m$ and $1 \leq t \leq n-m$, we obtain from Proposition 1.12 and Proposition 1.13 that

$$\big\{\alpha \in OI(X) \mid |\operatorname{dom}\alpha \cap Y| = s, |\operatorname{dom}\alpha \cap (X \setminus Y)| = t, (\operatorname{dom}\alpha \cap Y)\alpha \subseteq Y,$$
$$\max((\operatorname{dom}\alpha \cap Y)\alpha) = k, (\operatorname{dom}\alpha \cap (X \setminus Y))\alpha \subseteq \{k+1, \ldots, n\}\big\}$$
$$= \big\{\alpha_1 \cup \alpha_2 \mid \alpha_1 \in OI(Y), |\operatorname{dom}\alpha_1| = s, \max(\operatorname{ran}\alpha_1) = k \text{ and}$$
$$\alpha_2 \in OI(X \setminus Y, \{k+1, \ldots, n\}) \text{ and } |\operatorname{dom}\alpha_2| = t\big\}$$

$$= \bigl\{\alpha_1 \cup \alpha_2 \mid \alpha_1 \in OP(Y), |\operatorname{dom}\alpha_1| = |\operatorname{ran}\alpha_1| = s, \max(\operatorname{ran}\alpha_1) = k \text{ and } \alpha_2 \in OP(X \smallsetminus Y, \{k+1,\ldots,n\}) \text{ and } |\operatorname{dom}\alpha_2| = |\operatorname{ran}\alpha_2| = t\bigr\}.$$

It follows that

$$\Bigl|\bigl\{\alpha \in OI(X) \mid |\operatorname{dom}\alpha \cap Y| = s, |\operatorname{dom}\alpha \cap (X \smallsetminus Y)| = t, (\operatorname{dom}\alpha \cap Y)\alpha \subseteq Y, \max((\operatorname{dom}\alpha \cap Y)\alpha) = k \text{ and } (\operatorname{dom}\alpha \cap (X \smallsetminus Y))\alpha \subseteq \{k+1,\ldots,n\}\bigr\}\Bigr|$$

$$= \bigl|\{\alpha \in OP(Y) \mid |\operatorname{dom}\alpha| = |\operatorname{ran}\alpha| = s \text{ and } \max(\operatorname{ran}\alpha) = k\}\bigr| \cdot \bigl|\{\alpha \in OP(X \smallsetminus Y, \{k+1,\ldots,n\}) \mid |\operatorname{dom}\alpha| = |\operatorname{ran}\alpha| = t\}\bigr|$$

$$= \binom{m}{s}\binom{k-1}{s-1}\binom{s-1}{s-1}\binom{n-m}{t}\binom{n-k}{t}\binom{t-1}{t-1} \quad \text{by Lemma 3.2.1 and Lemma 2.2.1}$$

$$= \binom{m}{s}\binom{k-1}{s-1}\binom{n-m}{t}\binom{n-k}{t}.$$

This shows that for $1 \le s \le k \le m$ and $1 \le t \le n-m$,

$$\Bigl|\bigl\{\alpha \in OI(X) \mid |\operatorname{dom}\alpha \cap Y| = s, |\operatorname{dom}\alpha \cap (X \smallsetminus Y)| = t, (\operatorname{dom}\alpha \cap Y)\alpha \subseteq Y, \max((\operatorname{dom}\alpha \cap Y)\alpha) = k \text{ and } (\operatorname{dom}\alpha \cap (X \smallsetminus Y))\alpha \subseteq \{k+1,\ldots,n\}\bigr\}\Bigr|$$

$$= \binom{m}{s}\binom{k-1}{s-1}\binom{n-m}{t}\binom{n-k}{t}.$$

Consequently,

$$\Bigl|\{\alpha \in \overline{OI}(X,Y) \mid \operatorname{dom}\alpha \cap Y \neq \varnothing \text{ and } \operatorname{dom}\alpha \cap (X \smallsetminus Y) \neq \varnothing\}\Bigr| = \sum_{s=1}^{m}\sum_{t=1}^{n-m}\sum_{k=s}^{m}\binom{m}{s}\binom{k-1}{s-1}\binom{n-m}{t}\binom{n-k}{t}. \tag{4}$$

From (1), (2), (3) and (4) and using Result 1.20, we obtain

$$|\overline{OI}(X,Y)| = \binom{2n-m}{n} + \sum_{s=1}^{m}\sum_{k=s}^{m}\binom{m}{s}\binom{k-1}{s-1} + \sum_{s=1}^{m}\sum_{t=1}^{n-m}\sum_{k=s}^{m}\binom{m}{s}\binom{k-1}{s-1}\binom{n-m}{t}\binom{n-k}{t}$$

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$$= \binom{2n-m}{n} + \sum_{s=1}^{m}\sum_{k=s}^{m}\binom{m}{s}\binom{k-1}{s-1} +$$

$$\sum_{s=1}^{m}\sum_{k=s}^{m}\binom{m}{s}\binom{k-1}{s-1}\sum_{t=1}^{n-m}\binom{n-m}{t}\binom{n-k}{t}$$

$$= \binom{2n-m}{n} + \sum_{s=1}^{m}\sum_{k=s}^{m}\binom{m}{s}\binom{k-1}{s-1}\sum_{t=0}^{n-m}\binom{n-m}{t}\binom{n-k}{t}$$

$$= \binom{2n-m}{n} + \sum_{s=1}^{m}\sum_{k=s}^{m}\binom{m}{s}\binom{k-1}{s-1}\binom{2n-m-k}{n-k},$$

as required. □

To investigate the cardinality of $\mathrm{Reg}(\overline{OT}(X,Y))$, we prove the following two lemmas.

Lemma 3.2.6. *For* $\alpha \in \overline{OT}(X,Y), \alpha \in \mathrm{Reg}(\overline{OT}(X,Y))$ *if and only if* $\mathrm{ran}\,\alpha \cap Y = Y\alpha$.

Proof. Necessity follows immediately from Corollary 3.1.4. To prove sufficiency, suppose that $\mathrm{ran}\,\alpha \cap Y = Y\alpha$. If $u \in Y$ is an upper bound of $\mathrm{ran}\,\alpha$, then $1\alpha \le 2\alpha \le \ldots \le n\alpha \le u$. Since $1\alpha \in Y\alpha \subseteq Y$ and $u \in Y$, it follows from the property of Y that $\{1\alpha, 2\alpha, \ldots, n\alpha\} \subseteq Y$, i.e., $\mathrm{ran}\,\alpha \subseteq Y$, so $\max(\mathrm{ran}\,\alpha) \in Y$. We see that $\min(\mathrm{ran}\,\alpha) = 1\alpha \in Y\alpha \subseteq Y$. If $x \in \big(X \smallsetminus (\mathrm{ran}\,\alpha \cup \mathrm{ub}(\mathrm{ran}\,\alpha) \cup \mathrm{lb}(\mathrm{ran}\,\alpha))\big) \cap Y$, then $\{t \in \mathrm{ran}\,\alpha \mid t < x\} \subseteq \{1, 2, \ldots, x\} \subseteq Y$, so $\max(\{t \in \mathrm{ran}\,\alpha \mid t < x\}) \in Y$. It follows from Corollary 3.1.4 that $\alpha \in \mathrm{Reg}(\overline{OT}(X,Y))$. □

Lemma 3.2.7. *Let* $m < n$. *For* $\alpha \in \overline{OT}(X,Y)$, $\mathrm{ran}\,\alpha \cap Y = Y\alpha$ *if and only if* $(X \smallsetminus Y)\alpha \subseteq \{\max(Y\alpha), m+1, \ldots, n\}$.

Proof. Assume that $\mathrm{ran}\,\alpha \cap Y = Y\alpha$. Let $x \in X \smallsetminus Y$. Then $x\alpha \in Y$ or $x\alpha \in X \smallsetminus Y$. If $x\alpha \in X \smallsetminus Y$, then $x\alpha \in \{m+1, \ldots, n\}$. Assume that $x\alpha \in Y$. Then $x\alpha \in \mathrm{ran}\,\alpha \cap Y$, so $x\alpha = y\alpha$ for some $y \in Y$ by assumption. Since α is order-preserving, we have

$$\max(Y\alpha) = \big(\max(Y)\big)\alpha \le m\alpha \le x\alpha = y\alpha \le \big(\max(Y)\big)\alpha = \max(Y\alpha),$$

which implies that $x\alpha = \max(Y\alpha)$. This shows that $(X \setminus Y)\alpha \subseteq \{\max(Y\alpha), m+1, \ldots, n\}$.

Conversely, assume that $(X \setminus Y)\alpha \subseteq \{\max(Y\alpha), m+1, \ldots, n\}$. Then $(X \setminus Y)\alpha \cap Y \subseteq \{\max(Y\alpha)\} \subseteq Y\alpha$. Using this and the fact that $Y\alpha \subseteq Y$, we obtain that $\operatorname{ran}\alpha \cap Y = (Y\alpha \cap Y) \cup ((X \setminus Y)\alpha \cap Y) = Y\alpha$. □

Theorem 3.2.8. $|\operatorname{Reg}(\overline{OT}(X,Y))| = \binom{2m-1}{m-1}\binom{2(n-m)}{n-m}$.

Proof. If $m = n$, then $\overline{OT}(X,Y) = OT(Y)$, so $\operatorname{Reg}(\overline{OT}(X,Y)) = OT(Y)$ by Theorem 1.4. Hence the result for $m = n$ is true by using Theorem 1.8. Next, assume that $m < n$. Let $\varnothing \neq Y' \subseteq Y$ be such that $|Y'| = r$ and let $k = \max(Y')$. It follows from Proposition 1.12 and Proposition 1.13 that

$$\{\alpha \in \overline{OT}(X,Y) \mid Y\alpha = Y' \text{ and } (X \setminus Y)\alpha \subseteq \{k, m+1, \ldots, n\}\}$$
$$= \big\{\alpha_1 \cup \alpha_2 \mid \alpha_1 \in OT(Y), \operatorname{ran}\alpha_1 = Y' \text{ and } \alpha_2 \in OT(X \setminus Y, \{k, m+1, \ldots, n\})\big\},$$

which implies that

$$\begin{aligned}
&\big|\{\alpha \in \overline{OT}(X,Y) \mid Y\alpha = Y' \text{ and } (X \setminus Y)\alpha \subseteq \{k, m+1, \ldots, n\}\}\big| \\
&\quad = \big|\{\alpha \in OT(Y) \mid \operatorname{ran}\alpha = Y'\}\big| \, \big|OT(X \setminus Y, \{k, m+1, \ldots, n\})\big| \\
&\quad = \binom{m-1}{r-1}\binom{(n-m+1)+(n-m)-1}{n-m} \qquad \text{by Proposition 1.19 and Theorem 2.2.3(i)} \\
&\quad = \binom{m-1}{r-1}\binom{2(n-m)}{n-m}.
\end{aligned}$$

We have from Lemma 3.2.7 and Lemma 3.2.6 that

$$\begin{aligned}
&\big\{\alpha \in \overline{OT}(X,Y) \mid Y\alpha = Y' \text{ and } (X \setminus Y)\alpha \subseteq \{k, m+1, \ldots, n\}\big\} \\
&\qquad = \{\alpha \in \overline{OT}(X,Y) \mid \operatorname{ran}\alpha \cap Y = Y\alpha = Y'\} \\
&\qquad = \{\alpha \in \operatorname{Reg}(\overline{OT}(X,Y)) \mid Y\alpha = Y'\}.
\end{aligned}$$

Hence

$$|\{\alpha \in \operatorname{Reg}(\overline{OT}(X,Y)) \mid Y\alpha = Y'\}| = \binom{m-1}{r-1}\binom{2(n-m)}{n-m}.$$

This implies that for $1 \leq r \leq m$,

$$|\{\alpha \in \operatorname{Reg}(\overline{OT}(X,Y)) \mid |Y\alpha| = r\}| = \binom{m}{r}\binom{m-1}{r-1}\binom{2(n-m)}{n-m}.$$

Consequently,

$$\begin{aligned} |\operatorname{Reg}(\overline{OT}(X,Y))| &= \sum_{r=1}^{m}\binom{m}{r}\binom{m-1}{r-1}\binom{2(n-m)}{n-m} \\ &= \binom{2m-1}{m}\binom{2(n-m)}{n-m} \qquad \text{by Result 1.26.} \end{aligned}$$

□

Next, to determine the number of regular elements in $\overline{OP}(X,Y)$, the following lemmas are required.

Lemma 3.2.9. *Let $m < n$. For $\alpha \in \overline{OP}(X,Y) \smallsetminus \{0\}$, $\operatorname{ran}\alpha \cap Y = (\operatorname{dom}\alpha \cap Y)\alpha$ if and only if one of the following statements holds.*

(i) $\operatorname{dom}\alpha \subseteq X \smallsetminus Y$ *and* $\operatorname{ran}\alpha \subseteq X \smallsetminus Y$.

(ii) $\operatorname{dom}\alpha \subseteq Y$.

(iii) $\operatorname{dom}\alpha \cap Y \neq \varnothing$, $\operatorname{dom}\alpha \cap (X \smallsetminus Y) \neq \varnothing$ *and* $(\operatorname{dom}\alpha \cap (X \smallsetminus Y))\alpha \subseteq \{\max((\operatorname{dom}\alpha \cap Y)\alpha), m+1, \ldots, n\}$.

Proof. Assume that $\operatorname{ran}\alpha \cap Y = (\operatorname{dom}\alpha \cap Y)\alpha$. If $\operatorname{dom}\alpha \cap Y = \varnothing$, then $\operatorname{ran}\alpha \cap Y = \varnothing$ which implies that $\operatorname{ran}\alpha \subseteq X \smallsetminus Y$, so we get (i) in this case. Suppose that $\operatorname{dom}\alpha \cap Y \neq \varnothing$. If $\operatorname{dom}\alpha \subseteq Y$, then (ii) holds. Next, assume that $\operatorname{dom}\alpha \nsubseteq Y$, i.e., $\operatorname{dom}\alpha \cap (X \smallsetminus Y) \neq \varnothing$. Let $x \in \operatorname{dom}\alpha \cap (X \smallsetminus Y)$. Then $x\alpha \in Y$ or $x\alpha \in X \smallsetminus Y$. If $x\alpha \in X \smallsetminus Y$, then $x\alpha \in \{m+1, \ldots, n\}$. If $x\alpha \in Y$, then $x\alpha \in \operatorname{ran}\alpha \cap Y = (\operatorname{dom}\alpha \cap Y)\alpha$. Since α is order-preserving,

$$\begin{aligned} \max((\operatorname{dom}\alpha \cap Y)\alpha) &= (\max(\operatorname{dom}\alpha \cap Y))\alpha \\ &\leq m\alpha \leq x\alpha \leq \max((\operatorname{dom}\alpha \cap Y)\alpha), \end{aligned}$$

and we deduce that $x\alpha = \max((\operatorname{dom}\alpha \cap Y)\alpha)$. This shows that $(\operatorname{dom}\alpha \cap (X \smallsetminus Y))\alpha \subseteq \{\max((\operatorname{dom}\alpha \cap Y)\alpha), m+1, \ldots, n\}$. Hence (iii) holds.

For the converse, if $\operatorname{dom}\alpha \subseteq X \smallsetminus Y$ and $\operatorname{ran}\alpha \subseteq X \smallsetminus Y$, then $\operatorname{dom}\alpha \cap Y =$

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$\varnothing$ and $\operatorname{ran}\alpha \cap Y = \varnothing$, so $\operatorname{ran}\alpha \cap Y = (\operatorname{dom}\alpha \cap Y)\alpha$. If $\operatorname{dom}\alpha \subseteq Y$, then $\operatorname{dom}\alpha \cap Y = \operatorname{dom}\alpha$, so $\operatorname{ran}\alpha = (\operatorname{dom}\alpha)\alpha = (\operatorname{dom}\alpha \cap Y)\alpha \subseteq Y$ which implies that $\operatorname{ran}\alpha \cap Y = \operatorname{ran}\alpha = (\operatorname{dom}\alpha \cap Y)\alpha$. Next, assume that (iii) holds. Then $(\operatorname{dom}\alpha \cap (X \smallsetminus Y))\alpha \cap Y \subseteq \{\max((\operatorname{dom}\alpha \cap Y)\alpha)\} \subseteq (\operatorname{dom}\alpha \cap Y)\alpha$. Also, we have $(\operatorname{dom}\alpha \cap Y)\alpha \subseteq Y$. It follows that

$$\begin{aligned}
\operatorname{ran}\alpha \cap Y &= \big((\operatorname{dom}\alpha \cap Y)\alpha \cup (\operatorname{dom}\alpha \cap (X \smallsetminus Y))\alpha\big) \cap Y \\
&= \big((\operatorname{dom}\alpha \cap Y)\alpha \cap Y\big) \cup \big((\operatorname{dom}\alpha \cap (X \smallsetminus Y))\alpha \cap Y\big) \\
&= (\operatorname{dom}\alpha \cap Y)\alpha.
\end{aligned}$$

The proof is thereby complete. $\square$

Lemma 3.2.10. *For $1 \le s \le m$ and $1 \le t \le n-m$,*

$$\Big|\big\{\alpha \in \overline{OP}(X,Y) \mid |\operatorname{dom}\alpha \cap Y| = s, |\operatorname{dom}\alpha \cap (X \smallsetminus Y)| = t \text{ and } (\operatorname{dom}\alpha \cap (X \smallsetminus Y))\alpha \subseteq \{\max((\operatorname{dom}\alpha \cap Y)\alpha), m+1, \ldots, n\}\big\}\Big|$$

$$= \binom{m}{s}\binom{n-m}{t}\binom{m+s-1}{s}\binom{n-m+t}{t}.$$

Proof. Let $\varnothing \neq Y_1 \subseteq Y$ and $\varnothing \neq Z \subseteq X \smallsetminus Y$ be such that $|Y_1| = s$ and $|Z| = t$. Let $\varnothing \neq Y_2 \subseteq Y$ be such that $|Y_2| = r$ where $1 \le r \le s$ and let $k = \max(Y_2)$. Then by Proposition 1.12 and Proposition 1.13, we have

$$\big\{\alpha \in \overline{OP}(X,Y) \mid \operatorname{dom}\alpha \cap Y = Y_1, \operatorname{dom}\alpha \cap (X \smallsetminus Y) = Z, (\operatorname{dom}\alpha \cap Y)\alpha = Y_2 \text{ and } (\operatorname{dom}\alpha \cap (X \smallsetminus Y))\alpha \subseteq \{k, m+1, \ldots, n\}\big\}$$

$$= \big\{\alpha_1 \cup \alpha_2 \mid \alpha_1 \in OT(Y_1, Y), \operatorname{ran}\alpha_1 = Y_2 \text{ and } \alpha_2 \in OT(Z, \{k, m+1, \ldots, n\})\big\}.$$

It follows that

$$\Big|\big\{\alpha \in \overline{OP}(X,Y) \mid \operatorname{dom}\alpha \cap Y = Y_1, \operatorname{dom}\alpha \cap (X \smallsetminus Y) = Z, (\operatorname{dom}\alpha \cap Y)\alpha = Y_2 \text{ and } (\operatorname{dom}\alpha \cap (X \smallsetminus Y))\alpha \subseteq \{k, m+1, \ldots, n\}\big\}\Big|$$

$$= \big|\{\alpha \in OT(Y_1, Y) \mid \operatorname{ran}\alpha = Y_2\}\big| \, \big|OT(Z, \{k, m+1, \ldots, n\})\big|$$

$$= \binom{s-1}{r-1}\binom{(n-m+1)+t-1}{t} \qquad \text{by Proposition 1.19 and Theorem 2.2.3(i)}$$

$$= \binom{s-1}{r-1}\binom{n-m+t}{t}.$$

This implies that for $1 \leq r \leq s$,

$$\Big|\{\alpha \in \overline{OP}(X,Y) \mid \operatorname{dom}\alpha \cap Y = Y_1, \operatorname{dom}\alpha \cap (X \smallsetminus Y) = Z, |(\operatorname{dom}\alpha \cap Y)\alpha| = r \text{ and } (\operatorname{dom}\alpha \cap (X \smallsetminus Y))\alpha \subseteq \{\max((\operatorname{dom}\alpha \cap Y)\alpha), m+1, \ldots, n\}\}\Big|$$

$$= \binom{m}{r}\binom{s-1}{r-1}\binom{n-m+t}{t}.$$

Hence

$$\Big|\{\alpha \in \overline{OP}(X,Y) \mid \operatorname{dom}\alpha \cap Y = Y_1, \operatorname{dom}\alpha \cap (X \smallsetminus Y) = Z, \text{ and } (\operatorname{dom}\alpha \cap (X \smallsetminus Y))\alpha \subseteq \{\max((\operatorname{dom}\alpha \cap Y)\alpha), m+1, \ldots, n\}\}\Big|$$

$$= \sum_{r=1}^{s} \binom{m}{r}\binom{s-1}{r-1}\binom{n-m+t}{t}$$

$$= \binom{m+s-1}{s}\binom{n-m+t}{t} \qquad \text{by Result 1.24.}$$

Consequently, for $1 \leq s \leq m$ and $1 \leq t \leq n-m$,

$$\Big|\{\alpha \in \overline{OP}(X,Y) \mid |\operatorname{dom}\alpha \cap Y| = s, |\operatorname{dom}\alpha \cap (X \smallsetminus Y)| = t, \text{ and } (\operatorname{dom}\alpha \cap (X \smallsetminus Y))\alpha \subseteq \{\max((\operatorname{dom}\alpha \cap Y)\alpha), m+1, \ldots, n\}\}\Big|$$

$$= \binom{m}{s}\binom{n-m}{t}\binom{m+s-1}{s}\binom{n-m+t}{t}.$$

□

Theorem 3.2.11.

$$|\operatorname{Reg}(\overline{OP}(X,Y))| = \sum_{r=0}^{n-m} \binom{n-m}{r}\binom{n-m+r-1}{r} + \sum_{s=1}^{m} \binom{m}{s}\binom{m+s-1}{s} \sum_{t=0}^{n-m} \binom{n-m}{t}\binom{n-m+t}{t}.$$

Proof. If $m = n$, then $\overline{OP}(X,Y) = OP(Y)$, so $\operatorname{Reg}(\overline{OP}(X,Y)) = OP(Y)$ by Theorem 1.5 and then using Theorem 1.9 to obtain the result for $m = n$. Next, assume that $m < n$. It follows from Theorem 3.1.10 and Lemma 3.2.9 that

$$\begin{aligned}
&\operatorname{Reg}(\overline{OP}(X,Y)) = \{\alpha \in \overline{OP}(X,Y) \mid \operatorname{ran}\alpha \cap Y = (\operatorname{dom}\alpha \cap Y)\alpha\} \\
&= \{0\} \cup \{\alpha \in \overline{OP}(X,Y) \smallsetminus \{0\} \mid \operatorname{dom}\alpha \subseteq X \smallsetminus Y \text{ and } \operatorname{ran}\alpha \subseteq X \smallsetminus Y\} \\
&\quad \cup \{\alpha \in \overline{OP}(X,Y) \smallsetminus \{0\} \mid \operatorname{dom}\alpha \subseteq Y\} \cup \\
&\quad \big\{\alpha \in \overline{OP}(X,Y) \mid \operatorname{dom}\alpha \cap Y \neq \varnothing, \operatorname{dom}\alpha \cap (X \smallsetminus Y) \neq \varnothing \text{ and} \\
&\qquad\qquad (\operatorname{dom}\alpha \cap (X \smallsetminus Y))\alpha \subseteq \{\max((\operatorname{dom}\alpha \cap Y)\alpha), m+1, \ldots, n\}\big\} \\
&= OP(X \smallsetminus Y) \cup (OP(Y) \smallsetminus \{0\}) \cup \\
&\quad \big\{\alpha \in \overline{OP}(X,Y) \mid \operatorname{dom}\alpha \cap Y \neq \varnothing, \operatorname{dom}\alpha \cap (X \smallsetminus Y) \neq \varnothing \text{ and} \\
&\qquad\qquad (\operatorname{dom}\alpha \cap (X \smallsetminus Y))\alpha \subseteq \{\max((\operatorname{dom}\alpha \cap Y)\alpha), m+1, \ldots, n\}\big\}.
\end{aligned} \tag{1}$$

By Theorem 1.9, we have

$$|OP(X \smallsetminus Y)| = \sum_{r=0}^{n-m} \binom{n-m}{r}\binom{n-m+r-1}{r} \tag{2}$$

and

$$|OP(Y) \smallsetminus \{0\}| = \sum_{s=1}^{m} \binom{m}{s}\binom{m+s-1}{s}. \tag{3}$$

Also, we have

$$\begin{aligned}
&\Big|\big\{\alpha \in \overline{OP}(X,Y) \mid \operatorname{dom}\alpha \cap Y \neq \varnothing, \operatorname{dom}\alpha \cap (X \smallsetminus Y) \neq \varnothing \text{ and} \\
&\qquad (\operatorname{dom}\alpha \cap (X \smallsetminus Y))\alpha \subseteq \{\max((\operatorname{dom}\alpha \cap Y)\alpha), m+1, \ldots, n\}\big\}\Big| \\
&= \sum_{s=1}^{m}\sum_{t=1}^{n-m} \Big|\big\{\alpha \in \overline{OP}(X,Y) \mid |\operatorname{dom}\alpha \cap Y| = s, |\operatorname{dom}\alpha \cap (X \smallsetminus Y)| = t \text{ and} \\
&\qquad (\operatorname{dom}\alpha \cap (X \smallsetminus Y))\alpha \subseteq \{\max((\operatorname{dom}\alpha \cap Y)\alpha), m+1, \ldots, n\}\big\}\Big| \\
&= \sum_{s=1}^{m}\sum_{t=1}^{n-m} \binom{m}{s}\binom{n-m}{t}\binom{m+s-1}{s}\binom{n-m+t}{t} \qquad \text{by Lemma 3.2.10} \\
&= \sum_{s=1}^{m}\binom{m}{s}\binom{m+s-1}{s}\sum_{t=1}^{n-m}\binom{n-m}{t}\binom{n-m+t}{t}.
\end{aligned} \tag{4}$$

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From (1), (2), (3) and (4), we obtain that

$$
\begin{aligned}
|\operatorname{Reg}(\overline{OP}(X,Y))| &= \sum_{r=0}^{n-m}\binom{n-m}{r}\binom{n-m+r-1}{r} + \sum_{s=1}^{m}\binom{m}{s}\binom{m+s-1}{s} \\
&\quad + \sum_{s=1}^{m}\binom{m}{s}\binom{m+s-1}{s}\sum_{t=1}^{n-m}\binom{n-m}{t}\binom{n-m+t}{t} \\
&= \sum_{r=0}^{n-m}\binom{n-m}{r}\binom{n-m+r-1}{r} + \\
&\quad \sum_{s=1}^{m}\binom{m}{s}\binom{m+s-1}{s}\sum_{t=0}^{n-m}\binom{n-m}{t}\binom{n-m+t}{t},
\end{aligned}
$$

as desired. □

Theorem 3.2.12. $|\operatorname{Reg}(\overline{OI}(X,Y))| = \binom{2m}{m}\binom{2(n-m)}{n-m}$.

Proof. By Theorem 3.1.12, we have

$$
\begin{aligned}
\operatorname{Reg}(\overline{OI}(X,Y)) &= \{\alpha \in \overline{OI}(X,Y) \mid (\operatorname{dom}\alpha \cap (X \smallsetminus Y))\alpha \subseteq X \smallsetminus Y\} \\
&= \{\alpha \in OI(X) \mid (\operatorname{dom}\alpha \cap Y)\alpha \subseteq Y \text{ and} \\
&\qquad\qquad (\operatorname{dom}\alpha \cap (X \smallsetminus Y))\alpha \subseteq X \smallsetminus Y\}.
\end{aligned}
$$

Proposition 1.12 and Proposition 1.13 imply that

$$
\begin{aligned}
&\{\alpha \in OI(X) \mid (\operatorname{dom}\alpha \cap Y)\alpha \subseteq Y \text{ and } (\operatorname{dom}\alpha \cap (X \smallsetminus Y))\alpha \subseteq X \smallsetminus Y\} \\
&\qquad = \{\alpha_1 \cup \alpha_2 \mid \alpha_1 \in OI(Y) \text{ and } \alpha_2 \in OI(X \smallsetminus Y)\}.
\end{aligned}
$$

Consequently,

$$
\operatorname{Reg}(\overline{OI}(X,Y)) = \{\alpha_1 \cup \alpha_2 \mid \alpha_1 \in OI(Y) \text{ and } \alpha_2 \in OI(X \smallsetminus Y)\}.
$$

Hence

$$
\begin{aligned}
|\operatorname{Reg}(\overline{OI}(X,Y))| &= |OI(Y)|\,|OI(X \smallsetminus Y)| \\
&= \binom{2m}{m}\binom{2(n-m)}{n-m} \qquad \text{by Theorem 1.10.}
\end{aligned}
$$

□

CHAPTER IV

REGULAR ELEMENTS OF GENERALIZED ORDER-PRESERVING TRANSFORMATION SEMIGROUPS

In the last chapter, the regular elements of the generalized order-preserving transformation semigroups $(OT(X,Y),\theta)$ where $\theta \in OT(Y,X)$, $(OP(X,Y),\theta)$ where $\theta \in OP(Y,X)$ and $(OI(X,Y),\theta)$ where $\theta \in OI(Y,X)$ are characterized when X and Y are chains. In addition, we provide the proofs of Theorem 1.16 , Theorem 1.17 and Theorem 1.18 by using these characterizations.

Throughout this chapter, let X and Y be any chains.

Before we determine the regular elements of $(OT(X,Y),\theta)$, $(OP(X,Y),\theta)$ and $(OI(X,Y),\theta)$, it is convenient to have the following preliminary result.

Lemma 4.1. *Let A and B be nonempty sets. If $\alpha, \beta \in P(A,B)$ and $\gamma \in P(B,A)$ are such that $\alpha = \alpha\gamma\beta\gamma\alpha$, then the following conditions hold.*

(i) $\operatorname{ran}\alpha = \operatorname{ran}(\gamma\alpha)$.

(ii) $\operatorname{ran}\alpha \subseteq \operatorname{dom}\gamma$.

(iii) *γ is 1-1 on* $\operatorname{ran}\alpha$.

Proof. Since $\operatorname{ran}\alpha = \operatorname{ran}(\alpha\gamma\beta\gamma\alpha) \subseteq \operatorname{ran}(\gamma\alpha) \subseteq \operatorname{ran}\alpha$, we obtain (i). Also, we have

$$\begin{aligned}\operatorname{ran}\alpha = (\operatorname{dom}\alpha)\alpha &= \big(\operatorname{dom}(\alpha\gamma\beta\gamma\alpha)\big)\alpha \\ &\subseteq \big(\operatorname{dom}(\alpha\gamma)\big)\alpha \\ &= \big((\operatorname{ran}\alpha \cap \operatorname{dom}\gamma)\alpha^{-1}\big)\alpha \\ &= \operatorname{ran}\alpha \cap \operatorname{dom}\gamma \subseteq \operatorname{dom}\gamma.\end{aligned}$$

This verifies (ii). Since $\alpha = \alpha\gamma\beta\gamma\alpha$, it follows that $z = z\gamma\beta\gamma\alpha$ for all $z \in \operatorname{ran}\alpha$,

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or in an other word, $\gamma\beta\gamma\alpha$ is the identity on $\operatorname{ran}\alpha$. If $y_1, y_2 \in \operatorname{ran}\alpha$ are such that $y_1\gamma = y_2\gamma$, then $y_1 = y_1\gamma\beta\gamma\alpha = y_2\gamma\beta\gamma\alpha = y_2$, so (iii) follows. □

First, we characterize the regular elements of $(OT(X,Y),\theta)$ where $\theta \in OT(Y,X)$.

Theorem 4.2. *For $\theta \in OT(Y,X)$ and $\alpha \in OT(X,Y)$, $\alpha \in \operatorname{Reg}((OT(X,Y),\theta))$ if and only if the following conditions hold.*

(i) $\alpha\theta \in \operatorname{Reg}(OT(X))$.

(ii) $\operatorname{ran}\alpha = \operatorname{ran}(\theta\alpha)$.

(iii) θ *is 1-1 on* $\operatorname{ran}\alpha$.

Proof. Assume that $\alpha \in \operatorname{Reg}((OT(X,Y),\theta))$. Then there exists $\beta \in OT(X,Y)$ such that $\alpha = \alpha\theta\beta\theta\alpha$. Thus $\alpha\theta, \beta\theta \in OT(X)$ and $\alpha\theta = (\alpha\theta)(\beta\theta)(\alpha\theta)$. This verifies (i) and, of course, (ii) and (iii) follow immediately from Lemma 4.1.

For the converse, assume that (i), (ii) and (iii) hold. Let $\beta \in OT(X)$ be such that $\alpha\theta = (\alpha\theta)\beta(\alpha\theta)$. Then $\alpha\big(\theta_{|\operatorname{ran}\alpha}\big) = \alpha\theta\beta\alpha\big(\theta_{|\operatorname{ran}\alpha}\big)$. Since $\theta_{|\operatorname{ran}\alpha}$ is 1-1, it follows that $\alpha = \alpha\theta\beta\alpha$. Then $\operatorname{ran}\alpha = \operatorname{ran}(\alpha\theta\beta\alpha) \subseteq \operatorname{ran}(\beta\alpha) \subseteq \operatorname{ran}\alpha$, so $\operatorname{ran}\alpha = \operatorname{ran}(\beta\alpha)$. Hence $\operatorname{ran}(\beta\alpha) = \operatorname{ran}\alpha = \operatorname{ran}(\theta\alpha)$. For each $y \in \operatorname{ran}(\beta\alpha) = \operatorname{ran}(\theta\alpha)$, choose an element $d_y \in y(\theta\alpha)^{-1}$. Then $d_y \in Y$ and $d_y(\theta\alpha) = y$ for all $y \in \operatorname{ran}(\beta\alpha)$. Note that $X = \dot{\bigcup\limits_{y \in \operatorname{ran}(\beta\alpha)}} y(\beta\alpha)^{-1}$. Define $\beta' : X \to Y$ by a bracket notation as follows:

$$\beta' = \begin{pmatrix} y(\beta\alpha)^{-1} \\ d_y \end{pmatrix}_{y \in \operatorname{ran}(\beta\alpha)}.$$

If $x \in X$, then $x\alpha \in \operatorname{ran}\alpha = \operatorname{ran}(\beta\alpha)$ and $x\alpha = x\alpha\theta\beta\alpha = (x\alpha\theta)\beta\alpha$, so $x\alpha\theta \in (x\alpha)(\beta\alpha)^{-1}$ which implies that $x\alpha\theta\beta'\theta\alpha = (x\alpha\theta)\beta'\theta\alpha = d_{x\alpha}(\theta\alpha) = x\alpha$. Hence $\alpha = \alpha\theta\beta'\theta\alpha$. To show that β' is order-preserving, let $x_1, x_2 \in X$ be such that $x_1 < x_2$. Then $x_1\beta\alpha \le x_2\beta\alpha$. If $x_1\beta\alpha = x_2\beta\alpha$, then $x_1, x_2 \in (x_1\beta\alpha)(\beta\alpha)^{-1}$, so $x_1\beta' = d_{x_1\beta\alpha} = x_2\beta'$. Assume that $x_1\beta\alpha < x_2\beta\alpha$. Since $\operatorname{ran}(\beta\alpha) = \operatorname{ran}(\theta\alpha)$, we get $x_1\beta\alpha, x_2\beta\alpha \in \operatorname{ran}(\theta\alpha)$. Since $\theta\alpha \in OT(Y)$, it follows from Proposition 1.11 that $(x_1\beta\alpha)(\theta\alpha)^{-1} < (x_2\beta\alpha)(\theta\alpha)^{-1}$. It follows that $d_{x_1\beta\alpha} < d_{x_2\beta\alpha}$. Since $((x_1\beta\alpha)(\beta\alpha)^{-1})\beta' = \{d_{x_1\beta\alpha}\}$ and $((x_2\beta\alpha)(\beta\alpha)^{-1})\beta' = \{d_{x_2\beta\alpha}\}$, we have that $x_1\beta' = d_{x_1\beta\alpha} < d_{x_2\beta\alpha} = x_2\beta'$.

The proof is thereby complete. □

We now use the above theorem to prove Theorem 1.16. To do this, the following series of lemmas is needed.

Lemma 4.3. *Let* $|X| > 1$*. If the semigroup* $(OT(X,Y),\theta)$ *is regular, then* θ *is 1-1.*

Proof. We will prove the lemma by contrapositive. Assume that θ is not 1-1. Then there are $c, d \in Y$ such that $c < d$ and $c\theta = d\theta$. Since $|X| > 1$, there exist $a, b \in X$ such that $a < b$. Define $\alpha : X \to Y$ by

$$\alpha = \begin{pmatrix} x & y \\ c & d \end{pmatrix}_{\substack{x<b \\ y \geq b}}.$$

Then $\alpha \in OT(X,Y)$ and $\operatorname{ran}\alpha = \{c, d\}$. Since $c, d \in \operatorname{ran}\alpha$ are such that $c\theta = d\theta$ and $c < d$, it follows that θ is not 1-1 on $\operatorname{ran}\alpha$. We conclude from Theorem 4.2 that α is not a regular element of $(OT(X,Y),\theta)$, and hence $(OT(X,Y),\theta)$ is not a regular semigroup. $\square$

Lemma 4.4. *Let* $|Y| \geq 2$*. If there is an element* $a \in X$ *such that* $a > \operatorname{ran}\theta$ *or* $a < \operatorname{ran}\theta$*, then* $(OT(X,Y),\theta)$ *is not a regular semigroup.*

Proof. Let $e, f \in Y$ be such that $e < f$. Let $\alpha : X \to Y$ be defined by

$$\alpha = \begin{pmatrix} u & v \\ e & f \end{pmatrix}_{\substack{u<a \\ v \geq a}} \text{ if } a > \operatorname{ran}\theta \quad \text{and} \quad \alpha = \begin{pmatrix} u & v \\ e & f \end{pmatrix}_{\substack{u \leq a \\ v > a}} \text{ if } a < \operatorname{ran}\theta.$$

Then $\alpha \in OT(X,Y), \operatorname{ran}\alpha = \{e, f\}, \operatorname{ran}(\theta\alpha) = \{e\}$ for $a > \operatorname{ran}\theta$ and $\operatorname{ran}(\theta\alpha) = \{f\}$ for $a < \operatorname{ran}\theta$. By Theorem 4.2, $\alpha \notin \operatorname{Reg}(OT(X,Y),\theta)$. Hence $(OT(X,Y),\theta)$ is not regular. $\square$

Lemma 4.5. *If* $\operatorname{ran}\theta \subsetneq X$ *and* $|Y| \geq 3$*, then the semigroup* $(OT(X,Y),\theta)$ *is not regular.*

Proof. Let $e, f, g \in Y$ be such that $e < f < g$ and let $a \in X \smallsetminus \operatorname{ran}\theta$. If $a > \operatorname{ran}\theta$ or $a < \operatorname{ran}\theta$, then Lemma 4.4 implies that $(OT(X,Y),\theta)$ is not regular. Assume that $a \not> \operatorname{ran}\theta$ and $a \not< \operatorname{ran}\theta$. Then $\{t \in \operatorname{ran}\theta \mid t < a\}$ and $\{t \in \operatorname{ran}\theta \mid t > a\}$ are nonempty sets. Define $\alpha : X \to Y$ by

$$\alpha = \begin{pmatrix} u & a & v \\ e & f & g \end{pmatrix}_{\substack{u<a \\ v>a}}.$$

Then $\alpha \in OT(X,Y)$ and $\operatorname{ran}\alpha = \{e,f,g\} \neq \{e,g\} = \operatorname{ran}(\theta\alpha)$. It follows immediately from Theorem 4.2 that α is not a regular element of $(OT(X,Y),\theta)$. This implies that $(OT(X,Y),\theta)$ is not a regular semigroup. □

Lemma 4.6. *Let* $|Y| = 2$. *Then* $(OT(X,Y),\theta)$ *is a regular semigroup if and only if* $\min(X)$ *and* $\max(X)$ *exist, and* $\operatorname{ran}\theta = \{\min(X), \max(X)\}$.

Proof. Let $Y = \{e,f\}$ be such that $e < f$. Assume that $(OT(X,Y),\theta)$ is regular. If $|X| = 1$, then $\min(X) = \max(X)$ and $\operatorname{ran}\theta = \{\min(X)\}$. Suppose that $|X| > 1$. We deduce from Lemma 4.3 that θ is 1-1. Then $e\theta < f\theta$ and $\operatorname{ran}\theta = \{e\theta, f\theta\}$. Also by Lemma 4.4, for every $a \in X$, $a \not> \operatorname{ran}\theta$ and $a \not< \operatorname{ran}\theta$. Thus $e\theta \leq a \leq f\theta$ for all $a \in X$. This implies that $e\theta = \min(X)$ and $f\theta = \max(X)$.

Conversely, assume that $\min(X)$ and $\max(X)$ exist, and $\operatorname{ran}\theta = \{\min(X), \max(X)\}$. To show that $(OT(X,Y),\theta)$ is regular, let $\alpha \in OT(X,Y)$. Then either $|\operatorname{ran}\alpha| = 1$ or $|\operatorname{ran}\alpha| = 2$ because $|Y| = 2$. If $|\operatorname{ran}\alpha| = 1$, then $\alpha\theta\alpha = \alpha$ because $\operatorname{ran}(\alpha\theta\alpha) \subseteq \operatorname{ran}\alpha$, so it is regular. Next, assume that $|\operatorname{ran}\alpha| = 2$. Then $\operatorname{ran}\alpha = \{e,f\}$, so $X = e\alpha^{-1} \dot\cup f\alpha^{-1}$. Since $e < f$ and α is order-preserving, we have $\min(X) \in e\alpha^{-1}$ and $\max(X) \in f\alpha^{-1}$. Then $\min(X)\alpha = e$ and $\max(X)\alpha = f$. Therefore we get $\operatorname{ran}(\theta\alpha) = (\operatorname{ran}\theta)\alpha = \{\min(X), \max(X)\}\alpha = \{e,f\} = \operatorname{ran}\alpha$. We now have $|\operatorname{dom}\theta| = |\operatorname{ran}\theta| = 2$ and it immediately follows that θ is 1-1. Since $\operatorname{ran}(\alpha\theta)$ is finite, by Corollary 1.7, $\alpha\theta \in \operatorname{Reg}(OT(X))$. By Theorem 4.2, $\alpha \in \operatorname{Reg}(OT(X,Y),\theta)$. □

Lemma 4.7. *If* $(OT(X,Y),\theta)$ *is a regular semigroup and* θ *is an order-isomorphism from* Y *onto* X, *then* $OT(X)$ *is a regular semigroup.*

Proof. Assume that $(OT(X,Y),\theta)$ is a regular semigroup and θ is an order-isomorphism from Y onto X. Then θ^{-1} is an order-isomorphism from X onto Y. To show that $OT(X)$ is regular, let $\alpha \in OT(X)$. Then $\alpha\theta^{-1} \in OT(X,Y)$. Since $(OT(X,Y),\theta)$ is regular, we have $\alpha\theta^{-1} = \alpha\theta^{-1}\theta\beta\theta\alpha\theta^{-1}$ for some $\beta \in OT(X,Y)$.

Thus $\beta\theta \in OT(X)$ and

$$\alpha = \alpha 1_X = \alpha\theta^{-1}\theta = \alpha\theta^{-1}\theta\beta\theta\alpha\theta^{-1}\theta = \alpha 1_X \beta\theta\alpha 1_X = \alpha\beta\theta\alpha.$$

This implies that $\alpha \in \mathrm{Reg}(OT(X))$. Hence $OT(X)$ is a regular semigroup. $\square$

Theorem 4.8. *The semigroup $(OT(X,Y),\theta)$ is regular if and only if one of the following statements holds.*

(i) *The semigroup $OT(X)$ is regular and θ is an order-isomorphism from Y onto X.*

(ii) $|X| = 1$.

(iii) $|Y| = 1$.

(iv) $|Y| = 2$, $\min(X)$ *and* $\max(X)$ *exist, and* $\operatorname{ran}\theta = \{\min(X), \max(X)\}$.

Proof. To prove necessity, assume that the semigroup $(OT(X,Y),\theta)$ is regular and suppose that (i), (ii) and (iii) are false. Then $|X| > 1$ and $|Y| > 1$ and (θ is not an order-isomorphism from Y into X or $OT(X)$ is not regular).

Case 1: $|X| > 1, |Y| > 1$ and θ is not an isomorphism from Y onto X. Since $(OT(X,Y),\theta)$ is regular, it follows from Lemma 4.3 that θ is 1-1. Then $\operatorname{ran}\theta \subsetneq X$. We therefore deduce from Lemma 4.5 that $|Y| \leq 2$, and thus $|Y| = 2$. Thus (iv) holds by Lemma 4.6.

Case 2: $|X| > 1, |Y| > 1$ and $OT(X)$ is not regular. Since $(OT(X,Y),\theta)$ is regular and $OT(X)$ is not regular, it follows from Lemma 4.7 that θ is not an order-isomorphism from Y onto X. As in the proof of Case 1, we have (iv) is true.

To prove sufficiency, we first assume that (i) is true. Let $\alpha \in OT(X,Y)$. Then $\alpha\theta \in OT(X)$, so $\alpha\theta \in \mathrm{Reg}(OT(X))$ because $OT(X)$ is regular. Since θ is an isomorphism from Y onto X, it follows that θ is 1-1 and $\operatorname{ran}\theta = X$. Then $\operatorname{ran}(\theta\alpha) = (\operatorname{ran}\theta)\alpha = X\alpha = \operatorname{ran}\alpha$. Since θ is 1-1 and $\operatorname{ran}\alpha \subseteq Y$, we have that θ is 1-1 on $\operatorname{ran}\alpha$. In view of Theorem 4.2, α is a regular element of $(OT(X,Y),\theta)$. Hence $(OT(X,Y),\theta)$ is a regular semigroup. Next, if $|X| = 1$, then for $\alpha \in OT(X,Y)$, $|\operatorname{ran}\alpha| = 1$, so $\alpha = \alpha\theta\alpha$ because $\operatorname{ran}(\alpha\theta\alpha) \subseteq \operatorname{ran}\alpha$ and it follows that $\alpha \in \text{Reg}(OT(X,Y),\theta)$. This shows that $(OT(X,Y),\theta)$ is a regular semigroup if $|X| = 1$. It is clear that $(OT(X,Y),\theta)$ is regular if $|Y| = 1$ since $|\,OT(X,Y)| = 1$. Finally, if (iv) is true, then Lemma 4.6 shows that $(OT(X,Y),\theta)$ is a regular semigroup.

Hence the theorem is completely proved. $\square$

Next, necessary and sufficient conditions for the elements of the semigroups $(OP(X,Y),\theta)$ where $\theta \in OP(Y,X)$ and $(OI(X,Y),\theta)$ where $\theta \in OI(Y,X)$ to be regular are provided.

Theorem 4.9. *For $\theta \in OP(Y,X)$ and $\alpha \in OP(X,Y)$, $\alpha \in \text{Reg}((OP(X,Y),\theta))$ if and only if the following conditions hold.*

(i) $\operatorname{ran}\alpha = \operatorname{ran}(\theta\alpha)$.

(ii) $\operatorname{ran}\alpha \subseteq \operatorname{dom}\theta$.

(iii) θ *is 1-1 on* $\operatorname{ran}\alpha$.

Proof. It is immediate from Lemma 4.1 that if $\alpha \in \text{Reg}((OP(X,Y),\theta))$, then (i), (ii) and (iii) hold.

Now suppose, conversely, that (i), (ii) and (iii) hold. Then $\operatorname{ran}(\alpha\theta) = (\operatorname{ran}\alpha \cap \operatorname{dom}\theta)\theta = (\operatorname{ran}\alpha)\theta$. Since $\operatorname{ran}\alpha = \operatorname{ran}(\theta\alpha)$, we get $y(\theta\alpha)^{-1} \neq \varnothing$ for every $y \in \operatorname{ran}\alpha$. For each $y \in \operatorname{ran}\alpha$, choose an element $d_y \in y(\theta\alpha)^{-1}$. Then $d_y \in Y$ and $d_y(\theta\alpha) = y$ for all $y \in \operatorname{ran}\alpha$. Define $\beta : \operatorname{ran}(\alpha\theta)(= (\operatorname{ran}\alpha)\theta) \to Y$ by

$$\beta = \begin{pmatrix} y\theta \\ d_y \end{pmatrix}_{y \in \operatorname{ran}\alpha}.$$

The mapping β is well-defined by (iii). To show that β is order-preserving, let $y_1, y_2 \in \operatorname{ran}\alpha$ be such that $y_1\theta < y_2\theta$. Since θ is order-preserving, it follows from (iii) that $y_1 < y_2$. Since $\theta\alpha \in OP(Y)$ and $y_1, y_2 \in \operatorname{ran}\alpha = \operatorname{ran}(\theta\alpha)$, by Proposition 1.11, $y_1(\theta\alpha)^{-1} < y_2(\theta\alpha)^{-1}$. But $d_{y_1} \in y_1(\theta\alpha)^{-1}$ and $d_{y_2} \in y_2(\theta\alpha)^{-1}$, so $d_{y_1} < d_{y_2}$. Then $(y_1\theta)\beta = d_{y_1} < d_{y_2} = (y_2\theta)\beta$. Hence $\beta \in OP(X,Y)$. It remains to show that $\alpha = \alpha\theta\beta\theta\alpha$. Since for $x \in \operatorname{dom}\alpha, x\alpha\theta \in \operatorname{dom}\beta$ and $x\alpha\theta\beta = d_{x\alpha} \in \operatorname{dom}(\theta\alpha)$, this implies that $\operatorname{dom}(\alpha\theta\beta\theta\alpha) = \operatorname{dom}\alpha$. If $x \in \operatorname{dom}\alpha$, then $x\alpha\theta\beta\theta\alpha = (x\alpha\theta)\beta\theta\alpha =$

$d_{x\alpha}(\theta\alpha) = x\alpha$. Hence $\alpha = \alpha\theta\beta\theta\alpha$. This shows that α is regular in $(OP(X,Y),\theta)$ and the verification is complete. $\square$

Theorem 4.10. For $\theta \in OI(Y,X)$ and $\alpha \in OI(X,Y)$, $\alpha \in \mathrm{Reg}((OI(X,Y),\theta))$ if and only if the following conditions hold.

(i) $\operatorname{dom}\alpha \subseteq \operatorname{ran}\theta$.

(ii) $\operatorname{ran}\alpha \subseteq \operatorname{dom}\theta$.

Proof. Assume that α is a regular element of $(OI(X,Y),\theta)$. Then there is $\beta \in OI(X,Y)$ such that $\alpha = \alpha\theta\beta\theta\alpha$. It follows from Lemma 4.1 that $\operatorname{ran}\alpha = \operatorname{ran}(\theta\alpha)$ and $\operatorname{ran}\alpha \subseteq \operatorname{dom}\theta$. Then $(\operatorname{dom}\alpha)\alpha = \operatorname{ran}\alpha = \operatorname{ran}(\theta\alpha) = (\operatorname{ran}\theta \cap \operatorname{dom}\alpha)\alpha$, so $\operatorname{dom}\alpha = \operatorname{ran}\theta \cap \operatorname{dom}\alpha$ because α is 1-1. Hence $\operatorname{dom}\alpha \subseteq \operatorname{ran}\theta$.

Conversely, assume that (i) and (ii) hold. Then $\operatorname{ran}(\theta\alpha) = (\operatorname{ran}\theta \cap \operatorname{dom}\alpha)\alpha = (\operatorname{dom}\alpha)\alpha = \operatorname{ran}\alpha$ and $\operatorname{dom}(\alpha\theta) = (\operatorname{ran}\alpha \cap \operatorname{dom}\theta)\alpha^{-1} = (\operatorname{ran}\alpha)\alpha^{-1} = \operatorname{dom}\alpha$. Define $\beta = (\alpha\theta)^{-1}\alpha(\theta\alpha)^{-1}$. It is evident that $\beta \in OI(X,Y)$. We also have that $\alpha\theta\beta\theta\alpha = \alpha\theta(\alpha\theta)^{-1}\alpha(\theta\alpha)^{-1}\theta\alpha = 1_{\operatorname{dom}(\alpha\theta)}\alpha 1_{\operatorname{ran}(\theta\alpha)} = 1_{\operatorname{dom}\alpha}\alpha 1_{\operatorname{ran}\alpha} = \alpha$, so $\alpha \in \mathrm{Reg}((OI(X,Y),\theta))$, as desired. $\square$

As in the proof of Theorem 4.10, we can see that Theorem 4.10(i) implies Theorem 4.9(i) and the converse holds if α is 1-1.

Finally, we shall apply Theorem 4.9 and Theorem 4.10 to prove Theorem 1.17 and Theorem 1.18, respectively. The following lemma is required.

Lemma 4.11. *Let $OS(X,Y)$ be $OP(X,Y)$ or $OI(X,Y)$ and $\theta \in OS(Y,X)$. If the semigroup $(OS(X,Y),\theta)$ is regular, then $\operatorname{dom}\theta = Y$ and $\operatorname{ran}\theta = X$.*

Proof. We prove the lemma by contrapositive. Assume that $\operatorname{dom}\theta \neq Y$ or $\operatorname{ran}\theta \neq X$.

Case 1: $\operatorname{dom}\theta \neq Y$. Let $y \in Y \smallsetminus \operatorname{dom}\theta$ and $x \in X$. Then $\binom{x}{y} \in OI(X,Y) \subseteq OP(X,Y)$. But $\operatorname{ran}(\binom{x}{y}) = \{y\} \nsubseteq \operatorname{dom}\theta$, so by Theorem 4.9 and Theorem 4.10, we have $\binom{x}{y}$ is not a regular element of $(OS(X,Y),\theta)$.

Case 2: $\operatorname{ran}\theta \neq X$. Let $x \in X \smallsetminus \operatorname{ran}\theta$ and $y \in Y$. Then $\binom{x}{y} \in OI(X,Y) \subseteq OP(X,Y)$. But $\theta\binom{x}{y} = 0, \operatorname{ran}(\binom{x}{y}) = \{y\}$ and $\operatorname{dom}(\binom{x}{y}) = \{x\} \nsubseteq \operatorname{ran}\theta$, so by

Theorem 4.9 and Theorem 4.10, $\binom{x}{y} \notin \text{Reg}((OS(X,Y),\theta))$.

Hence $(OS(X,Y),\theta)$ is not a regular semigroup, and hence the lemma is proved. $\square$

Theorem 4.12. *For $\theta \in OP(X,Y)$, the semigroup $(OP(X,Y),\theta)$ is regular if and only if*

(i) *θ is an order-isomorphism from Y onto X or*

(ii) $\operatorname{dom}\theta = Y$, $\operatorname{ran}\theta = X$ *and* $|X| = 1$.

Proof. To prove necessity, assume that $(OP(X,Y),\theta)$ is a regular semigroup. We have by Lemma 4.11 that $\operatorname{dom}\theta = Y$ and $\operatorname{ran}\theta = X$. If $|X| = 1$, then (ii) holds. Assume that $|X| > 1$. We will show that θ is an order-isomorphism from Y on to X. It remains to show that θ is 1-1. Suppose on the contrary that θ is not 1-1. Then there exist $a \in X, e, f \in Y$ such that $e < f$ and $e\theta = f\theta = a$. Since $|X| > 1$, there is $b \in X \setminus \{a\}$. Since X is a chain, we get $b < a$ or $a < b$. Define $\alpha : \{a,b\} \to Y$ by

$$\alpha = \begin{pmatrix} b & a \\ e & f \end{pmatrix} \quad \text{if} \quad b < a \quad \text{and} \quad \alpha = \begin{pmatrix} a & b \\ e & f \end{pmatrix} \quad \text{if} \quad a < b.$$

Then $\alpha \in OP(X,Y)$. Since $e, f \in \operatorname{ran}\alpha$, $e\theta = f\theta$ and $e < f$, it follows that θ is not 1-1 on $\operatorname{ran}\alpha$. In view of Theorem 4.9, α is not a regular element of $(OP(X,Y),\theta)$, which is contrary to the hypothesis. Hence we deduce that θ is 1-1, so (i) hold if $|X| > 1$.

To prove sufficiency, assume that (i) or (ii) holds.

Case 1: θ is an order-isomorphism from Y onto X. Then $\operatorname{dom}\theta = Y, \operatorname{ran}\theta = X$ and θ is 1-1. Let $\alpha \in OP(X,Y)$. Then $\operatorname{ran}\alpha \subseteq Y = \operatorname{dom}\theta$ and $\operatorname{ran}(\theta\alpha) = (\operatorname{ran}\theta \cap \operatorname{dom}\alpha)\alpha = (X \cap \operatorname{dom}\alpha)\alpha = (\operatorname{dom}\alpha)\alpha = \operatorname{ran}\alpha$. It follows from Theorem 4.9 that α is regular in $(OP(X,Y),\theta)$. Hence $(OP(X,Y),\theta)$ is a regular semigroup.

Case 2: $\operatorname{dom}\theta = Y$, $\operatorname{ran}\theta = X$ and $|X| = 1$. Let $\alpha \in OP(X,Y) \setminus \{0\}$. Then $|\operatorname{ran}\alpha| = 1$, so θ is 1-1 on $\operatorname{ran}\alpha$. Since $\operatorname{dom}\theta = Y$, $\operatorname{ran}\theta = X$ and $|X| = 1$, it follows that $\operatorname{ran}\alpha \subseteq \operatorname{dom}\theta$ and $\operatorname{ran}\alpha = \operatorname{ran}(\theta\alpha)$. Hence by Theorem 4.9, α is regular in $(OP(X,Y),\theta)$. This shows that $(OP(X,Y),\theta)$ is a regular semigroup. $\square$

Theorem 4.13. *For $\theta \in OI(Y,X)$, the semigroup $(OI(X,Y),\theta)$ is regular if and only if θ is an order-isomorphism from Y onto X.*

Proof. Assume that $(OI(X,Y),\theta)$ is a regular semigroup. By Lemma 4.11, $\operatorname{dom}\theta = Y$ and $\operatorname{ran}\theta = X$. Since $\theta \in OI(Y,X)$, it follows that θ is order-preserving and θ is 1-1. Therefore we deduce that θ is an order-isomorphism from Y onto X.

Conversely, assume that θ is an order-isomorphism from Y onto X. Then $\operatorname{dom}\theta = Y$ and $\operatorname{ran}\theta = X$. If $\alpha \in OI(X,Y)$, then $\operatorname{dom}\alpha \subseteq X = \operatorname{ran}\theta$ and $\operatorname{ran}\alpha \subseteq Y = \operatorname{dom}\theta$, so by Theorem 4.10, α is regular in $(OI(X,Y),\theta)$. Hence $(OI(X,Y),\theta)$ is a regular semigroup, as required. $\square$

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