CHAPTER III

TRANSFORMATION SEMIGROUPS

The characterization in the term of cardinality of a set X of each of P_X, T_X and I_X which has a proper dense subsemigroup was given by Higgins in [2] where P_X, T_X and I_X are the partial transformation semigroup on X, the full transformation semigroup on X and the symmetric inverse semigroup on X.

A continuation of this work in characterizing transformation semigroups having proper dense subsemigroups is given in this chapter. We characterize each of G_X, M_X, O_X, CP_X and CT_X which has a proper dense subsemigroup in term of the cardinality of the set X where G_X, M_X, O_X, CP_X and CT_X are the symmetric group on X, the transformation semigroup of all 1-1 transformations of X, the transformation semigroup of all onto transformations of X, the transformation semigroup of all constant partial transformations of X and the transformation semigroup of all constant transformations of X, respectively.

The next three theorems of this chapter show that if S is any one of G_X, M_X or O_X, then S has a proper dense subsemigroup if and only if X is infinite. To prove these theorems, we need the following fact: For any infinite set X, there exists a subset A of X such that $|A| = |X|$ and $X \setminus A$ is infinite countable. To show this, let X be an infinite set.

Case 1 : X is countable. Then there is a 1-1 correspondence between X and $\mathbb{N}$ (the set of positive integers). But $2\mathbb{N}$ ($=\{2n \mid n \in \mathbb{N}\}$) is a subset of $\mathbb{N}$ such that $|2\mathbb{N}| = |\mathbb{N}|$ and $\mathbb{N} \setminus 2\mathbb{N}$ is infinite and countable, so there exists a subset A of X such that $|A| = |X|$ and $X \setminus A$ is infinite and countable.

Case 2 : X is uncountable. Since X is an infinite set, X has an infinite countable subset, say C. Let $A = X \setminus C$. Then $|A| = |X|$ and $X \setminus A = C$ which is infinite and countable.

Also, the following lemmas are required.

Lemma 3.1. If G is a group and U is a subsemigroup of G containing 1, the identity of G, then $U^{-1} \subseteq \mathrm{Dom}(U,G)$.

Proof : Let $x \in U^{-1}$. Then $x^{-1} \in U$. Since

$$\begin{aligned} x &= 1x, && 1 \in U, \quad x \in G, \\ &= xx^{-1}x, && x^{-1} \in U, \quad x \in G, \quad 1 = xx^{-1}, \\ &= x1, && 1 \in U, \quad x^{-1}x = 1, \end{aligned}$$

it follows from Theorem 1.1 that $x \in \mathrm{Dom}(U,G)$. Hence $U^{-1} \subseteq \mathrm{Dom}(U,G)$. #

Lemma 3.2. Let X be an infinite set and let $A \subseteq X$ be such that $|A| = |X|$ and $X \setminus A$ is infinite countable. Let S denote any one of G_X, M_X or O_X where G_X, M_X and O_X are the symmetric group on X, the transformation semigroup of all 1-1 transformations of X and the transformation semigroup of all onto transformations of X, respectively, and let

$$U = \{\alpha \in S \mid A \subseteq A\alpha\}$$

and

$$U' = \{\alpha \in S \mid A\alpha \subseteq A\}.$$

Then U and U′ are proper subsemigroups of S containing 1_X, the identity map on X.

<u>Proof</u> : Clearly, $1_X \in U$ and $1_X \in U'$. Let $\alpha, \beta \in S$. If $A \subseteq A\alpha$ and $A \subseteq A\beta$, then $A\alpha\beta = (A\alpha)\beta \supseteq A\beta \supseteq A$. If $A\alpha \subseteq A$ and $A\beta \subseteq A$, then $A\alpha\beta = (A\alpha)\beta \subseteq A\beta \subseteq A$. This proves that U and U′ are subsemigroups of S containing 1_X.

To show that $U \neq S$ and $U' \neq S$, let x be a point in $X \smallsetminus A$. Then $|A \cup \{x\}| = |A|$ and $|X \smallsetminus (A \cup \{x\})| = |X \smallsetminus A|$. Thus there are bijections $\varphi : A \cup \{x\} \to A$ and $\varphi' : X \smallsetminus (A \cup \{x\}) \to X \smallsetminus A$. Let $\beta : X \to X$ be such that $\beta|_{A\cup\{x\}} = \varphi$ and $\beta|_{X \smallsetminus (A\cup\{x\})} = \varphi'$. Therefore $\beta \in G_X \subseteq S$, so $\beta^{-1}|_A = \varphi^{-1}$ and $\beta^{-1}|_{X \smallsetminus A} = \varphi'^{-1}$. Then $A\beta \subsetneq (A \cup \{x\})\beta = A$ and $A \subsetneq A \cup \{x\} = A\beta^{-1}$ which imply that $\beta \notin U$ and $\beta^{-1} \notin U'$. Hence U and U′ are proper subsemigroups of S containing 1_X. #

<u>Lemma 3.3</u>. Let X be an infinite countable set, A a subset of X such that $|A| = |X| = |X \smallsetminus A|$ and B a subset of X. Then the following statements hold.

(i) If $A \cap B$ is infinite, then there exists $\alpha \in G_X$ such that $A \subseteq A\alpha$ and $A \subseteq B\alpha$.

(ii) If $X \smallsetminus (A \cup B)$ is infinite, then there exists $\eta \in G_X$ such that $A\eta \subseteq A$ and $B\eta \subseteq A$.

(iii) If B is infinite and $X \smallsetminus (A \cup B)$ and $A \cap B$ are finite, then there exists $\lambda \in G_X$ such that $A\lambda \subseteq A$ and $A \cap B\lambda$ is infinite.

<u>Proof</u> : (i) By assumption, we have $|A \cap B| = |A| = |X \setminus (A \cap B)| = |X \setminus A|$, so there is an element α in G_X such that $(A \cap B)\alpha = A$ and $(X \setminus (A \cap B))\alpha = X \setminus A$. Then

$$\begin{aligned} A\alpha &= [(A \setminus B) \cup (A \cap B)]\alpha \\ &= (A \setminus B)\alpha \cup (A \cap B)\alpha \\ &= (A \setminus B)\alpha \cup A \end{aligned}$$

and

$$\begin{aligned} B\alpha &= [(A \cap B) \cup (B \cap (X \setminus A))]\alpha \\ &= (A \cap B)\alpha \cup (B \cap (X \setminus A))\alpha \\ &= A \cup (B \cap (X \setminus A))\alpha \end{aligned}$$

which imply that $A \subseteq A\alpha$ and $A \subseteq B\alpha$.

(ii) By assumption, we get $|A \cup B| = |A| = |X \setminus (A \cup B)| = |X \setminus A|$, so there exists an element η in G_X such that $(A \cup B)\eta = A$ and $(X \setminus (A \cup B))\eta = X \setminus A$. Since $A = (A \cup B)\eta = A\eta \cup B\eta$, we obtain that $A\eta \subseteq A$ and $B\eta \subseteq A$.

(iii) Assume that B is infinite and $X \setminus (A \cup B)$ and $A \cap B$ are finite. Let $C = B \cap (X \setminus A)$. Then C is infinite since B is infinite and $B = (B \setminus C) \cup C = (A \cap B) \cup C$. Since C is infinite, there exists an infinite subset D of C such that $|D| = |C| = |C \setminus D|$. Then $|A \cup (C \setminus D)| = |A|$. From the facts that $X \setminus (A \cup (C \setminus D)) = (X \setminus (A \cup C)) \cup D$, $X \setminus (A \cup C) = X \setminus (A \cup B)$ and $X \setminus (A \cup B)$ is finite, we obtain $|X \setminus (A \cup (C \setminus D))| = |D| = |X \setminus A|$. Let λ be an element of G_X such that $(A \cup (C \setminus D))\lambda = A$ and $(X \setminus (A \cup (C \setminus D)))\lambda = X \setminus A$. Then $A = (A \cup (C \setminus D))\lambda = A\lambda \cup (C \setminus D)\lambda$, so $A\lambda \subseteq A$ and $(C \setminus D)\lambda \subseteq A$.

Next, we shall prove that $B\lambda \cap A$ is infinite. Since

$$B\lambda \cap A = ((A\cap B)\cup C)\lambda \cap A$$
$$= [(A\cap B)\lambda \cup (C\lambda)] \cap A$$
$$= [(A\cap B)\lambda \cup (C\smallsetminus D)\lambda \cup D\lambda] \cap A$$
$$= [(A\cap B)\lambda \cap A] \cup [(C\smallsetminus D)\lambda \cap A] \cup [D\lambda \cap A],$$

we obtain that $|B\lambda \cap A| \geqslant |(C\smallsetminus D)\lambda \cap A|$. But $(C\smallsetminus D)\lambda \subseteq A$, so $|B\lambda \cap A| \geqslant |(C\smallsetminus D)\lambda| = |C\smallsetminus D| = |D|$. Hence $B\lambda \cap A$ is infinite. #

Theorem 3.4. For a set X, the symmetric group on X has a proper dense subsemigroup if and only if X is infinite.

Proof : Let G_X be the symmetric group on X. If X is finite, then G_X is a finite group, so by Theorem 2.5, G_X has no proper dense subsemigroup. Hence, if G_X has a proper dense subsemigroup, then X is infinite.

Conversely, assume that X is infinite. Let $A \subseteq X$ be such that $|A| = |X|$ and $X \smallsetminus A$ is infinite countable. Let

$$U = \{\alpha \in G_X \mid A \subseteq A\alpha\} .$$

By Lemma 3.2, we have that U is a proper subsemigroup of G_X containing 1_X.

To prove that U is dense in G_X, let $\alpha \in G_X$. Let $B = A\alpha \cap (X\smallsetminus A)$. Then B is countable, $A\alpha \cap A = A\alpha \smallsetminus B$ and $B = A\alpha \smallsetminus A$.

Case 1 : X is uncountable. Then A is uncountable and therefore $A\alpha$ is uncountable. This implies that $A\alpha \cap A \neq \emptyset$ since $X\smallsetminus A$ is countable. Then $|A| = |A\alpha| = |(A\alpha\smallsetminus B)\cup B| = |A\alpha\smallsetminus B|+|B| = |A\alpha\smallsetminus B|$ since $A\alpha$ is uncountable and B is countable. From the facts that $X\smallsetminus A$ is infinite countable and B is countable, we have $|X\smallsetminus A| = |(X\smallsetminus A\alpha)\cup B| = |X\smallsetminus (A\alpha\smallsetminus B)|$.

Then there is an element γ in G_X such that $(A\alpha \setminus B)\gamma = A$ and $(X \setminus (A\alpha \setminus B))\gamma = X \setminus A$. Thus

$$
\begin{aligned}
A\gamma &= [(A \setminus A\alpha) \cup (A\alpha \cap A)]\gamma \\
&= [(A \setminus A\alpha) \cup (A\alpha \setminus B)]\gamma \\
&= (A \setminus A\alpha)\gamma \cup A
\end{aligned}
$$

which implies that $A \subseteq A\gamma$, and hence $\gamma \in U$. By Lemma 3.1, $\gamma^{-1} \in \mathrm{Dom}(U, G_X)$. Also, we have that

$$
\begin{aligned}
A\alpha\gamma &= ((A\alpha \setminus B) \cup B)\gamma \\
&= (A\alpha \setminus B)\gamma \cup B\gamma \\
&= A \cup B\gamma \ .
\end{aligned}
$$

Therefore $A \subseteq A\alpha\gamma$, and hence $\alpha\gamma \in U$. Since $\mathrm{Dom}(U,G_X)$ is a subsemigroup of G_X and $\alpha\gamma$, $\gamma^{-1} \in \mathrm{Dom}(U,G_X)$, it follows that $\alpha = (\alpha\gamma)\gamma^{-1} \in \mathrm{Dom}(U,G_X)$.

<u>Case 2</u> : X is infinite countable. Then A is infinite countable and therefore $A\alpha$ is infinite countable.

<u>Subcase 2.1</u> : $X \setminus (A \cup A\alpha)$ is infinite. By Lemma 3.3 (ii), there exists η in G_X such that $A\eta \subseteq A$ and $A\alpha\eta \subseteq A$. Therefore $A \subseteq A\eta^{-1}$ and $A \subseteq A(\alpha\eta)^{-1}$ which imply that η^{-1}, $(\alpha\eta)^{-1} \in U$. We have by Lemma 3.1 that $\alpha\eta \in \mathrm{Dom}(U,G_X)$, and hence $\alpha = (\alpha\eta)\eta^{-1} \in \mathrm{Dom}(U,G_X)$.

<u>Subcase 2.2</u> : $X \setminus (A \cup A\alpha)$ is finite. If $A\alpha \cap A$ is infinite, then by Lemma 3.3 (i), there exists an element γ in G_X such that $A \subseteq A\gamma$ and $A \subseteq A\alpha\gamma$. Therefore γ, $\alpha\gamma \in U$. It follows from Lemma 3.1 that $\gamma^{-1} \in \mathrm{Dom}(U,G_X)$, and hence $\alpha = (\alpha\gamma)\gamma^{-1} \in \mathrm{Dom}(U,G_X)$. Assume that $A\alpha \cap A$ is finite. By Lemma 3.3 (iii), there exists λ in G_X such that $A\lambda \subseteq A$ and $A\alpha\lambda \cap A$ is infinite. Therefore by Lemma 3.1, $\lambda \in \mathrm{Dom}(U,G_X)$,

and we have by lemma 3.3 (i) that there exists ν in G_X such that $A \subseteq A\nu$ and $A \subseteq (A\alpha\lambda)\nu$. Then ν, $\alpha\lambda\nu \in U$. By Lemma 3.1, $\nu^{-1} \in Dom(U,G_X)$. Thus $\alpha\lambda = (\alpha\lambda\nu)\nu^{-1} \in Dom(U,G_X)$. Hence $\alpha = (\alpha\lambda)\lambda^{-1} \in Dom(U,G_X)$.

This proves that $Dom(U,G_X) = G_X$. Hence U is a proper dense subsemigroup of G_X. #

We need one more lemma to prove that for $S = M_X$ or O_X, S has a proper dense subsemigroup if and only if X is infinite.

Lemma 3.5. Let X be an infinite set, $A \subseteq X$ be such that $|A| = |X|$ and $X \setminus A$ is infinite countable. Let S denote any one of M_X or O_X. Let

$$U = \{\alpha \in S \mid A \subseteq A\alpha\}$$

and

$$U' = \{\alpha \in S \mid A\alpha \subseteq A\} .$$

Then $G_X \subseteq Dom(U,S)$ and $G_X \subseteq Dom(U',S)$.

Proof : By Lemma 3.2, we have that U and U′ are proper subsemigroups of S containing 1_X. Then $U \cap G_X \subseteq U$ and $(U \cap G_X)^{-1} \subseteq U'$. By the proof of Theorem 3.4, $U \cap G_X$ is a proper dense subsemigroup of G_X. Then by Lemma 2.2, $\langle(U \cap G_X) \cup (U \cap G_X)^{-1}\rangle = G_X$. To prove that $G_X \subseteq Dom(U,S)$ and $G_X \subseteq Dom(U',S)$, it suffices to prove that $(U \cap G_X)^{-1} \subseteq Dom(U,S)$ and $U \cap G_X \subseteq Dom(U',S)$ since $(U \cap G_X) \subseteq U$, $(U \cap G_X)^{-1} \subseteq U'$ and $G_X = \langle(U \cap G_X) \cup (U \cap G_X)^{-1}\rangle$. Let $\alpha \in U \cap G_X$. Then we have that

$$\alpha^{-1} = 1_X\alpha^{-1}, \qquad 1_X \in U,\ \alpha^{-1} \in S,$$

$$= \alpha^{-1}\alpha\alpha^{-1}, \qquad \alpha \in U,\ \alpha^{-1} \in S,\ 1_X = \alpha^{-1}\alpha.$$

$$= \alpha^{-1}1_X, \qquad 1_X \in U,\ \alpha\alpha^{-1} = 1_X$$

and

$$\alpha = 1_X\alpha, \qquad 1_X \in U',\ \alpha \in S,$$

$$= \alpha\alpha^{-1}\alpha, \qquad \alpha^{-1} \in U',\ \alpha \in S,\ 1_X = \alpha\alpha^{-1},$$

$$= \alpha 1_X, \qquad 1_X \in U',\ \alpha^{-1}\alpha = 1_X$$

which imply by Theorem 1.1 that $\alpha^{-1} \in \mathrm{Dom}(U,S)$ and $\alpha \in \mathrm{Dom}(U',S)$. Hence we prove that $G_X \subseteq \mathrm{Dom}(U,S)$ and $G_X \subseteq \mathrm{Dom}(U',S)$. #

Theorem 3.6. For a set X, the transformation semigroup of all 1-1 transformations of X has a proper dense subsemigroup if and only if X is infinite.

Proof : Let M_X be the transformation semigroup of all 1-1 transformations of X. If X is finite, then $M_X = G_X$, so by Theorem 3.4, M_X has no proper dense subsemigroup. Hence, if M_X has a proper dense subsemigroup, then X is infinite.

Conversely, assume that X is infinite. Let $A \subseteq X$ be such that $|A| = |X|$ and $X \smallsetminus A$ is infinite countable. Let

$$U' = \{\alpha \in M_X \mid A\alpha \subseteq A\}.$$

By Lemma 3.2, we have that U' is a proper subsemigroup of M_X containing 1_X. It follows from Lemma 3.5 that $G_X \subseteq \mathrm{Dom}(U',M_X)$.

First, we claim that $\{\alpha \in M_X \mid A \subseteq A\alpha\} \subseteq \mathrm{Dom}(U',M_X)$. To prove the claim, let $\alpha \in M_X$ be such that $A \subseteq A\alpha$. Let $B = A\alpha \cap (X \smallsetminus A)$. Then

B is countable since $X \setminus A$ is countable, and also $A\alpha = A \cup B$. Since α is 1-1 and $X \setminus A$ is infinite countable, we have that $(X \setminus A)\alpha$ is infinite countable. We have $A\alpha \cap (X \setminus A)\alpha = \emptyset$ since α is 1-1, so $(X \setminus A)\alpha \cap (A \cup B) = \emptyset$. Hence $(X \setminus A)\alpha \subseteq X \setminus (A \cup B)$, so $X \setminus (A \cup B)$ is infinite countable. Then $|X \setminus (A \cup B)| = |X \setminus A|$. But $|A \cup B| = |A|$, so there is an element β in G_X such that $(A \cup B)\beta = A$ and $(X \setminus (A \cup B))\beta = X \setminus A$. Then $\beta^{-1} \varepsilon \text{Dom}(U', M_X)$. Since $A\alpha = A \cup B$ and $(A \cup B)\beta = A$, we obtain $A\alpha\beta = A$, so $\alpha\beta \varepsilon U'$. Thus $\alpha = (\alpha\beta)\beta^{-1} \varepsilon \text{Dom}(U', M_X)$.

Next, we shall prove that U' is dense in M_X. Let $\alpha \varepsilon M_X$ and let $B = A\alpha \cap (X \setminus A)$. Then B is countable, $A\alpha \cap A = A\alpha \setminus B$ and $B = A\alpha \setminus A$.

Case 1: X is uncountable. Then A is uncountable, and $|A| = |A\alpha|$ since α is 1-1. Since B is countable, $A\alpha \setminus B$ is infinite. Then there is a subset C of $A\alpha \setminus B$ such that $|C| = |B|$. This follows that $|A\alpha \setminus B| = |A\alpha \setminus (B \cup C)|$. Since α is 1-1, we have that $A\alpha \cap (X \setminus A)\alpha = \emptyset$, so $(X \setminus A)\alpha \subseteq X \setminus A\alpha$. Then $X \setminus A\alpha$ is infinite since $X \setminus A$ is infinite and α is 1-1.

Subcase 1.1: $X \setminus A\alpha$ is uncountable. Since $X \setminus A\alpha = ((X \setminus A) \setminus A\alpha) \cup (A \setminus A\alpha)$ and $X \setminus A$ is countable, we have that $A \setminus A\alpha$ is infinite. Hence $|A \setminus A\alpha| = |(A \setminus A\alpha) \cup B|$ since B is countable. Now we have $|A\alpha \setminus B| = |A\alpha \setminus (B \cup C)|$, $|A \setminus A\alpha| = |(A \setminus A\alpha) \cup B|$ and $|B| = |C|$. Moreover, $(A\alpha \setminus B)$, $(A \setminus A\alpha)$ and B are pairwise disjoint, $A\alpha \setminus (B \cup C)$, $(A \setminus A\alpha) \cup B$ and C are pairwise disjoint and $(A\alpha \setminus B) \cup (A \setminus A\alpha) \cup B = A \cup B = (A\alpha \setminus (B \cup C)) \cup ((A \setminus A\alpha) \cup B) \cup C$. Then there exists $\lambda \varepsilon G_X$ such that $(A\alpha \setminus B)\lambda = A\alpha \setminus (B \cup C)$, $(A \setminus A\alpha)\lambda = (A \setminus A\alpha) \cup B$ and $B\lambda = C$. Then $\lambda^{-1} \varepsilon \text{Dom}(U', M_X)$ and $A\alpha\lambda = ((A\alpha \setminus B) \cup B)\lambda = (A\alpha \setminus B)\lambda \cup B\lambda = (A\alpha \setminus (B \cup C)) \cup C = A\alpha \setminus B \subseteq A$. Hence $\alpha\lambda \varepsilon U'$, so $\alpha = (\alpha\lambda)\lambda^{-1} \varepsilon \text{Dom}(U', M_X)$.

Subcase 1.2: $X \setminus A\alpha$ is infinite countable. Then $|X \setminus (A\alpha \setminus B)| = |X \setminus A|$ and $|A\alpha \setminus B| = |A\alpha| = |A|$ since B is countable. Let $\gamma \varepsilon G_X$ be such that $(A\alpha \setminus B)\gamma = A$ and $(X \setminus (A\alpha \setminus B))\gamma = X \setminus A$. Then $\gamma^{-1} \varepsilon \text{Dom}(U', M_X)$ and $A\alpha\gamma = (A\alpha \setminus B)\gamma \cup B\gamma = A \cup B\gamma$, and therefore $A \subseteq A\alpha\gamma$. By the preceeding claim, $\alpha\gamma \varepsilon \text{Dom}(U', M_X)$. Hence $\alpha = (\alpha\gamma)\gamma^{-1} \varepsilon \text{Dom}(U', M_X)$.

<u>Case 2</u> : X is infinite countable. By assumption, A, $A\alpha$, $X \setminus A$ and $(X \setminus A)\alpha$ are all infinite countable.

<u>Subcase 2.1</u> : $X \setminus (A \cup A\alpha)$ is infinite. Then by Lemma 3.3 (ii), we have that there exists η in G_X such that $A\eta \subseteq A$ and $A\alpha\eta \subseteq A$. Thus $\eta^{-1} \in \mathrm{Dom}(U', M_X)$ and $\alpha\eta \in U'$, and hence $\alpha = (\alpha\eta)\eta^{-1} \in \mathrm{Dom}(U', M_X)$.

<u>Subcase 2.2</u> : $X \setminus (A \cup A\alpha)$ is finite. If $A \cap A\alpha$ is infinite, then by Lemma 3.3 (i), there exists $\beta \in G_X$ such that $A \subseteq A\beta$ and $A \subseteq A\alpha\beta$, so β^{-1}, $\alpha\beta \in \mathrm{Dom}(U', M_X)$, and hence $\alpha = (\alpha\beta)\beta^{-1} \in \mathrm{Dom}(U', M_X)$. Assume that $A \cap A\alpha$ is finite. By Lemma 3.3 (iii), there exists λ in G_X such that $A\lambda \subseteq A$ and $A\alpha\lambda \cap A$ is infinite. Then $\lambda^{-1} \in \mathrm{Dom}(U', M_X)$. It follows from Lemma 3.3 (i) that there exists $\mu \in G_X$ such that $A \subseteq A\mu$ and $A \subseteq (A\alpha\lambda)\mu$. Then μ^{-1}, $\alpha\lambda\mu \in \mathrm{Dom}(U', M_X)$, and hence $\alpha\lambda = (\alpha\lambda\mu)\mu^{-1}$ is an element in $\mathrm{Dom}(U', M_X)$. Since $\lambda^{-1} \in \mathrm{Dom}(U', M_X)$, $\alpha = (\alpha\lambda)\lambda^{-1} \in \mathrm{Dom}(U', M_X)$.

This proves that $\mathrm{Dom}(U', M_X) = M_X$. Hence U' is a proper dense subsemigroup of M_X. #

<u>Theorem 3.7.</u> For a set X, the transformation semigroup of all onto transformations of X has a proper dense subsemigroup if and only if X is infinite.

<u>Proof</u> : Let O_X be the transformation semigroup of all onto transformations of X. If X is finite, then $O_X = G_X$, so by Theorem 3.4, O_X has no proper dense subsemigroup. Hence, if O_X has a proper dense subsemigroup, then X is infinite.

For the converse, assume that X is infinite. Let $A \subseteq X$ be such that $|A| = |X|$ and $X \setminus A$ is infinite countable.

Set

$$U = \{\alpha \in O_X \mid A \subseteq A\alpha\}$$

and

$$U' = \{\alpha \in O_X \mid A\alpha \subseteq A\}.$$

By Lemma 3.2, we have that U and U' are proper subsemigroups of O_X containing 1_X. It follows from Lemma 3.5 that $G_X \subseteq Dom(U, O_X)$ and $G_X \subseteq Dom(U', O_X)$. Let

$$U^* = U' \cup \{\alpha \in O_X \mid A\alpha \text{ is finite}\}.$$

To show that U^* is a proper subsemigroup of O_X, let $\alpha, \beta \in U^*$. If $A\alpha$ and $A\beta$ are finite, then $A\alpha\beta$ is finite. Assume that $A\alpha \subseteq A$ and $A\beta$ is finite. Then $A\alpha\beta = (A\alpha)\beta \subseteq A\beta$ and $A\beta\alpha = (A\beta)\alpha$. Since $A\beta$ is finite, we have that $A\alpha\beta$ and $A\beta\alpha$ are finite. This proves that U^* is a subsemigroup of O_X. Next, we shall show that $U^* \neq O_X$. By Lemma 3.2, $U' \cap G_X \neq G_X$, so there exists $\beta \in G_X$ such that $A\beta \nsubseteq A$. Then $\beta \notin U'$ and $A\beta$ is infinite. Thus $\beta \in O_X \setminus U^*$. Hence U^* is a proper subsemigroup of O_X. Since $U' \subseteq U^*$, $G_X \subseteq Dom(U', O_X) \subseteq Dom(U^*, O_X)$.

Finally, we shall prove that

(i) if X is uncountable, then U is dense in O_X and

(ii) if X is infinite countable, then U^* is dense in O_X.

To prove (i), assume that X is uncountable and let $\alpha \in O_X$. Then A is uncountable and $X = (A \cup (X \setminus A))\alpha = A\alpha \cup (X \setminus A)\alpha$. Since $X \setminus A$ is countable, $A\alpha$ is uncountable. Since $X = A\alpha \cup (X \setminus A)\alpha$, $X \setminus A\alpha \subseteq (X \setminus A)\alpha$ which implies that $X \setminus A\alpha$ is countable. Then $|A| = |X| = |A\alpha| + |(X \setminus A\alpha)| = |A\alpha|$. Let $B = A\alpha \cap (X \setminus A)$. Then B is countable and $A\alpha \setminus B = A\alpha \cap A$. From the facts that $A\alpha$ is an uncountable set and B is a countable set, we obtain that $|A\alpha| = |A\alpha \setminus B|$. Since $X \setminus A\alpha$ is countable, $|X \setminus (A\alpha \setminus B)| = |X \setminus (A\alpha \cap A)| = |(X \setminus A\alpha) \cup (X \setminus A)| = |X \setminus A|$. Let $\gamma \in G_X$ be such that $(A\alpha \setminus B)\gamma = A$ and $(X \setminus (A\alpha \setminus B))\gamma = X \setminus A$. Then $\gamma^{-1} \in Dom(U,O_X)$ since $G_X \subseteq Dom(U,O_X)$. Because

$$\begin{aligned} A\alpha\gamma &= [(A\alpha \setminus B) \cup B]\gamma \\ &= (A\alpha \setminus B)\gamma \cup B\gamma \\ &= A \cup B\gamma \quad , \end{aligned}$$

we obtain $A \subseteq A\alpha\gamma$. Therefore $\alpha\gamma \in U$. Thus $\alpha = (\alpha\gamma)\gamma^{-1} \in Dom(U,O_X)$.

To prove (ii), assume that X is infinite countable. Then A is infinite countable. First, we claim that $\{\alpha \in O_X \mid A \subseteq A\alpha\} \subseteq Dom(U',O_X)$. To prove the claim, let $\alpha \in O_X$ be such that $A \subseteq A\alpha$. Since $A \subseteq A\alpha$, $A\alpha$ is infinite countable.

<u>Case 1</u> : $X \setminus A\alpha$ is infinite. Then $|X \setminus A\alpha| = |X \setminus A| = |A\alpha| = |A|$. Let $\beta \in G_X$ be such that $(A\alpha)\beta = A$ and $(X \setminus A\alpha)\beta = X \setminus A$. Then $\beta^{-1} \in Dom(U',O_X)$. Since $A\alpha\beta = (A\alpha)\beta = A$, we have that $\alpha\beta \in U'$. Thus $\alpha = (\alpha\beta)\beta^{-1} \in Dom(U',O_X)$.

<u>Case 2</u> : $X \setminus A\alpha$ is finite. Then $(X \setminus A) \setminus (X \setminus A\alpha)$ is infinite. But $(X \setminus A) \setminus (X \setminus A\alpha) = (X \setminus A) \cap A\alpha$, so $(X \setminus A) \cap A\alpha$ is infinite. Let $B = A\alpha \cap (X \setminus A)$. Then $A\alpha \cap A = A\alpha \setminus B$. Let $C = B\alpha^{-1} \cap A$. Then $C\alpha = B$ and $(A \setminus C)\alpha \cap B = \emptyset$. From the facts that $A\alpha = ((A \setminus C) \cup C)\alpha = (A \setminus C)\alpha \cup C\alpha$ and $C\alpha = B$, we obtain that $A\alpha = (A \setminus C)\alpha \cup B$, so $A\alpha \setminus B = ((A \setminus C)\alpha \cup B) \setminus B = (A \setminus C)\alpha$. Thus $(A \setminus C)\alpha = A\alpha \setminus B = A\alpha \cap A$. Since $A \subseteq A\alpha$, $A = A\alpha \cap A = (A \setminus C)\alpha$. Then $A \setminus C$ is infinite countable, and hence $|A \setminus C| = |A| = |X \setminus A| = |X \setminus (A \setminus C)|$. Let $\gamma \in G_X$ be such that $A\gamma = A \setminus C$

and $(X \setminus A)\gamma = X \setminus (A \setminus C)$. Then $\gamma^{-1} \in \mathrm{Dom}(U', O_X)$. Since $(A \setminus C)\alpha = A$ and $A\gamma = A \setminus C$, we obtain $A\gamma\alpha = (A\gamma)\alpha = (A \setminus C)\alpha = A$, so $\gamma\alpha \in U'$. Thus $\alpha = \gamma^{-1}(\gamma\alpha) \in \mathrm{Dom}(U', O_X)$.

Hence $\{\alpha \in O_X \mid A \subseteq A\alpha\} \subseteq \mathrm{Dom}(U', O_X)$. Since $\mathrm{Dom}(U', O_X) \subseteq \mathrm{Dom}(U^*, O_X)$, $\{\alpha \in O_X \mid A \subseteq A\alpha\} \subseteq \mathrm{Dom}(U^*, O_X)$.

Next, we shall prove that U^* is dense in O_X, let $\alpha \in O_X$. If $A\alpha$ is finite, then $\alpha \in U^* \subseteq \mathrm{Dom}(U^*, O_X)$. Assume that $A\alpha$ is infinite countable. Let $B = A\alpha \cap (X \setminus A)$. Then $A\alpha \cap A = A\alpha \setminus B$.

<u>Case 1</u> : $X \setminus (A \cup A\alpha)$ is infinite. Then by Lemma 3.3 (ii), there exists $\eta \in G_X$ such that $A\eta \subseteq A$ and $A\alpha\eta \subseteq A$. Then $\eta^{-1} \in \mathrm{Dom}(U^*, O_X)$ and $\alpha\eta \in U^*$, and hence $\alpha = (\alpha\eta)\eta^{-1} \in \mathrm{Dom}(U^*, O_X)$.

<u>Case 2</u> : $X \setminus (A \cup A\alpha)$ is finite. If $A \cap A\alpha$ is infinite, then by Lemma 3.3 (i), there exists $\beta \in G_X$ such that $A \subseteq A\beta$ and $A \subseteq A\alpha\beta$. Then $\beta^{-1} \in \mathrm{Dom}(U^*, O_X)$ and $\alpha\beta \in \mathrm{Dom}(U^*, O_X)$. Hence $\alpha = (\alpha\beta)\beta^{-1} \in \mathrm{Dom}(U^*, O_X)$. Assume that $A \cap A\alpha$ is finite. By Lemma 3.3 (iii), there exists $\lambda \in G_X$ such that $A\lambda \subseteq A$ and $A\alpha\lambda \cap A$ is infinite. Then $\lambda^{-1} \in \mathrm{Dom}(U^*, O_X)$. It follows from Lemma 3.3 (i) that there exists $\beta \in G_X$ such that $A \subseteq A\beta$ and $A \subseteq (A\alpha\lambda)\beta$. Then β^{-1}, $\alpha\lambda\beta \in \mathrm{Dom}(U^*, O_X)$, and hence $\alpha\lambda = (\alpha\lambda\beta)\beta^{-1} \in \mathrm{Dom}(U^*, O_X)$. Since $\lambda^{-1} \in \mathrm{Dom}(U^*, O_X)$, $\alpha = (\alpha\lambda)\lambda^{-1} \in \mathrm{Dom}(U^*, O_X)$.

This prove that $\mathrm{Dom}(U^*, O_X) = O_X$. Hence U^* is a proper dense subsemigroup of O_X. #

Let X be a set. For $A \subseteq X$, $A \neq \emptyset$ and $x \in X$, let A_x denote the partial transformation of X with domain A and range $\{x\}$. Then

$$CP_X = \{A_x \mid \emptyset \neq A \subseteq X \text{ , } x \in X\} \cup \{0\}$$

and

$$CT_X = \begin{cases} \{X_x \mid x \in X\} & \text{if } X \neq \emptyset, \\ \{0\} & \text{if } X = \emptyset. \end{cases}$$

It is easily seen that for $\emptyset \neq A \subseteq X$, $\emptyset \neq B \subseteq X$, $x,y \in X$,

$$A_x B_y = \begin{cases} A_y & \text{if } x \in B, \\ 0 & \text{if } x \in B. \end{cases}$$

In particular, $X_x X_y = X_y$ for all $x,y \in X$. Therefore CT_X is a right zero semigroup.

We shall prove in the following lemma that every right [left] zero semigroup has no proper dense subsemigroup. This implies that for any set X, CT_X has no proper dense subsemigroup.

<u>Lemma 3.8</u>. Every right [left] zero semigroup has no proper dense subsemigroup.

<u>Proof</u> : Let S be a right zero semigroup and U a dense subsemigroup of S. Let $d \in S$. If $d \notin U$, then by Theorem 1.1, there is a zigzag in U over S with value d. This implies that $d = xu$ for some $x \in S$, $u \in U$. Since S is a right zero semigroup, we have that $xu = u$, so $d = u \in U$, a contradiction. Then $d \in U$. Hence $U = S$. This shows that S has no proper dense subsemigroup. #

Theorem 3.9. For any set X, the transformation semigroup of all constant transformations of X has no proper dense subsemigroup.

The last Theorem of this chapter characterizes the transformation semigroup CP_X which has a proper dense subsemigroup in term of the cardinality of X as follows :

Theorem 3.10. For a set X, the transformation semigroup of all constant partial transformations of X has a proper dense subsemigroup if and only if $|X| > 1$.

Proof : Let CP_X be the transformation semigroup of all constant partial transformations of X. If $|X| = 0$, then $CP_X = \{0\}$ which has no proper dense subsemigroup. If $|X| = 1$, then $CP_X = \{0, 1_X\} = P_X$ which implies by Theorem 1.3 that CP_X has no proper dense subsemigroup. Hence, if CP_X has a proper dense subsemigroup, then $|X| > 1$.

Conversely, assume that $|X| > 1$. Let p be a fixed point of X and let

$$U = \{A_x \mid \emptyset \neq A \subseteq X,\ x \in X \smallsetminus \{p\}\} \cup \{\{p\}_p\ , 0\}.$$

Let $q \in X \smallsetminus \{p\}$. Then $\{q\}_p \notin U$, so $U \neq CP_X$. If $x, y \in X \smallsetminus \{p\}$ and $\emptyset \neq A \subseteq X$, $\emptyset \neq B \subseteq X$, then

$$A_x B_y = \begin{cases} A_y & \text{if } x \in B, \\ 0 & \text{if } x \notin B, \end{cases}$$

$$A_x\{p\}_p = 0$$

and

$$\{p\}_p A_x = \begin{cases} \{p\}_x & \text{if } p \in A , \\ 0 & \text{if } p \notin A . \end{cases}$$

This proves that U is a subsemigroup of CP_X, let $x \in X$, $\emptyset \neq A \subseteq X$. Then $A_q \in U$. If $x \neq p$, then $A_x \in U \subseteq Dom(U,CP_X)$. If $x = p$, then

$$\begin{aligned} A_x = A_p &= A_q\{q\}_p , && A_q \in U, \quad \{q\}_p \in CP_X , \\ &= A_p\{p\}_q\{q\}_p , && \{p\}_q \in U, \quad A_p \in CP_X, A_q = A_p\{p\}_q , \\ &= A_p\{p\}_p , && \{p\}_p \in U, \quad \{p\}_q\{q\}_p = \{p\}_p \end{aligned}$$

which implies by Theorem 1.1 that $A_p \in Dom(U,CP_X)$.

Hence U is a proper dense subsemigroup of CP_X. #