Chapter IV

Quantum Probabilities

4.1 Introduction

The probabilities of a classical system, namely, classical probabilities, obey Kolmogorov's axioms. However, experiments have led some physicists to believe that quantum probabilities, that is, probabilities of a quantum system, is radically different. Accardi (1981) uses transition probabilities to distinguish between Kolmogorovian, complex Hilbert space and real Hilbert space probability models. Aerts (1986) adopts some of Accardi's definitions but modifies his definition of conditional probability. He proposes that quantum probabilities come from our lack of knowledge about the measurements while classical probabilities come from our lack of knowledge about the state of the system. Pitowsky (1986) studies the pair distribution and finds that classical probability is more restrictive than quantum probability.

4.2 Axiomatic Quantitative Probability

The axiomatic quantitative probability was developed by Andrei Nikolaevich Kolmogorov in 1929 (Kolmogorov, 1956) and has since been widely accepted. The Kolmogorov setup for probability consists of a probability space $(\Omega, \mathcal{F}, P)$ having as components a sample space, Ω; a σ-field (also called a σ-algebra) $\mathcal{F}$ of selected subsets of Ω; and a probability measure or assignment, P (Fine, 1973, pp.58-59). The sample space Ω has elements ω called the elementary events. The σ-field of subsets of Ω, $\mathcal{F}$, has the following three properties :

1) $\mathcal{F} \in \Omega$

2) If $F \in \mathcal{F}$, then $\overline{F} \in \mathcal{F}$ (closure under complementation)

3) If for countably many i, $F_i \in \mathcal{F}$, then $\cup_i F_i \in \mathcal{F}$ (closure under countable unions)

The <u>probability measure</u> P is a set function from $\mathcal{F}$ to the interval [0, 1], $P : \mathcal{F} \to [0,1]$ and it satisfies the following three axioms (Helstrom, 1984, pp.8-10), (several different but mathematically equivalent forms of the axioms can be given) :

1. $P(A) \geq 0 \qquad \forall A \in \mathcal{F}$
2. $P(\Omega) = 1$
3. If $A_i \cap A_j = \emptyset$ for all i and j, $i \neq j$, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Corollary 1. With $\overline{A}$ the complement of A,

$$P(\overline{A}) = 1 - P(A)$$

Corollary 2. $0 \leq P(A) \leq 1$

Corollary 3. If $A_i \cap A_j = \emptyset$ for all i and j, $i \neq j$, $1 \leq i \leq n$, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

Corollary 4. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Conditional Probability

The conditional probability of event A, given event M, is defined as (see, for example, Helstrom, 1984, p.22)

$$P(A \mid M) = P(A \cap M) \;/\; P(M) \qquad (4.1)$$

in terms of the probability measure assigned to the experiment.

Independence

Events A and B are said to be stochastic or statistically independent if (see, for example, Helstrom, 1984, p.27)

$$P(A \mid B) = P(A) \;; \qquad (4.2)$$

that is, the probability of event A is the same whether event B has occurred or not. By the definition of conditional probability (4.1), an equivalent condition for independence is

$$P(A \cap B) = P(A)\, P(B) \qquad (4.3)$$

4.3 Interpretation of the Concept of Probability

We can interpret the probability concept in many ways (Helstrom, 1984, pp.13-20).

4.3.1 Classical Concept

For a chance experiment which has n finite outcomes, we may assign probability 1/n to each atomic event equally, i.e., all events are equally probable. We say that we are using the principle of sufficient reason or the principle of indifference which states that there are no reasons to favour one outcome over the others.

For a composite event, its probability is then equal to the number of outcomes it contains divided by n.

Clearly, this classical concept of probability fails in the case of chance experiments with continuum outcomes (i.e. outcomes which are not countable).

4.3.2 Relative Frequency

We define relative frequency of outcome k, $q_N(k)$ as

$$q_N(k) = n_N(k) \,/\, N$$

Where $n_N(k)$ is the number of times the outcome k (k = 1, 2, 3, ... l) turns up in our chance experiment with N trials.

Then,

$$\sum_{k=1}^{l} n_N(k) = N,$$

and $$\sum_{k=1}^{1} q_N(k) = 1.$$

We believe that as the number of trials N increases, the relative frequency of outcome k approaches the probability of that outcome.

$$q_N(k) \rightarrow P\{k\} \quad \text{as } N \rightarrow \infty$$

For composite events we can likewise define the relative frequency $q_N(A)$ of event A as

$$q_N(A) = n_N(A) \;/\; N$$

where $n_N(A)$ is the number of times the event A turns up in N trials.

So, in some chance experiments which the principle of sufficient reason is inapplicable, we may use the empirical approach and take the relative frequency of event as the probability of that event, provided that N is large.

4.3.3 Subjective or Personal Probability

The probability concepts that we have described so far is called objective probability. It is universal and replicable. Different people always agree on the probability of the same event. In contrast, subjective probability (Roberts, 1979, 1979, pp.371-372) represents "the degree of certainty" that an individual thinks some event may or may not occur. For example, we may say that a nuclear war will occur in ten years. But different people often have different subjective probability of this event and even we can sometimes change our mind. Due to this personal aspect of the subjective probability, it is not widely used in physics research.

4.3.4 Propensity Interpretation

The propensity interpretation of probability argues that in a chance experiment of a physical system the probability of an event is the property of that system itself. It is the propensity of that system to behave in the way we choose to observe during our chance experiment. Propensities and their probabilities are related to the phase space of the system, which represents the motion or behaviour of the system.

Let R_A be a region in phase space so defined that if phase point (i.e. the point representing the state in the phase space) is in R_A, event A occurs.

As the number of trials N increases, the relative frequency of event A, $q_N(A)$, defined as $n_N(A)/N$ (where $n_N(A)$ is the number of times event A occurs in N trials) approaches V_A/V, the relative volume of region R_A. We can thus identify the probability of a chance experiment of a physical system with the property of the system itself.

Remarks

1. There are some schools which regard the conditional probability as the basic concept (Ballentine, 1986). They always write $P(A \mid C)$ (the probability of A conditional on C) instead of $P(A)$, say.

2. We can interpret the argument A of $P(A)$ (and C of $P(A \mid C)$) as events or propositions.

Events : the probability that event A will occur (under the conditions specified by the occurrence of event C)

Propositions : the probability that A is true (provided C is true)

Corresponding to event A is the proposition "event A has occurred." But there are propositions that do not correspond to events.

4.4 Quantum Probabilities

Some physicists believe that classical probabilities, i.e., probabilities that obey the (Kolmogorov's) axioms of probability theory, do not apply to quantum mechanics. We must use "quantum probabilities" instead.

Let us now see how some major schools look at this problem.

4.4.1 Accardi

Luigi Accardi and his colleagues (Accardi and Fedullo, 1982; Accardi, 1981) have distinguished between Kolmogorovian, complex Hilbert space and real Hilbert space probability models, using transition (i.e. conditional) probabilities.

Accardi defines transition probabilities $P(A = a_\alpha \mid B = b_\beta)$ as the probability that A takes the value a_α conditioned by the fact that B is known to assume the value b_β, where A, B, C,... denote some observable quantities with values (a_α), (b_β), (c_γ)... respectively. Assume $\alpha, \beta, \gamma, \ldots = 1, 2, \ldots, n$ for some $n < +\infty$, independent of A, B, C,... and that $a_\alpha, b_\beta, c_\gamma, \ldots \in R$. Moreover, we assume the symmetry condition

$$P(A = a_\alpha \mid B = b_\beta) = P(B = b_\beta \mid A = a_\alpha), \ldots \qquad (4.4)$$

and that for each $\alpha, \beta, \ldots$

$$P(A = a_\alpha \mid B = b_\beta) > 0, \ldots \qquad (4.5)$$

A Kolmogorovian (classical) model for the transition probabilities is defined by a probability space $(\Omega, \mathcal{O}, \mu)$ and a measurable partition of Ω - (A_α), (B_β), (C_γ),... for each A, B, C such that

$$P(A = a_\alpha \mid B = b_\beta) = \mu(A_\alpha \cap B_\beta) / \mu(B_\beta), \ldots \qquad (4.6)$$

A complex (respectively, real) Hilbert space model for the transition probabilities is defined by a complex (respectively, real) Hilbert space $\mathcal{K}$ and an orthonormal basis such that

$$P(A = a_{\alpha} \mid B = b_{\beta}) = |\langle \phi_A, \Psi_B \rangle|^2 , \ldots \qquad (4.7)$$

if $\{\phi_A\}$ and $\{\Psi_B\}$ are the bases corresponding to A and B and so on.

Accardi considers two-valued observables A, B, C with this notation.

$$P(A \mid B) = P = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

$$P(B \mid C) = Q = \begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix} \qquad (4.8)$$

$$P(C \mid A) = R = \begin{pmatrix} r & 1-r \\ 1-r & r \end{pmatrix}$$

where $0 < p, q, r < 1$.

His results can be summarized in Table 4.1

P, Q, R will admit	if and only if
Kolmogorovian model	$\lvert p+q-1\rvert \leq r \leq 1-\lvert p-q\rvert$
complex Hilbert space model	$[\sqrt{pq}-\sqrt{(1-p)(1-q)}]^2 \leq r \leq [\sqrt{pq}+\sqrt{(1-p)(1-q)}]^2$
real Hilbert space model	$\sqrt{r}=\sqrt{pq}+\sqrt{(1-p)(1-q)}$ or $\sqrt{r}=\lvert\sqrt{pq}-\sqrt{(1-p)(1-q)}\rvert$

Table 4.1

Accardi's criteria for distinguishing between Kolmogorovian, complex Hilbert space and real Hilbert space probability models.

He denotes by (Kolm), (C-Hilbert). and (R-Hilbert), the family of triples P, Q, R of matrices of the form (4.8) which admit, respectively, a Kolmogorovian, a complex Hilbert space, and a real Hilbert space model, and concludes that :

1. (R-Hilbert) $\subseteq$ (C-Hilbert) -- strict inclusion
2. (Kolm) $\subseteq$ (C-Hilbert) -- strict inclusion (for 2-valued observables)
3. If $[\sqrt{pq} - \sqrt{(1-p)(1-q)}]^2 < r < \lvert p+q-1\rvert$ or

$$1-\lvert p-q\rvert < r < [\sqrt{pq} + \sqrt{(1-p)(1-q)}]^2$$

then P, Q, R will admit neither a Kolmogorovian nor a real Hilbert space model but admit a complex Hilbert space model.

4.4.2 Aerts

Dirk Aerts (1986) proposes that quantum probabilities come from our lack of knowledge about the measurements while the lack of knowledge about the state of the system produces classical probabilities.

He begins by summarizing the differences between hidden variable theory (see 2.5 and Appendix C) and quantum theory in Table 4.2

	hidden variable theory	quantum theory
observables (algebraic approach)	commutative algebra	noncommutative algebra
properties (lattice theoretic approach)	Boolean lattice	non-Boolean lattice
probability models (probability approach)	Kolmogorovian model	quantum model

Table 4.2

Differences between hidden variable theory and quantum theory, according to Aerts.

Aerts adopts Accardi's definitions of the Kolmogorovian model, and of the complex and real Hilbert space models (see 4.4.1) but he modifies conditional probability. Conditional probability $P(e = e_i \mid f = f_j)$, Aerts says, is the probability of finding the outcome e_i when we perform the measurement e, when the state of the system is such that if we performed the measurement f we would find the outcome f_i. For measurements of the first kind (see Appendix F) Aerts' definition and the conventional definition (eq. 4.1) are equivalent, but in general they differ.

So equations (4.6) and (4.7) become

$$P(e = e_i \mid f = f_j) = \mu(E_i \cap F_j) / \mu(F_j)$$

(where E_i is the set of states for which the measurement e gives outcome e_i, etc.)

and $P(e = e_i \mid f = f_i) = |\langle \phi_i, \Psi_j \rangle|^2$

(where $\{\phi_i\}$ is an orthonormal basis corresponding to e, etc.).

He then gives these two examples.

Example A : a macroscopic system that admits a quantum probability model.

Consider a particle with positive charge q on a sphere radius r, at a point (r, Θ, ø). Let q_1 and q_2 be two negative charges and $q_1 + q_2 = Q$. The measurement e : choose q_1 randomly between 0 and Q (this introduces the lack of knowledge about the measurement) then put q_1 at (r, α, β) and q_2 at (r, π - α, π + β) on the sphere (see Fig. 4.1 and 4.2)

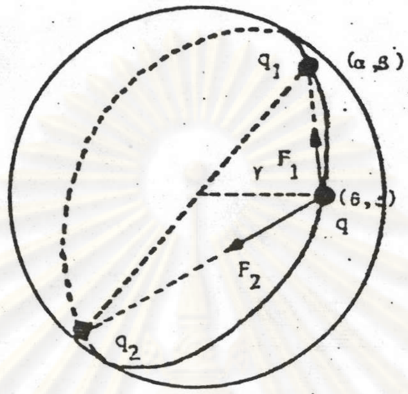

Fig 4.1 A positive charge q is located on the sphere at (r, Θ, ø) and two negative charges q_1 and q_2 are chosen as explained in the text and located on the sphere at points (r, α, β) and (r, π - α, π + β) (Fig 4.1-4.5 are after Aerts, 1986).

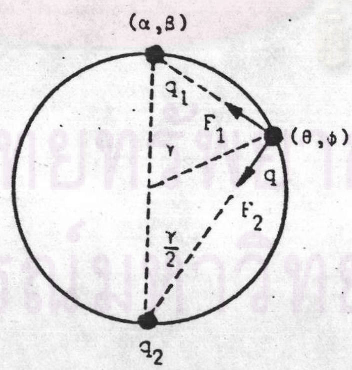

Fig 4.2 We consider the three charges of Fig. 4.1 as they are located in one plane.

We say that the measurement has an outcome e_1 if $|F_1| > |F_2|$

and outcome e_2 if $|F_2| > |F_1|$. Where F_1 is the Coulomb force of q_1 on q and F_2 is the Coulomb force of q_2 on q. He finds that $P(|F_1| > |F_2|) = \cos^2\gamma/2$ (γ is the angle between (r, Θ, ϕ) and (r, α, β) which is the same as the probability resulting from the measurement of the spin in the (α, β) direction of a (θ, ϕ) direction spin -1/2 particle.

Suppose the charge q is in every direction (θ, ϕ) with equal probability. Consider measurements e, f, g such that $e = e_{0,0}$, $f = e_{\pi/3,0}$, and $g = e_{2\pi/3,0}$ (see Fig. 4.3) (this introduces the lack of knowledge about the state of the system).

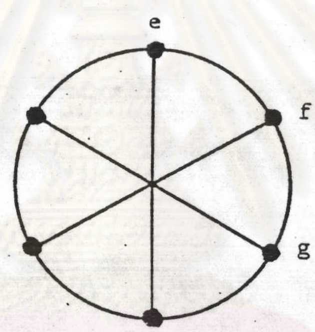

Fig 4.3 The three measurements e, f and g that are considered to show that the system of Fig. 4.1 does not allow a Kolmogorovian probability model.

Then he shows that this example cannot be replaced by a Kolmogorovian probability model.

<u>Example B</u> : a macroscopic system that admits neither a Kolmogorovian nor a quantum probability model.

Consider almost the same setup as example A, but now the charge q can move on the sphere. Let F'_1 be the projection of F_1 on the tangent plane at (r, θ, ϕ) and F'_2 the projection of F_2 on the same plane (see Fig. 4.4). If $|F'_1| > |F'_2|$, q will move towards q_1 call it outcome e_1, if $|F'_1| < |F'_2|$, q will move towards q_2 and it is outcome e_2.

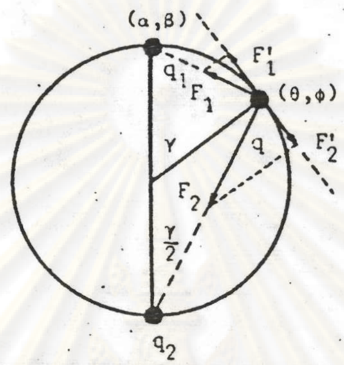

Fig. 4.4 The same physical situation as Fig. 4.1 but now the charge q is only allowed to move on the sphere.

Consider the measurements e, f, g such that $e = e_{0,0}$, $f = e_{2\pi/3,0}$ and $g = e_{-2\pi/3,0}$ (see Fig. 4.5).

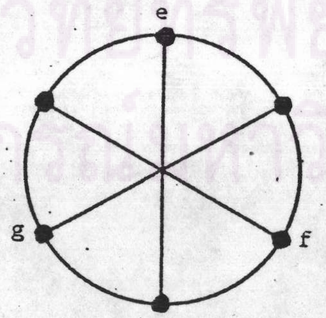

Fig 4.5 The three measurements e, f and g that are considered to show that the system of Fig. 4.4 does not allow a Kolmogorovian or a quantum probability model.

He finds that this system cannot be described by a Kolomogorovian or a quantum probability model. Aerts then concludes that a lack of knowledge about the measurements not only can give rise to a quantum probability model (as in example A) but also a non-Kolmogorovian, non-quantum probability model.

4.4.3 Pitowsky

Itamar Pitowsky (1986) studies the pair distribution in quantum mechanics and compares it with the classical one.

He calls L_n the set of all phenomenal pair distributions of order n, C_n the set of all phenomenal pair distributions of order n that have a classical representation, Q_n the set of all phenomental pair distributions that have a quantum mechanical representation.

A matrix $p \in SR_n$ (the set of all $n \times n$ real symmetric matrices) is called a phenomenal pair distribution of order n if, for all i, j = 1, 2,..., n,

$$0 \leq p_{ij} \leq \min (p_{ii}, p_{jj}) \leq \max (p_{ii}, p_{jj}) \leq 1$$

Let $s_1, s_2, \ldots, s_n$ be "events" or "states" of some system and put p_{ii} = prob (s_i) and p_{ij} = prob (s_i & s_j) then surely $p = (p_{ij}) \in L_n$.

A phenomenal pair distribution $p \in L_n$ has a classical

<u>representation</u> if there exists a probability space (X, Σ, μ) and events $A_1, \ldots, A_n \in \Sigma$ such that $p_{ij} = \mu(A_i \cap A_j)$, $i, j = 1, 2, \ldots, n$.

A phenomenal pair distribution $p \in L_n$ has a <u>quantum mechanical representation</u> if there exists a separable complex Hilbert space H, a density operator (statistical operator) W on H and (continuous) projections $E_1, \ldots, E_n$ (not necessarily pairwise commutative) such that

$$p_{ij} = \hbar\,[W\,(E_i \wedge E_j)], \quad i, j = 1, 2, \ldots, n$$

where $E_i \wedge E_j$ denotes the projection onto the closed subspace $E_i\,(H) \cap E_j\,(H)$.

He then finds that Q_n contains the whole interior of L_n and that all phenomenal pair distributions have quantum representation (except those that lie on the faces of L_n). Various interference phenomena yield pair distributions that are not classical. He calls Q_n/C_n, that is, the set of all the matrices that have a quantum mechanical representation but not a classical representation, "<u>the interference region</u>."

Examples of pair distribution that lies in the interference region :

1. <u>EPR experiment</u> : measurement of spin on a pair of

electrons in the singlet state while the particles are sufficiently separated (see Appendix C).

He states "Bell's Theorem" : There is a choice of directions x, y, z such that $p \notin C_3$. $p \in C_3$ only if

$$1/2 \sin^2(\hat{xy}/2) + 1/2 \sin^2(\hat{yz}/2) \geq 1/2 \sin^2(\hat{xz}/2)$$

which is violated for, say $\theta = 60^\circ$.

2. Scattering of identical particles : (appropriate normalization assumed) a completely elastic proton-proton scattering in which only Coulomb forces play an effective role (spin independent interaction)

s_1 : the left proton is scattered into the upper half of the scattering plane.

s_2 : the left proton is scattered into the lower half of the scattering plane.

Let $0 < \theta < \pi/2$ and let $\Delta\theta$ be a small angle.

S_3 : a proton is detected at $(\theta - (1/2)\Delta\theta, \theta + (1/2)\Delta\theta)$,

He shows that

$$p_{33} = C\,|f(\theta) - f(\pi - \theta)|^2 \Delta\theta\,,$$

$$p_{13} = C\,|f(\theta)|^2\,\Delta\theta$$

$$p_{23} = C\,|f(\pi - \theta)|^2\,\Delta\theta$$

$p_{33} \neq p_{13} + p_{23}$ even though $p_{11} + p_{22} = 1$, $p_{12} = 0$, so that $p \notin C_3$.

He concludes that :

1. classical probability is more restrictive than quantum probability, and

2. save for some boundary cases, every pair (and in fact every multiple) phenomenal distribution has a quantum representation.