CHAPTER III

TOPOLOGICAL VECTOR SPACES OVER THE QUATERNIONS

<u>Definition 3.1</u> <u>A topological vector space X over $\mathbb{H}$</u> (abbreviated by $TVS(\mathbb{H})$) is a topological space and a vector space over $\mathbb{H}$ such that the vector operations are continuous, that is the map $(t, x) \to tx$ and $(x, y) \to x+y$ which carry $\mathbb{H} \times X$ and $X \times X$ to X respectively are continuous. If X is a vector space over $\mathbb{H}$, any topology T which makes (X, T) a $TVS(\mathbb{H})$ will be called a <u>vector topology</u>.

<u>Example 3.2</u> Every paranormed space over $\mathbb{H}$ is a $TVS(\mathbb{H})$.

<u>Proof</u> : Let $(X, \|.\|)$ be a paranormed space over $\mathbb{H}$. Since $\{B(0; r) \mid r \in \mathbb{Q}^+\}$ is a countable base of neighberhoods of 0 in X, X is first countable; hence sequences can be used. X has a natural pseudometric induced by its paranorm; it is defined by $d(x, y) = \|x-y\|$.

We must show that the map $(x, y) \to x+y$ is continuous. Let $(x, y) \in X \times X$. Let (x_n), (y_n) be sequences in X such that $x_n \to x$ and $y_n \to y$. Then $d(x_n+y_n, x+y) = \|x_n-x+y_n-y\| \leq \|x_n-x\| + \|y_n-y\| \to 0$ as $n \to \infty$. So $x_n+y_n \to x+y$. Hence the map $(x, y) \to x+y$ is continuous. By property (PN4) of a paranorm, the map $(t, x) \to tx$ is continuous. #

<u>Notation</u> The set of all neighberhoods of 0 in (X, T), a $TVS(\mathbb{H})$, will be denoted by N, $N(X)$, $N(T)$ or $N(X, T)$. The set of all neighberhoods of a point x in (X, T), a $TVS(\mathbb{H})$, will be denoted by N_x.

Theorem 3.3 Let X be a TVS($\mathbb{H}$), $a \in X$ and $G \subseteq X$. Then $G \in N_a$ if and only if $G - a \in N(X)$. In other words $a + U \in N_a$ if and only if $U \in N(X)$.

Proof : Let X be a TVS($\mathbb{H}$), $a \in X$ and $G \subseteq X$. Define $f : X \to X$ by $f(x) = x + a$. Clearly f is a bijection. We must show that f is continuous on X. Let $b \in X$ be arbitrary. Let W be an open set containing $b + a$. We must show that there exists an open set $V \ni b$ such that $f(V) \subseteq W$. Since the map $(x, y) \overset{\varphi}{\to} x + y$ is continuous, there exists an open set $M \times N \ni (b, a)$ such that $\varphi(M \times N) \subseteq W$. Choose $V = M$. We must show that $f(V) \subseteq W$. Let $y \in f(V)$. Then $y = f(c)$ for some $c \in V = M$. Hence $(c, a) \in M \times N$ so $\varphi(c, a) = c + a \in W$. But $f(c) = c + a = y$ therefore $y \in W$. so $f(V) \subseteq W$. Hence f is continuous at $x = b$. But $b \in X$ was arbitrary therefore f is continuous on X. We can show in a similarway that the inverse map $x \to x - a$ is also continuous. Hence f is a homeomorphism of X onto itself.

Suppose that $G \in N_a$. Then there exists an open set $U \ni a$ such that $U \subseteq G$. Since $a \in G$, $0 = a - a \in G - a = \{g - a \mid g \in G\}$ and $0 \in U - a$ also. Hence $f^{-1}(U) = U - a$ is open in X. But $0 \in U - a$ so $U - a \in N$. Since $G - a \supseteq U - a \ni 0$, $G - a \in N$. To prove the converse, we use a similar proof. #

Notation : Let X, Y be TVS($\mathbb{H}$)'s. Let $B(X, Y)$ denote the set of all continuous linear maps from X into Y. If $Y = \mathbb{H}$, $B(X, Y)$ is denoted by X'; that is X' is the set of all continuous linear functionals on X ; X' is called the dual space of X.

Theorem 3.4 If X and Y are seminormed spaces over $\mathbb{H}$, so is $B(X, Y)$. Moreover if Y is a normed space over $\mathbb{H}$, so is $B(X, Y)$.

Proof : Let (X, p) and (Y, q) be seminormed spaces over $\mathbb{H}$. Define a map $\|.\| : B(X, Y) \to \mathbb{R}$ by $\|T\| = \sup\{q(T(x)) \mid p(x) \leq 1\}$, where $T \in B(X, Y)$. We must show that $\|.\|$ is a seminorm on B(X, Y). Let $T \in B(X, Y)$ and $\lambda \in \mathbb{H}$. Then

$$\begin{aligned} \lambda T &= \sup\{ q((\lambda T)(x) \mid p(x) \leq 1 \} \\ &= \sup\{ q(\lambda T(x)) \mid p(x) \leq 1 \} \\ &= \sup\{ |\lambda|\ q(T(x)) \mid p(x) \leq 1 \} \\ &= |\lambda| \sup\{ q(T(x)) \mid p(x) \leq 1 \} \\ &= |\lambda|\ \|T\| . \end{aligned}$$

Let $T, T' \in B(X, Y)$. We must show that $\|T + T'\| \leq \|T\| + \|T'\|$.

$$\begin{aligned} \|T + T'\| &= \sup\{ q((T + T')(x)) \mid p(x) \leq 1 \} \\ &= \sup\{ q(T(x) + T'(x)) \mid p(x) \leq 1 \} \\ &\leq \sup\{ q(T(x)) + q(T'(x)) \mid p(x) \leq 1 \} \\ &\leq \sup\{ q(T(x)) \mid p(x) \leq 1 \} + \sup\{ q(T'(x)) \mid p(x) \leq 1 \} \\ &= \|T\| + \|T'\|. \end{aligned}$$

Clearly, $\|0\| = 0$. Thus $\|.\|$ is a seminorm on B(X, Y). Suppose (Y, q) is a normed space over $\mathbb{H}$. We must show that B(X, Y) is also a normed space. We have shown that $\|.\|$ is a seminorm on B(X, Y). We must show that $\|T\| = 0$ implies that T = 0. Let $T \in B(X, Y)$ be such that $\|T\| = 0$. We must show that T = 0. Let $x \in X$. If x = 0. Since T is linear, T(0) = 0. Assume $x \neq 0$. If $p(x) \leq 1$ then $q(T(x)) \leq \sup\{ q(T(x)) \mid p(x) \leq 1 \} = \|T\| = 0$ so q(T(x)) = 0. Since q is a norm. T(x) = 0. If p(x) > 1. Then $p\left(\frac{x}{p(x)}\right) = \frac{p(x)}{p(x)} = 1$ so $q\left(T\left(\frac{x}{p(x)}\right)\right) \leq \sup\{ q(T(y)) \mid p(y) \leq 1 \} = \|T\| = 0$. Hence $q\left(T\left(\frac{x}{p(x)}\right)\right) = 0$. Since q is a norm, $T\left(\frac{x}{p(x)}\right) = 0$ so $({}^{1}/p(x))\ T(x) = 0$. But p(x) > 1 therefore T(x) = 0 so T = 0. Hence $\|.\|$ is a norm on B(X, Y). #

Theorem 3.5 Let X be a seminormed space over $\mathbb{H}$ and $f \in S'$, where S is a vector subspace of X. Then f can be extended to $F \in X'$ such that $\|F\| = \|f\|$.

Proof : Let $(X, \|.\|)$ be a seminormed space over $\mathbb{H}$ and let S be a subspace of X. Let $f \in S'$. We must show that there exists an $F \in X'$ such that $\|F\| = \|f\|$ and $F = f$ on S.

Define $p : X \to \mathbb{R}$ by $p(x) = \|f\|\|x\|$ for $x \in X$. Since f is continuous on S, $\|f\| < \infty$; hence p is well-defined. Claim that p is a seminorm on X. $p(0) = \|f\|\|0\| = \|f\| \cdot 0 = 0$. Let $x \in X$ and $\lambda \in \mathbb{H}$. Then $p(\lambda x) = \|f\|\|\lambda x\| = \|f\| \, |\lambda| \, \|x\| = |\lambda| \, \|f\| \, \|x\| = |\lambda| \, p(x)$. Let $x, y \in X$. Then $p(x+y) = \|f\| \, \|x+y\| \leq \|f\| (\|x\| + \|y\|) = \|f\| \, \|x\| + \|f\| \, \|y\| = p(x) + p(y)$. So we have the claim. Since f is continuous on S, by Theorem 2.9, $|f(x)| \leq \|f\| \, \|x\|$ for all $x \in S$. Hence $|f(x)| \leq p(x)$ for all $x \in S$. By Theorem 2.7, f can be extended to $F \in X'$ such that $|F(x)| \leq p(x)$ for all $x \in X$. We must show that $\|F\| = \|f\|$. $\|f\| = \sup \{ |f(x)| \mid \|x\| \leq 1, x \in S \} \leq \sup \{ |F(x)| \mid \|x\| \leq 1 \} = \|F\|$. So $\|f\| \leq \|F\|$. $\|F\| = \sup \{ |F(x)| \mid \|x\| \leq 1 \} \leq \sup \{ p(x) \mid \|x\| \leq 1 \} = \sup \{ \|f\| \, \|x\| \mid \|x\| \leq 1 \} \leq \sup \{ \|f\| \mid \|x\| \leq 1 \} = \|f\|$. Hence $\|F\| = \|f\|$. #

Theorem 3.6 Let $(X, \|.\|)$ be a seminormed space over $\mathbb{H}$ and let S be a subspace of X. Let $x_0 \in X \setminus \bar{S}$. Then there exists an $f \in X'$ with $f(x_0) = 1$, $f = 0$ on S and $\|f\| = 1/d(x_0, S)$ where $\bar{S}$ is the closure of S and $d(x_0, S) = \inf \{ \|x_0 - s\| \mid s \in S \}$.

Proof : Define $g : S + \langle x_0 \rangle \to \mathbb{H}$ by $g(s + tx_0) = t$ where $s \in S$ and $t \in \mathbb{H}$. We must show that $g \in (S + \langle x_0 \rangle)'$. Let $u, v \in S + \langle x_0 \rangle$

and a, b $\in \mathbb{H}$. We must show that $g(au + bv) = ag(u) + bg(v)$. Since $u, v \in S + \langle x_0 \rangle$, $u = s' + t' x_0$, $v = s'' + t'' x_0$ for some $s', s'' \in S$ and $t', t'' \in \mathbb{H}$. Then $g(au + bv) = g(a(s' + t' x_0) + b(s'' + t'' x_0))$ $= g(as' + bs'' + (at' + bt'')x_0) = at' + bt'' = a(g(s' + t'x_0))$ $+ bg(s'' + t''x_0) = ag(u) + bg(v)$. Hence g is a linear functional on $S + \langle x_0 \rangle$. Let $d = d(x_0, S)$. Since $x_0 \notin \bar{S}$, $d = d(x_0, S) > 0$. Let $z \in S + \langle x_0 \rangle$. Then $z = s + tx_0$ for some $s \in S$ and $t \in \mathbb{H}$ so $\|z\| =$ $\|s + tx_0\| = \|t(x_0 - (-s/t))\| = |t| \, \|x_0 - (-s/t)\| \geq |t| \inf \{ \|x_0 - y\| \mid y \in S \}$ $= |t| d = |g(z)| d$. Thus $\|g\| = \sup \{ |g(z)| \mid \|z\| \leq 1 \} \leq \|z\|/d \leq 1/d$ so g is bounded on the unit disc. By Theorem 2.9, g is continuous on $S + \langle x_0 \rangle$ hence $g \in (S + \langle x_0 \rangle)'$. Let $s \in S$. Then $1 = g(x_0 - s)$ $\leq \|g\| \, \|x_0 - s\|$ so $1 \leq \inf \{ \|g\| \, \|x_0 - s\| \mid s \in S \} = \|g\| \inf$ $\{ \|x_0 - s\| \mid s \in S \} = \|g\| d$ therefore $\|g\| \geq 1/d$. Hence $\|g\| = 1/d$ so we have that $g \in (S + \langle x_0 \rangle)'$ is such that $g(x_0) = 1$, $g = 0$ on S and $\|g\| = 1/d$. By Theorem 3.5, there exists $f \in X'$ such that $f = g$ on $S + \langle x_0 \rangle$ and $\|f\| = \|g\|$. #

<u>Definition 3.7</u> A set $S \subseteq X$ is called <u>fundamental</u> if the span of S is dense in X.

<u>Theorem 3.8</u> Let X be a seminormed space, $S \subseteq X$. Suppose that for all $f \in X'$, $f = 0$ on S implies that $f = 0$. Then S is fundamental. If S is a subspace of X then S is dense.

<u>Proof</u> : Let $\langle S \rangle$ be the span of S. Let $x_0 \in X \setminus \overline{\langle S \rangle}$. By Theorem 3.6, there exists an $f \in X'$ such that $f(x_0) = 1$, $f = 0$ on $\langle S \rangle$ and $\|f\| = 1/d$ where $d = d(x_0, \langle S \rangle) = \inf \{ \|x_0 - m\| \mid m \in \langle S \rangle \}$. Since $f = 0$ on $\langle S \rangle$, $f = 0$ on S. By assumption, $f = 0$ on X,

so $f(x_0) = 0$, a contradiction. Hence $X \setminus \overline{\langle S \rangle} = \emptyset$: that is $\overline{\langle S \rangle} = X$. So S is fundamental.

Assume that S is a vector subspace of X. Suppose that $\bar{S} \neq X$. Let $x_0 \in X \setminus \bar{S}$. By Theorem 3.6, there exists an $f \in X'$ such that $f(x_0) = 1$, $f = 0$ on S. By assumption, $f = 0$ on X ; so $f(x_0) = 0$, a contradiction. Hence $X \setminus \bar{S} = \emptyset$. Then $\bar{S} = X$. Thus S is dense in X. #

<u>THeorem 3.9</u> Let Φ be a collection of vector topologies for a vector space X over $\mathbb{H}$. Let $v\Phi$ be the set of all unions of finite intersections of members in $\cup\Phi$. Then $v\Phi$ is a vector topology for X and a net $x_\delta \to 0$ in $v\Phi$ if and only if $x_\delta \to 0$ in T, for each $T \in \Phi$.

<u>Proof</u> : $v\Phi$ is the smallest topology generated by $\cup\Phi$ and $v\Phi \supseteq T$ for each $T \in \Phi$. We must show that $v\Phi$ is a vector topology for X.

1. Continuity of multiplication. Let $t \in \mathbb{H}$ and $x \in X$. Let $(t_\delta)_{\delta \in D}$, $(x_\delta)_{\delta \in D}$ be nets in X with respect to the same index set D such that $t_\delta \to t$ and $x_\delta \to a$ in $v\Phi$. We must show that $t_\delta x_\delta \to tx$ in $v\Phi$. Since $x_\delta \to x$ in $v\Phi$, by Theorem 1.18, $x_\delta \to x$ in (x, T) for each $T \in \Phi$. But T is a vector topology and $t_\delta x_\delta \to tx$ in each (X, T) so by Theorem 1.18, $t_\delta x_\delta \to tx$ in $v\Phi$. Hence the multiplication is continuous.

2. Continuity of addition. The proof is similar so the proof of the continuity of multiplication. The rest of the proof follows easily from Theorem 1.18. #

<u>Definition 3.10</u> Let P be a collection of paranorms on a vector space X over $\mathbb{H}$. Then σP denotes $v\{T_p | p \in P\}$ where T_p is the topology induced by a paranorm p.

<u>Theorem 3.11</u> Let X be a vector space over $\mathbb{H}$ and let P be a collection of paranorms on X. Then σP is a vector topology for X. Moveover, a net $x_\delta \to 0$ in σP if and only if $p(x_\delta) \to 0$ for each $p \in P$.

<u>Proof</u> : It follows from Example 3.2 and Theorem 3.9 . #

<u>Theorem 3.12</u> Let X be a vector space over $\mathbb{H}$. For each $\alpha \in I$, (I an index set) let Y_α be a TVS($\mathbb{H}$) and let $f_\alpha : X \to Y_\alpha$ be a linear map. Let $F = \{ f_\alpha | \alpha \in I \}$. Then wF is a vector topology. Moreover, a net $x_\delta \to 0$ in wF if and only if $f_\alpha(x_\delta) \to 0$ for each $\alpha \in I$.

<u>Proof</u> : Suppose that $F = \{ f \}$ where $f : X \to Y$ is a linear map. Let $t \in \mathbb{H}$ and $x \in X$. Let $(t_\delta)_{\delta \in D}$ and $(x_\delta)_{\delta \in D}$ be nets in $\mathbb{H}$ and X respectively with respect to the same index set D such that $t_\delta \to t$ and $x_\delta \to x$. We must show that $t_\delta x_\delta \to tx$. By Theorem 1.19, $f(x_\delta) \to f(x)$ in Y, so $f(t_\delta x_\delta) = t_\delta f(x_\delta) \to t f(x) = f(tx)$. By Theorem 1.19 again, $t_\delta x_\delta \to tx$. Hence the multiplication is continuous. Using a similar proof we can show that the addition is also continuous. Hence wF is a vector topology. The last property comes from Theorem 1.19 . #

<u>Definition 3.13</u> Let X be a vector space over $\mathbb{H}$. We say that $A \subseteq X$ has <u>finite codimension in X</u> if and only if there exists a finite dimensional subspace B of X such that $A + B = X$.

<u>Theorem 3.14</u> Let X be a vector apace over $\mathbb{H}$ and let $F \subseteq X$. In (X, wF), a TVS($\mathbb{H}$), every $U \in N(X)$ includes a vector subspace of X of finite codimension.

<u>Proof</u> : Let (X, wF) be a TVS($\mathbb{H}$). Let $U \in N(X)$. We must show that there exists a vector subspace A of X such that A has finite

codimension and $A \subseteq U$. Since $U \in wF$, there exists $V_i \in N(X, wf_i)$, $f_i \in F$, $i = 1, 2, \ldots, n$ such that $\bigcap_{i=1}^{n} V_i \subseteq U$. Since $V_i \in N(X, wf_i)$ for all $i = 1, 2, \ldots, n$, $V_i \supseteq \{ x \mid |f_i(x)| < \varepsilon_i \}$ for some $\varepsilon_i > 0$, $i = 1, 2, \ldots, n$. Let $A = \bigcap \{ \ker f_i \mid i = 1, 2, \ldots, n \}$. Since $\ker f_i$ is vector subspace of X for all $i = 1, 2, \ldots, n$, A is also a vector subspace of X. We must show that $A \subseteq U$. Let $x \in A$. Then $x \in \ker f_i$ for all $i = 1, 2, \ldots, n$. Therfore, $f_i(x) = 0$ for all i $i = 1, 2, \ldots, n$; hence $|f_i(x)| = 0$ for all $i = 1, 2, \ldots, n$ so $x \in \bigcap_{i=1}^{n} \{ x \mid |f_i(x)| < \varepsilon_i \} \subseteq \bigcap_{i=1}^{n} V_i \subseteq U$. Next, we must show that A has finite codimension, that is, there exists a finite dimensional vector subspace C of X such that $A + C = X$.

Case 1 X is finite dimensional. The result is clear.

Case 2 X is infinite dimensional. Claim that $A \neq \{0\}$. To prove this suppose not. Define $G' : X \to \mathbb{H}^n$ by $G'(x) = (f_1(x), f_2(x), \ldots, f_n(x))$. Since f_i is a linear functional forall $i = 1,2,\ldots,n$, G' is linear. Since $A = \{0\}$ $\bigcap \{ x \mid f_i(x) = 0 \} = \{0\}$ for some $i = 1,2,\ldots,n$ so, $x \neq 0$ implies that $f_i(x) \neq 0$ for some $i = 1,2,\ldots,n$. Let $x, y \in X$ be such that $x \neq y$. Then $x - y \neq 0$. Hence $G'(x) - G'(y) = G'(x-y) = (f_1(x-y), f_2(x-y), \ldots, f_n(x-y)) \neq (0,0,\ldots,0)$. So $G'(x) \neq G'(y)$, Hence G is injective so $\dim X \leq \dim \mathbb{H}^n = n$, a contradiction so we have the claim.

Let $B = X/A$. Then B is a vector space over $\mathbb{H}$ with respect to the addition and scalar multiplication defined by $(x+A) + (y+A) = (x+y) + A$ and $t(x+A) = tx + A$ where $x, y \in A$ and $t \in \mathbb{H}$. Define $G : B \to \mathbb{H}^n$ as follows. Let $\alpha \in B$. Choose $x \in \alpha$.

Define $G(\alpha) = (f_1(x), f_2(x),\dots,f_n(x))$. We must show that G is well-difined. Choose $y \in \alpha$ also. Then $G(\alpha) = (f_1(y), f_2(y),\dots,f_n(y))$. Since $y \in \alpha$, $x - y \in A$. Hence $f_i(x-y) = 0$ for all $i = 1,2,\dots,n$ so $f_i(x) = f_i(y)$ for all i. Hence $(f_1(x), f_2(x),\dots,f_n(x))$ $= (f_1(y),f_2(y),\dots,f_n(y))$ so G is well-defined. To show that G is injective let α , $\alpha' \in B$ be such that $G(\alpha) = G(\alpha')$. We must show that $\alpha = \alpha'$. Choose $x \in \alpha$ and $y \in \alpha'$. Then $(f_1(x),f_2(x),\dots,f_n(x))$ $= (f_1(y),f_2(y),\dots f_n(y))$. Since f_i is linear for all $i = 1,2,\dots,n$, $f_i(x-y) = 0$ for all i; hence $x - y \in A$. Thus $\alpha = \alpha'$ so G is injective. Hence $\dim B \leq \dim \mathbb{H}^n = n < \infty$. Suppose that $\dim B = m$. Let $\{\alpha_1, \alpha_2,\dots,\alpha_m\}$ be a basis of B. Choose $\beta_i \in \alpha_i$, $i = 1,2,\dots,m$. Let $< \beta_1,\beta_2,\dots,\beta_m >$ be the set of all linear combination of elements in $\{\beta_1,\beta_2,\dots,\beta_m\}$. Claim that $\{ \beta_1,\beta_2,\dots \beta_m \}$ is linearly independent. Suppose that $\sum_{j=1}^{m} \lambda_j\beta_j = 0, \lambda_j \in \mathbb{H}$, $j = 1,2,\dots,n$. We must show that $\lambda_j = 0$ for all j. Since $\sum_{j=1}^{m} \lambda_j\beta_j = 0$, $[\sum_{j=1}^{m} \lambda_j\beta_j] = [0]$ so $\sum_{j=1}^{m} \lambda_j[\beta_j]$ $= [0]$. Hence $\sum_{j=1}^{m} \lambda_j\alpha_j = 0$ (in B). But $\{ \alpha_1,\alpha_2,\dots,\alpha_m \}$ is linearly independent, so $\lambda_j = 0$ for all $j = 1,2,\dots m$ and thus we have the claim. Hence $<\beta_1,\beta_2,\dots,\beta_m>$ is a vector subspace of X with the basis $\{ \beta_1,\beta_2,\dots,\beta_m \}$. We must show that $A + <\beta_1,\beta_2,\dots,\beta_m> = X$. Let $x \in X$. Then $x + A \in B$ so $x + A = \sum_{i=1}^{m} \lambda_j\beta_j$ for some $\lambda_j \in \mathbb{H}$, $j = 1,2,\dots,m$. Hence $x + A = \sum_{j=1}^{m} \lambda_j(\beta_j + A) = \sum_{j=1}^{m} \lambda_j\beta_j + A$ therefore there exist a, $b \in A$ such that $x + a = \sum_{j=1}^{m} \lambda_j\beta_j + b$. Hence $x = (b-a) + \sum_{j=1}^{m} \lambda_j\beta_j \in A + <\beta_1,\beta_2,\dots,\beta_m>$ so $X = A + <\beta_1,\beta_2,\dots,\beta_m>$

and thus A has finite codimension. #

<u>Definition 3.15</u> A collection F of subsets of vector space is called <u>additive</u> if and only if for each $U \in F$ there exists $V \in F$ such that $V + V \subseteq U$.

<u>Theorem 3.16</u> Let X be a vector space over $\mathbb{H}$ and a topological space. Define $u : X \times X \to X$ by $u(x, y) = x + y$. Then u is continuous at 0 (=(0, 0) if and only if N(X) is additive.

<u>Proof</u> : ($\Longrightarrow$) Let $U \in N(X)$. Since u is continuous at 0, there exists a $W \in N(X \times X)$ such that $u(W) \subseteq U$. Since $W \in N(X \times X)$, there exist $V_1, V_2 \in N(X)$ such that $V_1 \times V_2 \subseteq W$. Let $V = V_1 \cap V_2$. Then $V + V \subseteq V_1 + V_2 \subseteq u(W) \subseteq U$.

($\Longleftarrow$) Let $U \in N(X)$. Since N(X) is additive, there exists a $V \in N(X)$ such that $V + V \subseteq U$. So $V \times V \in N(X \times X)$ and $u(V \times V) = V + V \subseteq U$. Hence u is continuous at 0. #

<u>Theorem 3.17</u> Let X be a TVS($\mathbb{H}$) and $U \in N(X)$. Then $tU \in N(X)$ for every $t \neq 0$.

<u>Proof</u> : Since Scalar multiplication is continuous, the map $x \to tx$ is continuous and bijective with inverse $x \to (\frac{1}{t})x$ which is also continuous and bijective. So the map $x \mapsto tx$ is a homeomorphism of X on itself and hence preserves open sets. #

<u>Theorem 3.18</u> Let X be a TVS($\mathbb{H}$). Then every neighborhood U of 0 in X is absorbing.

<u>Proof</u> : Let $U \in N(X)$. Let $x \in X$. Define $u_x : \mathbb{H} \to X$ by $u_x(t) = tx$. Then u_x is continuous at 0 so there exists an $\varepsilon > 0$ such that $|t| < \varepsilon$ implies that $u_x(t) \in U$; that is, $tx \in U$. Hence U is absorbing . #

<u>Theorem 3.19</u> Let X be a TVS($\mathbb{H}$). Then every $U \in N(X)$ includes a balanced neighberhood of 0.

<u>Proof</u> : Let $U \in N(X)$. Since the map $u : \mathbb{H} \times X \to X$ given by $u(t, x) = tx$ is continuous, there exists a $W \in N(\mathbb{H} \times X)$ such that $u(W) \subseteq U$. Since $W \in N(\mathbb{H} \times X)$, there exist $P \in N(\mathbb{H})$ and $Q \in N(X)$ such that $P \times Q \subseteq W$. Since $P \in N(\mathbb{H})$ then there exists an $\varepsilon > 0$ such that $P \supseteq \{ t \mid |t| \leq \varepsilon\}$. Let $V = \bigcup\{tQ \mid |t| \leq \varepsilon \}$. We must show that V is balanced. Let $v \in V$ and $t \in \mathbb{H}$ be such that $|t| \leq 1$. Since $v \in V$, $= sq$ for some $q \in Q$ and $|s| \leq \varepsilon$. Then $tv = t(sq) = (ts)q$. But $|ts| = |t||s| \leq 1 \cdot \varepsilon = \varepsilon$. so $tv \in V$. Hence V is balanced. #

<u>Theorem 3.20</u> Let X be a TVS($\mathbb{H}$) and $S \subseteq X$. Then $\bar{S} = \bigcap\{ S + U \mid U \in N(X)\}$ In particular, $\bar{S} \subseteq S + U$ for every $U \in N(X)$.

<u>Proof</u> : Let $x \in \bar{S}$ and $U \in N(X)$. By Theorem 3.19; there exists a balanced set $V \in N(X)$ such that $0 \in V \subseteq U$. Since $x + V$ is a neighberhood of x, $(x+V) \cap S \neq \emptyset$. SO $x \in S - V = S + V$. Since $V \subseteq U$ and $x \in S+V$, $x \in S+U$; hence $\bar{S} \subseteq \bigcap\{S+U \mid U \in N(X)\}$. Conversely, let $x \notin \bar{S}$. By Theorem 3.19, there exists a balanced set $U \in N(X)$ such that $(x+U) \cap S = \emptyset$. Hence $x \notin S - U = S+U$ so $x \notin \bigcap\{ S+U \mid U \in N(X)\}$. #

<u>Theorem 3.21</u> Let X be a TVS($\mathbb{H}$). Then every $U \in N(X)$ includes a closed balanced neighberhood of 0.

Proof : Let $U \in N(X)$. By Theorem 3.16, since addition is continuous, there exists a set $V \in N(X)$ such that $V + V \subseteq U$. By Theorem 3.19, there exists a balanced set $W \in N(X)$ such that $W \subseteq V$. Then $\bar{W} \subseteq W + V \subseteq V + V \subseteq U$. We must show that $\bar{W}$ is balanced. Let $t \in \mathbb{H}$ be such that $|t| \leq 1$. Since the map $f : X \to X$ given by $f(x) = tx$ is continuous, $f(\bar{W}) \subseteq \overline{f(W)}$, so $t\bar{W} \subseteq \overline{tW}$. Since W is balanced and $|t| \leq 1$, $tW \subseteq W$. Hence $\overline{tW} \subseteq \bar{W}$; so $t\bar{W} \subseteq \bar{W}$. Thus $\bar{W}$ is balanced. #

Theorem 3.22 Every TVS($\mathbb{H}$) is a regular topological space. The following conditions on X, a TVS($\mathbb{H}$), are equivalent :

(a) X is a T_3- space .

(b) $\{0\}$ is a closed set .

(c) For each $x \neq 0$, there exists a $U \in N(X)$ such that $x \notin U$.

Proof : Let X be a TVS($\mathbb{H}$). We must show that X is reqular ; that is, for all $x \in X$, $U \in N_x$ contains a closed neighberhood of x. Let $x \in X$ and $U \in N_x$. Then $U = x + V$ for some $V \in N(X)$. By Theorem 3.21 there exists a closed set $W \in N(X)$ such that $W \subseteq V$. Then $x + W$ is a closed neighberhood of x such that $x + W \subseteq x + V = U$. So X is regular.

To show that (a) $\Longrightarrow$ (b). Assume that X is T_3. Since X is T_3, X is T_2 ; hence every singleton is closed so $\{0\}$ is closed in X.

(b) $\Longrightarrow$ (c). Assume that $\{0\}$ is closed. Let $x \neq 0$. Since $\{0\}$ is closed and $x \neq 0$ then there exists $U \in N(X)$ such that $(x - U) \cap \{0\} = \emptyset$. Then $x \notin U$.

(c) $\Longrightarrow$ (a). We have shown that X is reqular. It remains to show that X is T_1. Let $x, y \in X$ be such that $x \neq y$. Then $x - y \neq 0$. Hence there exists a set $U \in N(X)$ such that $x - y \notin U$.

Since $U \in N(X)$, by Theorem 3.19, U contains a balanced set $V \in N(X)$. Claim that $\mathrm{Int}(V)$ is balanced. Let $t \in \mathbb{H}$ be such that $|t| \leq 1$. We shall show that $t\,\mathrm{Int}(V) = \mathrm{Int}(tV)$. If $t = 0$ the result is obvious. Assume $t \neq 0$. Define $f_t : X \to X$ by $f_t(x) = tx$. It is clear that f_t is a homeomorphism of X onto itself. So f_t preserves open sets. Let $x \in t\,\mathrm{Int}(V)$. Then there exists a $y \in \mathrm{Int}(V)$ such that $x = ty$. Since $y \in \mathrm{Int}(V)$, there exists an open set W in X such that $y \in W \subseteq V$. Hence $ty \in tW \subseteq tV$. Since W is open and f_t is a homeomorphism, tW is open so $x = ty \in tW \subseteq tV$, therefore $x \in \mathrm{Int}(tV)$. Conversely, let $x \in \mathrm{Int}(tV)$. Then there exists an open set G in X such that $x \in G \subseteq tV$. Since $t \neq 0$, $(\frac{1}{t})x \in (\frac{1}{t})G \subseteq V$. Since f_t is a homeomorphism, $(\frac{1}{t})G$ is open so $(\frac{1}{t})x \in \mathrm{Int}(V)$. Hence $\frac{x}{t} = y$ for some $y \in \mathrm{Int}(V)$ so $x = ty$. Thus $x \in t\,\mathrm{Int}(V)$ so we have the claim. Since V is balanced and $|t| \leq 1$, $t\mathrm{Int}(V) = \mathrm{Int}(tV) \subseteq \mathrm{Int}(V)$ so $\mathrm{Int}(V)$ is balanced. Let $P = \mathrm{Int}(V)$. Then P is open balanced set in $N(X)$ such that $P \subseteq U$. Since $x - y \notin U$, $x - y \notin P$ therefore $x \notin y + P$ and $y \notin x + P$ Hence $y + P$ is an open set containing y and $x + P$ is on open set containing x. Thus X is T_1 so X is T_1 and regular ; hence X is T_3. #

<u>Definition 3.23</u> A $TVS(\mathbb{H})$ is called <u>separated</u> if the three equivalent conditions of Theorem 3.22 hold. We also say that the topology is separated.

<u>Theorem 3.24</u> Let X be a $TVS(\mathbb{H})$, $A, B, \subseteq X$. Then $\bar{A} + \bar{B} \subseteq \overline{A + B}$.

<u>Proof</u> : Let $x \in \bar{A} + \bar{B}$. Then $x = a + b$ for some $a \in \bar{A}$ and $b \in \bar{B}$. Hence there exist filter bases Q, P in A, B respectively such that $Q \to a$ and $P \to b$. It is clear that $Q + P$ is a filter base in $A + B$. Since $Q \to a$, $P \to b$ and $+$ is continuous, $Q + P \to a + b = x$

So $x \in \overline{A+B}$. #

<u>Theorem 3.25</u> Let X be a TVS($\mathbb{H}$) and $S \subseteq X$. Then the following hold.

(a) If S is a subspace of X, then so is $\bar{S}$.

(b) If S is a convex set in X, then so is $\bar{S}$.

(c) If S is a balanced in X, then so is $\bar{S}$.

<u>Proof</u> : (a) Let $t \in \mathbb{H}$. Since the map $f : X \to X$ given by $f(x) = tx$ is continuous, $f(\bar{S}) \subseteq \overline{f(S)}$; that is $t\bar{S} \subseteq \overline{tS}$. Let $a,b \in \mathbb{H}$. Then $a\bar{S} + b\bar{S} \subseteq \overline{aS + bS} \subseteq \bar{S}$ so $\bar{S}$ is a subspace of X.

(b) Let $a,b \in [0,1]$ be such that $a+b = 1$. Since $\bar{S}$ is a vector subspace of X, $a\bar{S} + b\bar{S} \subseteq \bar{S}$.

(c) Let $x \in \bar{S}$ and $t \in \mathbb{H}$ be such that $|t| \leq 1$. Then there exists a filter Q in S such that $Q \to x$. Let $A \in Q$. So $tA \subseteq tS \subseteq S$. Hence tQ is a filter base in S. Since scalar multiplication is continuous, $tQ \to tx$ therefore $tx \in \bar{S}$. Hence $\bar{S}$ is balanced. #

<u>Theorem 3.26</u> Let X be a TVS($\mathbb{H}$). Let β be an additive filterbase of balanced absorbing subsets of X. Then there exists a unique vector topology on X for which β is a local base of neighborhoods of O.

<u>Proof</u> : Let $T = \{G \subseteq X \mid G = \emptyset$ or for each $x \in G$ there exists $U \in \beta$ such that $x + U \subseteq G\}$. We must show that T is a topology on X. Let $G_1, G_2 \in T$. To show that $G_1 \cap G_2 \in T$, suppose that $G_1 \cap G_2 \neq \emptyset$ Let $x \in G_1 \cap G_2$. Then there exist $U_1, U_2 \in \beta$ such that $x + U_1 \subseteq G_1$ and $x + U_2 \subseteq G_2$. Since β is a filter base, there exists a $U \in \beta$ such that $U \subseteq U_1 \cap U_2$. Hence $x + U \subseteq (x + U_1) \cap (x + U_2) \subseteq G_1 \cap G_2$ so $G_1 \cap G_2 \in T$.

Let $\{G_\alpha\}_{\alpha \in I}$ be a family of open sets in T. We must show that $\bigcup_{\alpha \in I} G_\alpha \in T$. Suppose that $\bigcup_{\alpha \in I} G_\alpha \neq \emptyset$. Let $x \in \bigcup_{\alpha \in I} G_\alpha$. Then $x \in G_{\alpha_0}$ for some $\alpha_0 \in I$. Since $G_{\alpha_0} \in T$, there exists a $U_{\alpha_0} \in \beta$ such that $x + U_{\alpha_0} \subseteq G_{\alpha_0} \subseteq \bigcup_{\alpha \in I} G_\alpha$. Hence $\bigcup_{\alpha \in I} G_\alpha \in T$.

Let $x \in X$. Then $x + U \subseteq X$ for all $U \in \beta$ so $X \in T$. Hence T is a topology for X. Next, we must show that β is a local base of neighberhoods of 0. Let $U \in \beta$. Let $G = \{ x | x + V \subseteq U$ for some $V \in \beta\}$. Since $0 + U \subseteq U$, $0 \in G$. Let $x \in G$. Then there exists a $V \in \beta$ such that $x + V \subseteq U$. Since V is absorbing, $0 \in V$; hence $x \in x + V \subseteq U$. So $G \subseteq U$. We must show that G is open. Let $x \in G$. Then there exists a $V \in \beta$ such that $x + V \subseteq U$. Let $W \in \beta$ be such that $W + W \subseteq V$. Let $w \in W$. Then $x + w + W \subseteq x + W + W \subseteq x + V \subseteq U$ so $x + w \in G$. Hence $x + W \subseteq G$ so G is open and $0 \in G \subseteq U$. Hence $U \in N(X)$ so $\beta \subseteq N(X)$. Let $U \in N(X)$. Then there exists a $G \in T$ such that $0 \in G \subseteq U$. Since $G \in T$, there exists a $U' \in \beta$ such that $0 + U' \subseteq G \subseteq U$ so $0 \in U' \subseteq U$. Hence N(X) is the filter generated by β so β is a local base of neighberhood of 0.

We must show that T is a vector topology.

1. <u>Continuity of addition</u>. Let $a, b \in X$. Let $(x_\delta)_{\delta \in D}$ and $(y_\delta)_{\delta \in D}$ be nets in X with respect to the directed set D such that $x_\delta \to a$ and $y_\delta \to b$. Let $U \in N(X)$. Choose $V \in \beta$ such that $V + V \subseteq U$. Since $x_\delta \to a$ and $y_\delta \to b$ then there exists a $\delta' \in D$ such that $\delta \geq \delta'$ implies that $x_\delta - a \in V$ and $y_\delta - b \in V$. Hence $x_\delta + y_\delta - a - b \in V + V \subseteq U$; so $x_\delta + y_\delta \in a + b + U$ so $x_\delta + y_\delta \to a + b$ therefore the addition is continuous.

2. Continuity of multiplication. Let $x \in X$ and $t \in \mathbb{H}$. Let $U \in \beta$. Then $tx + U$ is a neighborhood of tx. Hence there exists a $V \in \beta$ such that $V + V + V + V \subseteq U$. Let $n \in \mathbb{N}$ be such that $2^n > |t|$. Choose $W \in \beta$ such that $W + W + \ldots + W \subseteq V$ (2^n terms in the sum). Since W is balanced, $tW \subseteq 2^n W \subseteq V$. Since W is absorbing, $0 \in W$; hence $W \subseteq 2^n W \subseteq V$; so $W \subseteq V$. Since V is absorbing, there exists an $\varepsilon > 0$ such that $0 < \varepsilon < 1$ and $ax \in V$ for $|a| < \varepsilon$. Let P be the ε - neighborhood of t in $\mathbb{H}$. We must show that $P(x+W) \subseteq tx + U$. Let $p \in P$ and $y \in x + W$. Then $py = tx + t(y-x) + (p-t)x + (p-t)(y-x)$. Since $|p-t| < \varepsilon < 1$ and W is balanced, $(p-t)(y-x) \in (p-t)W \subseteq W \subseteq V$. Since $t(y-x) \in tW \subseteq V$, $(p-t)x \in V$ and $(p-t)(y-x) \in V$ so $py \in tx + V + V + V \subseteq tx + U$. Hence the multiplication is continuous. Since β generates a unique filter $N(X)$, the topology T is unique. #

Definition 3.27 Let X be a TVS ($\mathbb{H}$). Then $S \subseteq X$ is called bounded if and only if every $U \in N(X)$ there exists an $\varepsilon > 0$ such that $tS \subseteq U$ whenever $|t| < \varepsilon$; that is S is absorbed by every neighborhood of 0.

Remark 3.28

(a) Every singleton in a TVS($\mathbb{H}$) is bounded.

(b) The union of two bounded sets in a TVS($\mathbb{H}$) is bounded.

Theorem 3.29 The following are equivalent for a set S in X, a TVS($\mathbb{H}$) :

(a) S is bounded.

(b) For every sequence $(x_n)_{n \in \mathbb{N}} \subseteq S$ and every sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ of scalars in $\mathbb{H}$ with $\varepsilon_n \to 0$ we have $\varepsilon_n x_n \to 0$.

(c) For every sequence $(x_n)_{n \in \mathbb{N}} \subseteq S$, $(\frac{1}{n})x_n \to 0$.

Proof : (a) $\Rightarrow$ (b). Let $U \in N(X)$. Since S is bounded, there exists an $\varepsilon > 0$ such that $tS \subseteq U$ for all $t \in \mathbb{H}$ such that $|t| < \varepsilon$. Since $\varepsilon_n \to 0$, choose $N' \in \mathbb{N}$ such that for all $n > N'$, $|\varepsilon_n| < \varepsilon$. Let $n \in \mathbb{N}$ be such that $n \geq N'$. Then $|\varepsilon_n| < \varepsilon$; hence $\varepsilon_n x_n \in \varepsilon_n S \subseteq U$ so $\varepsilon_n x_n \to 0$.

(b) $\Rightarrow$ (c). Obvious.

(c) $\Rightarrow$ (a) Suppose S is not bounded. Then there exists a balanced set $U \in N(X)$ such that for all $\varepsilon > 0$ there exists a $t \in \mathbb{H}$ such that $0 < |t| < \varepsilon$ and $tS \not\subseteq U$. Since $tS \not\subseteq U$, there exists a $y \in tS$ such that $y \notin U$. Now $y = ts$ for some $s \in S$. Since $|\frac{t}{\varepsilon}| < 1$ and U is balanced, $\frac{t}{\varepsilon} U \subseteq U$. Since $y = ts \notin U$, $y = ts \notin \frac{t}{\varepsilon} U$ so $\varepsilon s \notin U$ i.e $\varepsilon S \not\subseteq U$ for all $\varepsilon > 0$. In particular, $\frac{1}{n} S \not\subseteq U$ for all $n = 1,2,\ldots$. Choose $x_n \in S \setminus nU$ for $n = 1,2,\ldots$. Then $\frac{1}{n} x_n \notin U$ for all $n = 1,2,\ldots$ so $\frac{1}{n} x_n \not\to 0$. #

<u>Definition 3.30</u> A map between TVS($\mathbb{H}$) s which preserves bounded sets is called <u>bounded</u>.

<u>Theorem 3.31</u> A continuous linear map between TVS($\mathbb{H}$)'s is bounded.

Proof : Let X, Y be TVS($\mathbb{H}$)'s. Let $u : X \to Y$ be a continuous linear map. We must show that u is bounded. Let S be a bounded set of X. Let $(y_n)_{n \in \mathbb{N}} \subseteq u(S)$. Then $y_n = u(x_n)$ for some $x_n \in S$, $n \in \mathbb{N}$. Hence $(\frac{1}{n})y_n = \frac{1}{n}u(x_n) = u(\frac{1}{n} x_n)$. Since $x_n \in S$ for all $n \in \mathbb{N}$ and S is bounded, $\frac{1}{n} x_n \to 0$ as $n \to \infty$. Since u is continuous and $\frac{1}{n} x_n \to 0$ as $n \to \infty$, $u(\frac{1}{n} x_n) \to u(0) = 0$ so u(S) is bounded by Theorem 3.29. #

Corollary 3.32 If T and T′ are vector topologies for a vector space X over $\mathbb{H}$ and $T' \supseteq T$, then each set which is bounded in (X, T′) is bounded in (X, T).

Proof : Let S be a bounded set in (X, T′). Let i be the identity map from (X, T′) into (X, T). Then i is a continuous linear map. By Theorem 3.31, i(S) = S is bounded in (X, T). #

Theorem 3.33 Let Φ be a collection of vector topologies for a vector space X over $\mathbb{H}$ and $S \subseteq X$. Then S is bounded in (X, $v\Phi$) if and only if it is bounded in (X, T) for each $T \in \Phi$.

Proof : ($\Longrightarrow$) Suppose S is bounded in (X, $v\Phi$). Since $v\Phi \supseteq T$ for each $T \in \Phi$, by Corollary 3.32, S is bounded in (X, T) for each $T \in \Phi$.

($\Longleftarrow$) Suppose S is bounded in (X, T) for each $T \in \Phi$. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of S. We must show that $(\frac{1}{n})x_n \to 0$ in $v\Phi$. Since S is bounded in (X, T) for each $T \in \Phi$. By Theorem 3.29, $(\frac{1}{n})x_n \to 0$ in (X, T) for ecah $T \in \Phi$. By Theorem 1.18, $(\frac{1}{n})x_n \longrightarrow 0$ in $v\Phi$. By Theorem 3.29, S is bounded in (X, $v\Phi$). #

Definition 3.34 Let X be a vector space over $\mathbb{H}$. $B \subseteq X$ is called a bornivore if and only if for each bounded set $S \subseteq X$, there exists an $\varepsilon > 0$ such that $tS \subseteq B$ for $|t| < \varepsilon$.

Remark 3.35 Let (X, d) be a pseudometric space. Then every bornivore is a neighberhood of 0.

Proof : Let $B \in N(X)$. Then $nB \notin N(X)$ for all $n \in \mathbb{N}$. Choose x_n with $d(x_n, 0) < \frac{1}{n}$ and $x_n \notin nB$. Let $S = (x_n)_{n \in \mathbb{N}}$. We must show that S is bounded. Let $(z_n)_{n \in \mathbb{N}}$ be a sequence in S.

We must show that $\frac{1}{n} z_n \to 0$. Let $\varepsilon < 0$ be given. Choose $n \in \mathbb{N}$ such that $\frac{1}{n^2} < \varepsilon$. Then $d(\frac{1}{n} z_n, o) = \frac{1}{n} d(z_n, 0) < \frac{1}{n}.\frac{1}{n} < \varepsilon$. Hence $\frac{1}{n} z_n \to 0$ so S is bounded. Since $x_n \notin nB$ for each n, $\frac{1}{n} S \not\subseteq B$ for all $n \in \mathbb{N}$; hence B does not absorb S so B is not a bornivore. #

<u>Lemma 3.36</u> Let X, Y be TVS($\mathbb{H}$)'s, $u : X \to Y$ a bounded linear map and B a bornivore in Y. Then $u^{-1}(B)$ is a bornivore in X.

<u>Proof</u> : Let $S \subseteq X$ be bounded. Since u is bounded, u(S) is also bounded. Since B is bornivore in Y, there exists an $\varepsilon > 0$ such that $tu(S) \subseteq B$ for all $t \in \mathbb{H}$ such that $|t| < \varepsilon$ so $u(tS) \subseteq B$ therefore $tS \subseteq u^{-1}(u(tS)) \subseteq u^{-1}(B)$. Hence $u^{-1}(B)$ is bornivore in X.

<u>Theorem 3.37</u> Let X be a TVS($\mathbb{H}$) in which every bornivore is a neighberhood of 0. Then every bounded linear map $u : X \to Y$, Y a TVS($\mathbb{H}$) is continuous.

<u>Proof</u> : Let U be an open neighberhood of 0 in Y. By Lemma 3.36, U is bornivore ; hence $u^{-1}(U) \in N(X)$. Thus u is continuous at 0. Let $x \in X$. Let $(x_\delta)_{\delta \in D}$ be a net in X such that $x_\delta \to x$. Then $x_\delta - x \to 0$. But u is continuous at 0 therefore $u(x_\delta) - u(x)$ $= u(x_\delta - x) \to u(0) = 0$. Hence $u(x_\delta) \to u(x)$ so u is continuous at x But x was arbitrarry, so u is continuous on X. #

<u>Lemma 3.38</u> Let (X, T) be a TVS($\mathbb{H}$), B a balanced convex absorbing set and p the gauge of B. Then the following hold :

(a) If B is bounded then $\sigma p \supseteq T$.

(b) If B is a neighborhood of 0 then $\sigma p \subseteq T$.

<u>Proof</u> : (a) Let $U \in N(X)$. Since B is bounded, there exists an $\varepsilon > 0$ such that $tB \subseteq U$ for all t such that $|t| < \varepsilon$. Hence there exists an $\varepsilon' > 0$ such that $|\varepsilon'| < \varepsilon$ and $U \supseteq \varepsilon' B$. Since p is the qauge of B, $U \supseteq \varepsilon' B \supseteq \varepsilon' \{ x | p(x) < 1 \} = \{ x | p(x) < \varepsilon' \}$ which is open in σp. Hence U is a σp-neighberhood of 0 so $\sigma p \supseteq T$.

(b) Let U be a σp-neighberhood of 0. Then there exists an $\varepsilon > 0$ such that $U \supseteq \{ x | p(x) \leq \varepsilon \} = \varepsilon \{ x | p(x) \leq 1 \} \supseteq \varepsilon B$. Thus $u \in N(T)$ therefore $\sigma p \supseteq T$. #

<u>Theorem 3.39</u> Let X be a TVS($\mathbb{H}$). Then X is a seminormed space if and only if X has a bounded convex neighberhood U of 0.

<u>Proof</u> : ($\Rightarrow$) Suppose that X is a seminormed space with a seminorm p. Set $U = \{ x \in X | p(x) \leq 1 \}$. To show that U is bounded, let $V \in N(X)$. Then there exists an $r > 0$ such that $V \supseteq B(0 ; r)$. Let $\varepsilon > 0$ be such that $\varepsilon < r$. Then if $|t| < \varepsilon$ we get that $tU = \{ x \in X | p(x) \leq |t| \} \subseteq \{ x \in X | p(x) < \varepsilon \} \subseteq \{ x \in X | p(x) < r \} = B(0, r) \subseteq V$ so U is a bounded convex neighberhood of 0.

($\Leftarrow$) Let U be a bounded convex neighberhood of 0.

<u>Case 1</u> : U is balanced. Then its gauge p_u gives the topology of X. By Lemma 3.38, $\sigma(p_u) = T$, the topology of X. Hence X is a seminormed space.

<u>Case 2</u> : U is not balanced. By Theorem 3.19, there exists a balanced neighberhood V of 0 such that $V \subseteq U$. let CH(V) be the convex hull of V ; that is $CH(V) = \{ \alpha x + \beta y \mid \alpha, \beta \in [0, 1], \alpha + \beta = 1$ and $x, y \in V \}$. Let $z \in CH(V)$. Then there exist $\alpha, \beta \in [0, 1], \alpha + \beta = 1$ and there exist $x, y \in V$ such that $z = \alpha x + \beta y$. Let $t \in \mathbb{H}$ be such

that $|t| \leq 1$. Then $tz = \alpha(tx) + \beta(ty) \in CH(V)$. So $CH(V)$ is balanced and convex. Since $CH(V) \supseteq V \ni 0$, $CH(V) \in N(X)$. Since $CH(V) \subseteq U$ and U is bounded, $CH(V)$ is bounded. Hence the gauge of $CH(V)$ is a seminorm on X so X is a seminormed space. #

<u>Lemma 3.40</u> Let X be a vector space over $\mathbb{H}$ and q a nonnegative real function defined on X. For $x \in X$, set

$\|x\| = \inf \{ \sum_{k=1}^{n} q(x_k - x_{k-1}) \mid x_0 = 0 ; x_n = x \}$ where the set

$(0, x_1, x_3, \ldots, x_{n-1}, x)$ is a chain ending at x. Then

(1) $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

(2) If $q(0) = 0$ then $\|0\| = 0$

(4) If $q(-x) = q(x)$ for all x then $\|-x\| = \|x\|$ for all x.

<u>Proof :</u> Let $x, y \in X$. We must show that $\|x+y\| \leq \|x\| + \|y\|$. Let $\varepsilon > 0$ be given. Choose chains $(x_i)_{i=0}^{n}$, $(y_i)_{i=0}^{m}$ ending at x and y respectively with $\sum_{k=1}^{n} q(x_k - x_{k-1}) < \|x\| + \frac{\varepsilon}{2}$ and $\sum_{k=1}^{m} q(y_k - y_{k-1}) < \|y\| + \frac{\varepsilon}{2}$. Let $(z_i)_{i=0}^{m+n}$ be the chain $(0, x_1, x_2, \ldots, x_{n-1}, x, x+y_1, x+y_2, \ldots, x+y_{m-1}, x+y)$. Then $(z_i)_{i=0}^{m+n}$ is a chain ending at $x+y$. Hence $\|x+y\| \leq \sum_{k=1}^{m+n} q(z_k - z_{k-1}) = \sum_{k=1}^{n} q(x_k - x_{k-1}) + \sum_{k=1}^{m} q(y_k - y_{k-1}) < \|x\| + \|y\| + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\|x+y\| \leq \|x\| + \|y\|$. Thus (2) holds. Suppose $q(0) = 0$. We must that $\|0\| = 0$. Let $n = 1$ in the definition of $\|0\|$.

Then $\|0\| = \inf \{ q(x_1 - x_0) \} = \inf \{ q(0-0) \} = \inf \{ q(0) \} = \inf \{ 0 \} = 0$ so (3) holds. Suppose $q(-x) = q(x)$ for all $x \in X$. We must show that

$\|-x\| = \|x\|$ for all $x \in X$. Let $x \in X$ be arbitrary. Let $(0, x_1, x_2, \ldots, x_n)$ be a chain ending at x and let $y_k = -x_k$. Then

$$\begin{aligned}
\sum_{k=1}^{n} q(y_k - y_{k-1}) &= q(y_n - y_{n-1}) + q(y_{n-1} - y_{n-2}) + \ldots + q(y_1 - y_0) \\
&= q(-x_n + x_{n-1}) + q(-x_{n-1} + x_{n-2}) + \ldots + q(-x_1 + x_0) \\
&= q(-(x_n - x_{n-1})) + q(-(x_{n-1} - x_{n-2})) + \ldots + q(-(x_1 - x_0)) \\
&= q(x_n - x_{n-1}) + q(x_{n-1} - x_{n-2}) + \ldots + q(x_1 - x_0) \\
&= \sum_{k=1}^{n} q(x_k - x_{k-1}).
\end{aligned}$$

Hence

$$\begin{aligned}
\|-x\| &= \inf \{ \sum_{k=1}^{n} q(x_k - x_{k-1}) \mid x_0 = 0 \text{ and } x_n = -x \} \\
&= \inf \{ \sum_{k=1}^{n} q(y_k - y_{k-1}) \mid y_0 = 0,\ y_n = -x_n = x \} \\
&= \|x\| \text{ thus (4) holds. } \#
\end{aligned}$$

Lemma 3.41 Let X be a vector space over $\mathbb{H}$ and q a nonnegative real function on X satisfying $q(0) = 0$ and $q(x+y+z) \leq 2\max\{ q(x), q(y), q(z) \}$ for all $x,y,z \in X$. Then for any $x_1, x_2, \ldots, x_n \in X$, we have that $q(\sum_{i=1}^{n} x_i) \leq 2 \sum_{i=1}^{n} q(x_i)$.

Proof : Let $x_1, x_2, \ldots, x_n \in X$. Let $u = \sum_{i=1}^{n} q(x_i)$.

Case 1 : $u = 0$. Since $q \geq 0$, $q(x_i) = 0$ for all $i = 1,2,\ldots,n$. We must show that $q(\sum_{i=1}^{n} x_i) = 0$. It is true for $n = 1$.

Suppose $q(\sum_{i=1}^{k} x_i) = 0$ for $1 \leq k < n$. We must show that

$\leq 2 \max\{ q(\sum_{i=1}^{k} x_i), q(x_{k+1}), q(0) \} = 2 \max\{0,0,0\} = 2.0 = 0,$

we get that $q(\sum_{i=1}^{k+1} x_i) = 0$. By mathematical induction, we conclude that $q(\sum_{i=1}^{n} x_i) = 0$ therefore $q(\sum_{i=1}^{n} x_i) \leq 2 \sum_{i=1}^{n} q(x_i)$.

Case 2 : $u \neq 0$. So $u > 0$. We must show that $q(\sum_{i=1}^{n} x_i) \leq 2u$

It is trivial for $n = 1,2,3$. Let $n \in \mathbb{N}$ be such that $n \geq 4$. Suppose that $q(\sum_{i=1}^{k} x_i) \leq 2 \sum_{i=1}^{k} q(x_i)$ for $4 \leq k < n$. We must show that $q(\sum_{i=1}^{n} x_i) \leq 2 \sum_{i=1}^{n} q(x_i)$. Let m be the largest integer such that $\sum_{i=1}^{m} q(x_i) \leq \frac{u}{2}$; if no such integer exists then let $m = 0$. Then, for $0 \leq m < n$, $\sum_{i=1}^{m+1} q(x_i) > \frac{u}{2}$. Hence $\sum_{i=m+2}^{n} q(x_i) \leq \frac{u}{2}$ (for $m = n-1$ let $\sum_{i=n+1}^{n} q(x_i) = 0$). Hence the sums of the left-hand side of the inequalities have fewer than n terms (or are 0 ; that is for $m = 0$ and $m = n-1$) By the induction hypothesis, we have that

$$q(\sum_{i=1}^{m} x_i) \leq 2 \sum_{i=1}^{m} q(x_i) \leq u,$$

$$q(\sum_{i=m+2}^{n} x_i) \leq 2 \sum_{i=m+2}^{n} q(x_i) \leq u \text{ and}$$

$$q(x_{m+1}) \leq \sum_{i=1}^{n} q(x_i) = u.$$

By assumption. $q(\sum_{i=1}^{n} x_i) = q(\sum_{i=1}^{m} x_i + x_{m+1} + \sum_{i=m+2}^{n} x_i)$

$$\leq 2 \max \{ q(\sum_{i=1}^{m} x_i),\ q(x_{m+1}),\ q(\sum_{i=m+2}^{n} x_i)\}$$

$$\leq 2u = 2 \sum_{i=1}^{n} q(x_i). \quad \#$$

Theorem 3.42 Let (X, T) be a first countable TVS(IH). Then there exists a paranorm $\|.\|$ on X such that $T = T_{\|.\|}$ where $T_{\|.\|}$ is the topology induced by the paranorm $\|.\|$.

Proof : Let $\{0_n\}_{n \in N}$ be a countable basis of N(X). Let $U_0 = X$. Choose a balanced neighberhood W_1 of 0 such that $W_1 + W_1 + W_1 \subseteq U_0$ and $W_1 \subseteq O_1$. Let $W_1 = U_1$. Choose a balanced neighberhood W_2 of 0 such that $W_2 + W_2 + W_2 \subseteq U_1$, and $W_2 \subseteq O_2$. Let $W_2 = U_2$. Continuing in this way choose a balanced set $W_n \in N(X)$ such that $W_n + W_n + W_n \subseteq U_{n-1}$. and $W_n \subseteq D_n$, Let $W_n = U_n$. Then $\{U_n\}_{n \in \mathbb{N}}$ is a base for N(X). define $q : X \to \mathbb{R}^+ \cup \{0\}$ by

$$q(x) = \begin{cases} 0 & \text{if } x \in \overline{\{0\}}, \\ 2^{-k(x)} & \text{if } x \notin \overline{\{0\}} \text{ and } k(x) \text{ is the largest integer such that } x \in U_{k(x)}. \end{cases}$$

Since k(x) is the unique integer such that $x \in U_{k(x)}$, q is welldefined.

Claim 1 For any sequence $(x_n)_{n \in \mathbb{N}} \subseteq X$, $x_n \to 0$ if and only if $q(x_n) \to 0$.

Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X such that $x_n \to 0$ in X. We must show that $q(x_n) \to 0$. Let $\varepsilon > 0$ be given. Then there exists an $m \in \mathbb{N}$ such that $2^{-m} < \varepsilon$. Since $x_n \to 0$, there exists an $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies that $x_n \in U_m$. For such an x_n, either $x_n \in \overline{\{0\}}$

or $k(x_n) \geq m$. If $x_n \in \overline{\{0\}}$ then $q(x_n) = 0 < \varepsilon$. If $k(x_n) \geq m$ then $q(x_n) \leq 2^{-m} < \varepsilon$ so $q(x_n) \to 0$. Conversely, suppose that $q(x_n) \to 0$. We must show that $x_n \to 0$ in X. Let $\ell \in N \cup \{0\}$. We show that there exists an $n' \in \mathbb{N}$ such that $n \geq n'$ implies that $x_n \in U_\ell$. Since $q(x_n) \to 0$, there exsists an $n' \in \mathbb{N}$ such that $n \geq n'$ implies that $q(x_n) < 2^{-\ell}$. Foe such an x_n, either $x_n \in \overline{\{0\}}$ or $x_n \notin \overline{\{0\}}$. If $x_n \in \overline{\{0\}}$ then $x_n \in U_\ell$. If $x_n \notin \overline{\{0\}}$, let $k = k(x_n)$. Then $q(x_n) = 2^{-k} < 2^{-\ell}$. So $x \in U_k \subseteq U_\ell$. Hence there exists an $n' \in \mathbb{N}$ such that $n \geq n'$ implies that $x_n \in U_\ell$. But ℓ was arbitrary, $x_n \to 0$ in X so we have claim 1.

<u>Claim 2</u> : For any x, y, z $\in$ X, $q(x+y+z) \leq 2 \max \{g(x), g(y), g(z)\}$. Let x, y, z $\in$ X. <u>case 1</u> : All x, y, z $\in \overline{\{0\}}$. Since $\overline{\{0\}}$ is a vector subspace of X, $x+y+z \in \overline{\{0\}}$. Hence $q(x+y+z) = 0 \leq 2\max\{g(x), q(y), q(z)\}$. <u>case 2</u> : Not all x, y, z $\in \overline{\{0\}}$. Suppose that $x \notin \overline{\{0\}}$ and $q(x) = 2^{-k(x)} \geq q(y), q(z)$. Then x, y, z $\in U_k$ Since $U_k + U_k + U_k \subseteq U_{k-1}$, $x+y+z \in U_{k-1}$. Thus $q(x+y+z) \leq 2^{(-k(x)-1)} = 2.2^{-k(x)} = 2q(x) = 2 \max\{q(x), g(y), q(z)\}$ so we have claim 2.

Define $\|.\| : X \to \mathbb{R}^+ \cup \{0\}$ by $\|x\| = \inf \{ \sum_{k=1}^{n} q(x_k - x_{k-1}) | x_0 = 0, x_n = x\}$ where $(0. x_1. x_2, \ldots, x_n)$ is a chain ending at x.

<u>Claim 3</u> : For all $x \in X$, $\frac{1}{2} q(x) \leq \|x\| \leq q(x)$. Let $x \in X$. Since $q(x) \in \{ \sum_{k=1}^{n} q(x_k - x_{k-1}) | x_0 = 0, x_n = x \}$, $\|x\| = \inf \{ \sum_{k=1}^{n} q(x_k - x_{k-1}) | x_0 = 0, x_n = x \} \leq q(x)$. we must show that $\frac{1}{2} q(x) \leq \|x\|$. Let $(0. x_1. x_2. \ldots, x_n)$ be a chain ending at x. By Lemma 3.41,

$\sum_{k=1}^{n} q(x_k - x_{k-1}) \geq \frac{1}{2} q \left(\sum_{k=1}^{n} (x_k - x_{k-1}) \right) = \frac{1}{2} g(x)$. Hence $\|x\| = \inf$

$\{ \sum_{k=1}^{n} q(x_k - x_{k-1}) \mid x_0 = 0,\ x_n = x \} \geq \frac{1}{2} q(x)$ so we have claim 3.

<u>Claim 4</u> For any sequence $(x_n)_{n \in \mathbb{N}}$ in X, $x_n \to 0$ if and only if $x_n \to 0$. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X. Suppose that $x_n \to 0$. By claim 1, $q(x_n) \to 0$. Since $\frac{1}{2} q(x_n) \leq \|x_n\| \leq q(x_n)$ and $q(x_n) \to 0$, $\|x_n\| \to 0$ as $n \to \infty$. Coversely, suppose that $\|x_n\| \to 0$. Since $|x_n| \leq q(x_n)$, $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} q(x_n)$; hence $\lim_{n\to\infty} q(x_n) \geq 0$. Since $\frac{1}{2} q(x_n) \leq \|x_n\|$, $\lim_{n\to\infty} q(x_n) \leq \lim_{n\to\infty} 2\|x_n\| = 0$ so $q(x_n) \to 0$. By claim 1, $x_n \to 0$ so we have claim 4.

Let $x \in X$. We want to show that $q(-x) = q(x)$.

<u>Case 1</u> $x \in \overline{\{0\}}$. Since $\overline{\{0\}}$ is a vector subspace of X, $-x \in \overline{\{0\}}$. Hence $q(x) = 0 = q(-x)$.

<u>Case 2</u> $x \notin \overline{\{0\}}$. Then $q(x) = 2^{-k(x)}$ where $k(x)$ is the largest integer such that $x \in U_{k(x)}$. But $U_{k(x)}$ is balanced therefore $-x \in U_{k(x)}$; hence $q(-x) = 2^{-k(x)} = q(x)$. We must show that $\|.\|$ is a paranorm on X. We need only show that the multiplication is continuous.. Let $(t_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{H}$ such that $t_n \to t$ for some $t \in \mathbb{H}$ and let $(x_n)_{n \in \mathbb{N}} \subseteq X$ be such that $\|x_n - x\| \to 0$. We must show that $\|t_n x_n - tx\| \to 0$.

$\|t_n x_n - tx\| \leq \|t_n x_n - t_n x\| + \|t_n x - tx\| = |t_n| \|x_n - x\| + |t_n - t| \|x\|$. Since $\|x_n - x\| \to 0$ and $|t_n - t| \to 0$ as $n \to \infty$, $\|t_n x_n - tx\| \to 0$ as $n \to \infty$ so the multiplication is continuous. Hence $\|.\|$ is a paranorm on X.

Thus the paranorm $\|.\|$ induces a topolgy on X, say $T_{\|\cdot\|}$. Let $i : (X, T) \to (X, T_{\|\cdot\|})$ be the identity map. i is continuous. Hence for any sequence (x_n) in X, $x_n \to a$ in (x, T) if and only if $x_n \to a$ in $(x, T_{\|\cdot\|})$ therefore $T = T_{\|\cdot\|}$. #

<u>Theorem 3.43</u> Every TVS($\mathbb{H}$) is a completely reqular topological space.

<u>Proof</u> : Let (X, T) be a TVS($\mathbb{H}$). We must show that X is completely reqular ; that is, for each closed set $F \subseteq X$ and for each $x \notin F$ there is a continuous function f on X such that f = 0 on F and f(x) = 1.

Let F be a closed set and $x' \notin F$. Since F is closed, $X \setminus F$ is an open neighberhood of x'. Let (U_n) be a sequence of balanced neighberhoods of 0 such that $(x' + U_n) \cap F = \emptyset$ and $U_n + U_n \subseteq U_{n-1}$ for each n. Then $\{U_n\}_{n \in \mathbb{N}}$ is an additive filterbase of balanced absorbing sets so by Theorem 3.26, $\{U_n\}_{n \in \mathbb{N}}$ is a local base of neighberhoods of 0 for first countable vector topology T′ of X. By Theorem 3.42, there exists a paranorm $\|.\|$ on X such that $T' = T_{\|\cdot\|}$ is the topology induced by $\|.\|$. Defined $d : X \times X \to \mathbb{R}^+ \cup \{0\}$ by $d(x,y) = \|x - y\|$ so d is a pseudometric on X. Define $f : X \to \mathbb{R}$ by $f(x) = \dfrac{d(x,F)}{d(x,F)}$.

Then f is continuous on X and f(y) = 0 for all $y \in F$ and f(x) = 1. Since each U_n is a T-neighberhood of 0, $T \supseteq T'$ so f is T-continuous.

<u>Definition 3.44</u> Let X be a TVS($\mathbb{H}$). Then X is called <u>locally bounded</u> if and only if X has a bounded neighberhood of 0.

<u>Theorem 3.45</u> Every locally bounded TVS($\mathbb{H}$) is a paranormed space.

Proof : Let X be a locally bounded TVS(IH). We must show that X is first countable. Let U be a bounded neighberhood of 0. Then for any set $V \in N(X)$, there exists a positive integer n such that $V \supseteq \frac{1}{n} U$. Thus $\{(\frac{1}{n}) U\}$ is a countable local base of neighberhood of 0 so X is first countable. By Theorem 3.42, X is paranormed space. #

Theorem 3.46 Let X, Y be TVS(IH)'s and let $f : X \to Y$ be a linear map. Suppose that f(U) is bounded for some $U \in N(X)$. Then f is continuous. If Y is a locally bounded TVS(IH) and f is continuous then there exists a $U \in N(X)$ such that f(U) is bounded.

Proof : Let $V \in N(Y)$. Since f(U) is bounded, $t f(U) \subseteq V$ for some $t \neq 0$. Thus $f^{-1}(V) \supseteq tU$, a neighberhood of 0, so f is continuous.

Suppose that Y is a locally bounded TVS(IH) and f is continuous. Must show there exists a $U \in N(X)$ such that f(U) is bounded. Since Y is locally bounded, Y contains a bounded set $W \in N(Y)$. Since f is continuous, $f^{-1}(W) \in N(X)$. Choose $U = f^{-1}(W)$. Then $f(f^{-1}(W)) \subseteq W$ therefore $f(U) = f(f^{-1}(W))$ is bounded. #

Theorem 3.47 Let X be a TVS(IH), $f \in X^{\#}$, and assume that ker f is closed. Then f is continuous.

Proof : Case 1 f = 0. obvious.

Case 2 $f \neq 0$. Suppose that f is not continuous. Let $x \in X$ and U a balanced neighberhood of 0. By Theorem 3.46, f is unbounded on U ; that is f(U) is unbounded. Let $t \in$ IH be such that $|t| \leq 1$. Since U is balanced, $t f(U) = f(tU) \subseteq f(U)$. So f(U) is balanced. Claim that f(U) = IH. We must show that IH $\subseteq$ F(U). Let $h \in$ IH.

Case 1 $h = 0$. Since $f(U)$ is balanced and $|0| = 0 < 1$, $0 \in 0 \cdot f(U) \subseteq f(U)$.

Case 2 $h \neq 0$. Since $f(U)$ is unbounded, there exsits a $y \in U$ such that $|f(y)| > |h| > 0$ so $\frac{f(y)}{|f(y)|}$ and $\frac{x}{|x|} \in S$ where $S = \{ q \in \mathbb{H} \mid |q| = 1 \}$.

It is clear that $S = \{ q \in \mathbb{H} \mid |q| = 1 \}$ is a group with respect to multiplication. Hence there exsits a $V \in S$ such that $\frac{V f(y)}{|f(y)|} = \frac{h}{|h|}$ so $h = \frac{V|h|}{|f(y)|} \cdot f(y)$. Since $\left|\frac{V|h|}{|f(y)|}\right| = \frac{|V|\ |h|}{|f(y)|} < 1$ and $f(U)$ is balanced, $h \in f(U)$. So we have the claim. Hence there exists a $u \in U$ such that $f(u) + f(x) = 0$. Then $x + u \in (\ker f) \cap (x + U)$ so $\ker f$ is dense in X. But $\ker f$ is closed, so $\ker f = X$. Hence $f = 0$ on X, a contradiction. Thus f is continuous. #

Corollary 3.48 Let X be a separated TVS($\mathbb{H}$), $0 \neq y \in X$ and $f \in X^{\#}$. Let $g(x) = f(x)y$. Then if $g : X \to X$ is continuous, so is f.

Proof : It is clear that $\ker f = \ker g$. Since $\{ 0 \}$ is closed and g is continuous, $\ker g$ is closed ; hence $\ker f$ is closed. By Theorem 3.47, f is continuous. #

Completeness

Definition 3.49 Let X be a TVS($\mathbb{H}$). A net $(x_\delta)_{\delta \in D}$ in X is called a Cauchy net if and only if for all $U \in N(X)$ there exists a $\delta \in D$ such that $\alpha \geq \delta$ and $\beta \geq \delta$ imply that $x_\alpha - x_\beta \in U$.

Definition 3.50 Let X be a TVS($\mathbb{H}$) and $S \subseteq X$. S is complete if and only if every cauchy net in S converges to a point in S and S is called sequentially complete if and only if every cauchy sequence in S converges to a point in S.

Theorem 3.51 Let $(X, \|.\|)$ be a paranormed space. If X is sequentially complete then X is complete.

Proof : Let $(x_\delta)_{\delta\in D}$ be a cauchy net in X. We must show that $x_\delta \to x_0$ for some $x_0 \in X$. Let $n \in \mathbb{N}$. Since $(x_\delta)_{\delta\in D}$ is cauchy , there exists a $\delta_n' \in D$ such that $\alpha \geq \delta_n'$ and $\beta \geq \delta_n'$ imply that $\|x_\alpha - x_\beta\| < \frac{1}{n}$. Let $\delta_n = \max\{\delta_1', \delta_2', \ldots, \delta_n'\}$. Then $\delta_n \geq \delta_{n-1}$. Let $y_n = x_{\delta_n}$. Claim that (y_n) is a cauchy sequence in X. Let $\varepsilon > 0$ be given. Then there exists an $n' \in \mathbb{N}$ such that $\frac{1}{n'} < \varepsilon/2$. Let $m, n \in \mathbb{N}$ such that $m > n'$ and $n > n'$. Then $\|y_m - y_n\| \leq \|y_m - y_{n'}\| + \|y_{n'} - y_n\| < \frac{1}{n'} + \frac{1}{n'} = \frac{2}{n'} < \varepsilon$ so we have the claim. Since X is sequentially complete, $y_n \to x_0$ for some $x_0 \in X$. We must show that $x_\delta \to x_0$. Let $\varepsilon > 0$ be given Choose $\ell \in \mathbb{N}$ such that $\frac{1}{\ell} < \frac{\varepsilon}{2}$ and $\|y_\ell - x_0\| < \varepsilon/2$. Let $\delta \in D$ be such that $\delta \geq \delta_\ell$. Then $\|x_\delta - x_0\| \leq \|x_\delta - y_\ell\| + \|y_\ell - x_0\| < \frac{1}{\ell} + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon$ so we have the claim. Hence X is complete. #

Definition 3.52 Let X be a TVS($\mathbb{H}$) and $S \subseteq X$. $P : X \to S$ is called a topological projection if and only if P is a continuous linear map from X onto S satisfying $P^2 = P$ $(P^2 = P \circ P)$.

Lemma 3.53 Let X be a TVS($\mathbb{H}$) and $P : X \to S$ a continuous projection onto a subspace S. Let A be a subset of S such that $P^{-1}(A)$ is complete. Then A is complete.

Proof : Let $x = (x_\delta)_{\delta\in D}$ be a cauchy net in A. Then x is a cauchy net in X. Claim that x is cauchy in $P^{-1}(A)$. Let $W \in N(P^{-1}(A))$.

Then there exists a $U \in N(X)$ such that $W = N \cap P^{-1}(A)$. Since x is Cauchy in A, there exists a $\delta \in D$ such that $\alpha \geq \delta$ and $\beta \geq \delta$ imply that $x_\alpha - x_\beta \in U \cap A$. Since P is onto, $A \subseteq S$ and $P^2(x_\alpha - x_\beta) = P(x_\alpha - x_\beta)$, $P(x_\alpha - x_\beta) = x_\alpha - x_\beta \in A$; hence $x_\alpha - x_\beta \in P^{-1}(A)$. Thus $x_\alpha - x_\beta \in U \cap P^{-1}(A)$ $= W$ so we have the claim. Since $P^{-1}(A)$ is complete, $x \rightarrow x_0$ for some $x_0 \in P^{-1}(A)$ therefore $x = P(x) \rightarrow P(x_0) \in A$, so A is complete. #

Remark 3.54 A special case of Lemma 3.53 is that S is complete if X is.

Theorem 3.55 Let $\{X_\alpha\}_{\alpha \in I}$ be a collection of TVS($\mathbb{H}$)'s and $A_\alpha \subseteq X_\alpha$ for each $\alpha \in I$. Then ΠA_α is complete if A_α is complete for all $\alpha \in I$.

Proof : Let x be a Cauchy net in ΠA_α. Claim that $P_\alpha(x)$ is a Cauchy net in A_α for all $\alpha \in I$. Let $\alpha \in I$ be fixed and let $W_\alpha \in N(A_\alpha)$. Then there exists a $U_\alpha \in N(X_\alpha)$ such that $W_\alpha = U_\alpha \cap A_\alpha$. Since x is Cauchy in ΠA_α, there exists a $\delta \in I$ such that for all $\gamma \geq \delta$, $\beta \geq \delta$, $x_\gamma - x_\beta \in P_\alpha^{-1}(U_\alpha) \cap \Pi A_\alpha$. Now $P_\alpha(x_\gamma - x_\beta) = P_\alpha(x_\gamma) - P_\alpha(x_\beta) \in$ $U_\alpha \cap A_\alpha = W_\alpha$ so we have the claim. Since A_α is complete, $P_\alpha(x) \rightarrow a_\alpha$ for some $a_\alpha \in A_\alpha$. Since P_α is continuous, $x = (P_\alpha(x))_{\alpha \in I} \rightarrow (a_\alpha)_{\alpha \in I} \in \Pi A_\alpha$ therefore $x \rightarrow a$ where $a = (a_\alpha)_{\alpha \in I}$ so ΠA_α is complete. Conversely, suppose that ΠA_α is complete. We must show that A_α is complete for each $\alpha \in I$. For each $\beta \in I$, let $P_\beta : \Pi X_\beta \rightarrow X_\beta$ be the projection map. Let $\alpha \in I$ be arbitrary. Let $x = (x_\alpha^\delta)_{\delta \in D}$ be a Cauchy net in A_α. Given $\beta \in I \setminus \{\alpha\}$, fix $x_\beta^0 \in A_\beta$. Let $y = (x_\beta^0, x_\alpha^\delta)_{\substack{\alpha \neq \beta \\ \delta \in D}} \in \prod_{\beta \in I} A_\beta$. Claim that y is a

Cauchy net in $\prod_{\beta \in I} A_\beta$. Let $U \in N(\prod_{\beta \in I} X_\beta)$. Then U contains a basic open set $P^{-1}_{\beta_1}(U_{\beta_1}) \cap \ldots \cap P^{-1}_{\beta_n}(U_{\beta_n})$ where U_{β_i} is open in A_{β_i}, $i = 1, 2, \ldots, n$.

<u>Case 1</u> $\alpha = \beta_i$ for all i. Then for all $\delta \in D$, $\varepsilon, \gamma \geq \delta$ implies that $(x^0_\beta, x^\varepsilon_\alpha)_{\alpha \neq \beta} - (x^0_\beta, x^\gamma_\alpha)_{\alpha \neq \beta} = (0, x^\varepsilon_\alpha - x^\gamma_\alpha)_{\beta \neq \alpha} \in U$.

<u>Case 2</u> $\alpha = \beta_i$ for some i. Then there exists a $\delta \in D$ such that $\varepsilon, \gamma \geq \delta$ implies that $x^\varepsilon_\alpha - x^\gamma_\alpha \in U_\alpha$. Hence $(x^0_\beta, x^\varepsilon_\alpha)_{\alpha \neq \beta} - (x^0_\beta, x^\gamma_\alpha)_{\alpha \neq \beta} \in U$ so we have the claim.

Since $\prod_{\beta \in I} A_\beta$ is complete, y converges to some $a = (a_\beta)_{\beta \in I} \in \prod_{\beta \in I} A_\beta$. $a_\beta = P_\beta(a)$ so $x = P_\alpha(y) \to P_\alpha(a) = a_\alpha$ therefore $x \to a_\alpha$ so A_α is complete. Since $\alpha \in I$ was arbitrary, A_α is complete for all $\alpha \in I$.

<u>Definition 3.56</u> Let X be a TVS($\mathbb{H}$). X is called <u>boundedly complete</u> (or quasicomplete) if and only if every bounded closed set is complete

<u>Remark 3.57</u> Let X be a TVS($\mathbb{H}$). Then the following are clear :

(1) If X is complete then X must be boundedly complete.

(2) If X is boundedly complete then X must be sequentially complete.

<u>Definition 3.58</u> Let T, T′ be vector topologies for a vector space X over $\mathbb{H}$. We say that <u>T′ is F linked to T</u> if and only if T′ has a local base of neighborhoods of 0 each of which is T-closed.

<u>Remark 3.59</u> If $T' \subseteq T$ then T′ is F linked to T.

<u>Lemma 3.60</u> Let X be a vector space over $\mathbb{H}$ and let T, T$'$ be vector topologies with T$'$ being F linked to T and $x = (x_\delta)_{\delta \in D}$ a net in X. If x is a Cauchy net in (X, T$'$) and $x \to a$ in (X, T) then $x \to a$ in (X, T$'$).

<u>Proof :</u> Let U be a T$'$ neighberhood of 0 which is T-closed. Since x is Cauchy in (X, T$'$). there exists a $\delta \in D$ such that $\alpha \geq \delta$, $\beta \geq \delta$ imply that $x_\alpha - x_\beta \in U$. Fix $\alpha \geq \delta$. We have that $x_\alpha - x_\beta \in U$ for all $\beta \geq \delta$. Since U is T-closed, $x_\alpha - a \in U$. Since $x_\alpha - a \in U$ for all $\alpha \geq \delta$ and the set of such U is a base for N(X, T$'$), $x \to a$ in (X, T$'$). #

<u>Theorem 3.61</u> Let X be a vector space over $\mathbb{H}$ and let T, T$'$ be vector topologies for X such that $T' \supseteq T$ and T$'$ is F linked to T. Let $S \subseteq (X,T)$ be complete(sequentially complete). Then S is T$'$ complete (sequentially complete).

<u>Proof :</u> Suppose that S is T-complete. We must show that S is T$'$-complete. Let $x = (x_\delta)_{\delta \in D}$ be a Cauchy net in S with respect to T$'$. Claim that x is Cauchy in $S \subseteq (X, T)$. Let $U \in N(X, T)$ be closed. Then $U \cap S \in N(S, T)$. Since T$'$ is linked to T, $U \in N(X, T')$ hence $U \cap S \in N(S, T')$. Since x is Cauchy in (S, T$'$), there exists a $\delta \in D$ such that $\alpha \geq \delta$, $\beta \geq \delta$ imply that $x_\alpha - x_\beta \in U$; hence x is Cauchy in (S, T) so we have the claim. Since (S, T) is complete and x is Cauchy in (S, T), $x \to a$ for some $a \in (X, T)$. By Lemma 3.60, $x \to a$ in (x, T$'$) so S is T$'$-complete. We can use a similar proof in the case where S is T-sequentially complete. #

<u>Theorem 3.62</u> Let (X, T) be a TVS($\mathbb{H}$) which is complete, boundedly complete, or sequentially complete. Let T$'$ be a larger vector topology which is F linked to T. Then (X, T$'$) is, respectively,

complete, boundedly complete, or sequentially complete.

Proof : The first and the third are special cases of Theorem 3.61 where $S = X$. Suppose that (X, T) is boundedly complete. We must show that (X, T') is boundedly complete. Let S be a bounded closed set in (X, T). Let $\bar{S}$ be the T-closure of S. By Corollary 3.32, $\bar{S}$ is a bounded set in (X, T) and hence $\bar{S}$ is T-complete. By Theorem 3.61, $\bar{S}$ is T'-complete and S' is also T'-closed. Hence (X, T') is boundedly complete. #

Theorem 3.63 Let (X, T) be a separated TVS($\mathbb{H}$). Let A be a balanced, convex, bounded, sequentially complete set in X. Let Z be the span of A and p the gauge of A, defined on Z. Then (Z, p) is a Banach space over $\mathbb{H}$.

Proof : Claim that A is absorbing in Z. Let $z \in Z$. If $z = 0$, the result is obvious. Assume $z \neq 0$. Since Z is the span of A, $z = \sum_{j=1}^{n} t_j a_j$ for some $t_j \in \mathbb{H}$, $a_j \in A$, $j = 1,2,\ldots,n$. Let

$\alpha = \sum_{j=1}^{n} |t_j|$. Since $z \neq 0$, $\alpha > 0$. Let $\varepsilon = \frac{1}{\alpha} > 0$. Let $t \in \mathbb{H}$ be such that $|t| < \varepsilon$. We must show that $tz \in A$. Since $|t| < \varepsilon$, $|t| < \frac{1}{\sum_{j=1}^{n} |t_j|}$;

hence $\sum_{j=1}^{n} |t t_j| < 1$. Now $tz = t(\sum_{j=1}^{n} t_j a_j) = \sum_{j=1}^{n} (t t_j) a_j$. Since $\sum_{j=1}^{n} |t t_j| \leq 1$, A is balanced and convex,

$tz = \sum_{j=1}^{n} (t t_j) a_j \in A$ so we have the claim. By Lemma 3.38, $\sigma p \supseteq T|_Z$

where σp denotes the topolgy induced by the seminorm p and $T|_Z$ is the topology of X relative to Z. Since T is separated, σp is separated ; hence $(Z, \sigma p)$ is separated ; that is, p is a norm. Claim that σp is F linked to $T|_Z$. Let $B = \{ \varepsilon A \mid \varepsilon > 0 \}$. We must show that B is a local neigberhood base of 0 in (Z, p) ; that is, for each $U \in N(Z)$, $U \supseteq \varepsilon A$ for some $\varepsilon > 0$. Let $U \in N(Z)$. There exists a $W \in N(X)$ such that $U = Z \cap W$. Since A is bounded in X, there exists an $\varepsilon' > 0$ such that $tA \subseteq W$ all $t \in \mathbb{H}$ such that $|t| < \varepsilon'$. Choose $\varepsilon = \frac{\varepsilon'}{2} > 0$. Hence $\varepsilon A \subseteq W$. But Z is the span of A so $\varepsilon A \subseteq Z$; hence $\varepsilon A \subseteq Z \cap W = U$ so B is a local neighberhood base of 0 in (Z, p). We must show that εA is sequentially complete in X for all $\varepsilon > 0$. Let $\varepsilon > 0$ be given. Let $x = (x_n)$ be a Cauchy sequence in εA. Then $x = (\varepsilon y_n)$ where $y_n \in A$, $n \in \mathbb{N}$ so $x = \varepsilon(y_n)$. Since x is Cauchy in εA, $y = (y_n)$ is Cauchy in A so $x = \varepsilon y \to \varepsilon y_0$, $y_0 \varepsilon A$. Hence εA is requentially complete in X and also sequentially complete in $(Z, T|_Z)$ so εA is sequentially closed in $(Z, T|_Z)$. Thus σp is F linked to $T|_Z$ so we have the claim.

Since $\sigma p \supseteq T|_Z$ and σp is F linked to $T|_Z$, by Theorem 3.61, A is σp-sequentially complete. Thus $(Z, \sigma p)$ is a normed space with a sequentially complete neighberhood of 0. We must show that (Z, p) is sequentially complete. Let $x = (x_n)$ be a Cauchy sequence in Z. Then x is bounded and $x \in \varepsilon A$ for some $\varepsilon > 0$. Hence $x \to x_0$ for some $x_0 \in Z$ so (Z, p) is sequentially complete therefore (Z, p) is complete. As a result (Z, p) is a Banach space. #

Quotients

Theorem 3.64 Let X be a TVS($\mathbb{H}$), Y a vector space over $\mathbb{H}$, and $f: X \to Y$ a linear onto map. Let $\beta = \{ f(U) \mid U$ a balanced neighberhood of 0 in X $\}$. Then β is an additive filterbase of balanced absorbing sets.

Proof : We must show that β is a filterbase on Y. Since $U \neq \emptyset$ for all $U \in N(X)$, $f(U) \neq \emptyset$ for all $U \in N(X)$; hence $\emptyset \notin \beta$. Let U, V $\in N(X)$ be balanced. Let $W = U \cap V$. Then W is a balanced neighberhood of 0 and $f(W) = f(U \cap V) \subseteq f(U) \cap f(V)$. Hence β is a filterbase on Y. Let $U \in N(X)$. Since N(X) is additive, there exists a $V \in N(X)$ such that $V + V \subseteq U$. Hence $f(V) + f(V) = f(V+V) \subseteq f(U)$ so β is an additive filter base on Y. Let $U \in N(X)$. We must show that f(U) is balanced and absorbing. Let $t \in \mathbb{H}$ be such that $|t| \leq 1$. Since U is balanced, $t\, f(U) = f(tU) \subseteq f(U)$. To show that f(U) is absorbing, let $y \in Y$ be arbitrary. Then $y = f(x)$ for some $x \in X$. Since U is absorbing, there exists an $\varepsilon > 0$ such that $tx \in U$ for $|t| < \varepsilon$. Hence $ty = tf(x) = f(tx) \in f(U)$ so f(U) is absorbing. Hence β is an additive filter base of balanced absorbing sets. #

Definition 3.65 Let X be a TVS($\mathbb{H}$), Y a vector space over $\mathbb{H}$ and $f : X \to Y$ a linear onto map. The quotient topology Q_f is the vector topology generated by β defined in Theorem 3.64

By Theorem 3.64, β is an additve filter base of balanced and absorbing sets, by Theorem 3.26, X has a unique vector topology such that β is a local base of neighberhoods of 0.

Theorem 3.66 Let X be a TVS($\mathbb{H}$) and (Y, Q_f) be the quotient of X with respect to f where $f : X \to (Y, Q_f)$ is a linear onto map. Then f is continuous and open. Moreover, Qf is the only topology which makes f continuous and open.

Proof : Let $V \in N(Y, Qf)$. Then $V \supseteq f(U)$ for some $U \in N(X)$ so $f^{-1}(V) \supseteq U$. Hence f is continuous at 0. Let $x \in X$ and let $(x_\delta)_{\delta \in D}$ be anet in X such that $x_\delta \to x$. Then $x_\delta - x \to 0$. Since f is continuous at 0, $f(x_\delta) - f(x) = f(x_\delta - x) \to f(0) = 0$. So $f(x_\delta) \to f(x)$. Thus f is continuous at x. But $x \in X$ was arbitrary, therefore f is continuous on X. Let $U \in N(X)$ be open and balanced. Then $f(U) \in (Y, Qf)$ is open so f is open. Let T be a topology for Y which makes f continuous and open. We must show that T = Qf. Let $U \in T$. Since $f : X \to (Y, T)$ is continuous, $f^{-1}(U)$ is open in X. Since $f : X \to (Y, Qf)$ is open, $f(f^{-1}(U)) \in Qf$. Since f is onto, $U = f(f^{-1}(U)) \in Qf$. Hence $T \subseteq Qf$. Similarly, we can show that $Qf \subseteq T$; hence T = Qf. Thus Qf is the only topology which makes f continuous and open. #

Remark 3.67 The proof of theorem 3.64 shows that Qf is the largest topology making f continuous and the smallest topology making f open.

Definition 3.68 Let X, Y be TVS(IH)'s. A linear onto map $q : X \to Y$ is said to be a quotient map if and only if Y has the quotient topology with respect to q. If Y has the quotient topology induced by some map, say q, we call Y a quotient of X with respect to q.

Theorem 3.69 Let X be a TVS(IH) and let Y be a quotient of X with respect to the quotient map $q : X \to Y$ and let $f : Y \to Z$ be a linear map, Z a TVS(IH). Then $f : Y \to Z$ is continuous if and only if

$f \circ q : x \to z$ is continous.

Proof : ($\Rightarrow$) This statement is obvious. ($\Leftarrow$) Let $U \in N(Z)$. Since $f \circ q$ is continuous , there exists an open set $V \in N(X)$ such that $f(q(V)) \subseteq U$. Since q is open, $g(V) \in N(Y)$; hence f is continuous at 0 so f is continuous on X. #

Remark 3.70 Let X be a TVS($\mathbb{H}$) and S a vector subspace. Let $Y = \{ x+S | x \in X\}$. Define the addition and scalar multiplication as follow :

1. $(x+S) + (x'+S) = (x+x') + S$, for all $x, x' \in X$.
2. $t\cdot(x+S) = tx+S$, for $x \in X$ and $t \in \mathbb{H}$.

Then (Y, +,.) is a vector space over $\mathbb{H}$. Note that S is the zero of Y. Define $q : X \to Y$ by $q(x) = x+S$, for all $x \in X$. Then (Y, Qq) is the quotient of X by S, denoted by $X/_S$. Coversely, if $q : X \to Y$ is a quotient map, let S = ker q. Clearly, S is a vector subspace of X. Define $q : Y \to X/S$ as follows : Let $y \in Y$. Then $y = q(x)$ for some $x \in X$. Define $g(y) = x+S$. We will show g is well - defined. Suppose there exists an $x' \in X$ such that $q(x') = y$. Then $x-x' \in \ker q = S$. Hence $x+S = x'+S$ so g is well - defined. Next, we will show that g is linear and one-to-one. Let $y_1, y_2 \in Y$. Then $y_1 = q(x_1), y_2 = q(x_2)$ for some $x_1, x_2 \in X$. Hence $g(y_1+ y_2) = g\ (q(x_1) + g(x_2)$ $= x_1+ x_2+ S = (x_1+ S) + (x_2+ S) = g(y_1) + q(y_2)$. Let $t \in \mathbb{H}$ and $y \in Y$. Then $y = q(x)$ for some $x \in X$ so $g(ty) = g(tq(x)) = g(q(tx)) = tx + S$ $= t(x+S) = tg(y)$. Hence g is linear. Let $y \in Y$ be such that $g(y) = 0$. Now $y = q(x)$ for some $x \in X$ so $x+S = g(y) = S$; hence $x \in S$ therefore $y = q(x) = 0$

so $\ker g = \{0\}$. Hence g is one-to-one. It is clear that g is onto. Let $Q : X \to X|_S$ be defined by $Q(x) = x + S$. Then $Q = g \circ q$ is continuous. By Theorem 3.62, g is continuous. Since $q = g^{-1} \circ Q$ is continuous, g^{-1} is continuous. So $g : Y \to X|_S$ is a linear homeomorphism. Thus there exists a subspace S of X such that $Y = X|_S$ up to linear homeomorphism.

Theorem 3.71 Let X be a TVS($\mathbb{H}$). If $q : X \to Y$ is a quotient map then Y is separated if and only if ker q is closed.

Proof : ($\Rightarrow$) Since Y is separated, $\{0\}$ is closed. Since q is continuous, $q^{-1}(\{0\})$ is closed in X.

($\Leftarrow$) Let $y \in Y \setminus \{0\}$. We must show that there exists a $W \in N(Y)$ such that $y \notin W$. Since q is surjective and $y \neq 0$, there exists an $x \in X \setminus \ker q$ such that $y = q(x)$. Since $x \notin \ker q$ which is closed, there exists an open set $U \in N(X)$ such that $(x - U) \cap \ker q = \emptyset$ so $y \notin q(U)$. Since q is open and U is open, $q(U) \in N(Y)$ is an open set so Y is separated. #

Theorem 3.72 Let $(X, \|.\|)$ be a paranormed space over $\mathbb{H}$. Then the quotient Y of X is a paranormed space over $\mathbb{H}$.

Proof : Let $q : X \to Y$ be the quotient map. For $y \in Y$, let $p(y) = \inf\{\|x\| \mid y = q(x), x \in X\}$. We will show that p is a paranorm on Y. Obviously, $p(0) = 0$ and $p(y) \geq 0$ for all $y \in Y$. Let $y, z \in Y$ be arbitrary. We must show that $p(y+z) \leq p(y) + p(z)$. Let $\varepsilon > 0$ be given Then there exist $w, x \in X$ such that $y = q(w)$, $\|w\| < p(y) + \varepsilon/_2$ and $z = q(x)$ is such that $\|x\| < p(z) + \varepsilon/_2$. Now $y + z = q(w) + q(x) = q(w + x)$ so $p(y+z) \leq \|w+x\| \leq \|w\| + \|x\| \leq p(y) + p(z) + \varepsilon$. But $\varepsilon > 0$ was arbitrary therefore $p(y+z) \leq p(y)+p(z)$. Let $y \in Y$. Then $p(-y) = \inf\{ \|x\| \mid -y = q(x)\} = \inf\{ \|-x\| \mid y = q(-x)\} = \inf\{ \|m\| \mid y = q(m)\} = p(y)$. Define $d : Y \times Y \to \mathbb{R}$ by $d(y, z) = p(y-z)$ for all $y,z \in Y$. Then d is a pseudometric on Y. We shall now show that d induces the quotient topology ; hence the scalar multiplication and addition are continuous i.e. $t_n \to t$ and $p(y_n - y) \to 0$ implies that $p(t_n y_n - ty) \to 0$ as $n \to \infty$.

Claim that $q : X \to (Y, d)$ is continuous on X. Let $a \in X$ and let x be a net in X such that $x \to a$. Then $d(q(x), q(a)) = p(q(x-a)) \leq \|x-a\|$. But $\|x-a\| \to 0$; hence $q(x) \to q(a)$ so we have the claim. Next, we will show that $q : X \to (Y, d)$ is open. Let G be an open set is X. We must show that $q(G)$ is open in Y. Let $y \in q(G)$. Then $y = q(b)$ for some $b \in G$. Since G is open and $G \ni b$, there exists a $\delta > 0$ such that $\|x-a\| < \delta$ implies that $x \in G$. Let $z \in Y$ be such that $p(z-y) < \delta/_2$. Let $w \in X$ be such that $z-y = q(w)$ and $\|w\| < p(z-y) + \delta/_2$. Then $\|w+b-b\| = \|w\| < p(z-y)+\delta/_2 < \delta/_2 + \delta/_2 = \delta$; so $w+b \in G$. Now $z = q(w) + y = q(w+b) \in q(G)$ so $p(z-y) < \delta/_2$ which implies that $z \in q(G)$. Hence $y \in \operatorname{Int} q(G)$ therefore $q(G)$ is open so q is open. By Theorem 3.66, d induces the quotient topology. #

<u>Theorem 3.73</u> Let $(X, \|.\|)$ be a seminormed space over $\mathbb{H}$. Then the quotient of X is also a seminormed space over $\mathbb{H}$.

<u>Proof</u> : Let Y be the quotient of X with respect to the quotient map q. For $y \in Y$, let $p(y) = \inf\{\|x\| \mid y = q(x), x \in X\}$. We have shown that p is paranorm on Y in the proof of Theorem 3.72. Hence we must show that $p(ty) = |t|p(y)$ for all $t \in \mathbb{H}$ and $y \in Y$.

Case 1 $t = 0$, in this case the result is obvious.

Case 2 $t \neq 0$. $p(ty) = \inf \{ \|x\| \mid ty = q(x),\ x \in X \}$

$$= \inf \{ \|x\| \mid y = q(x/t),\ x \in X \}$$

$$= \inf \{ |t| \| \tfrac{x}{t} \| \mid y = q(x/t),\ x \in X \}$$

$$= |t| \inf \{ \| x/t \| \mid y = q(x/t),\ x \in X \}$$

$$= |t|\, p(y).$$

Thus p is a seminorm on Y so (Y, p) is a seminormed space over $\mathbb{H}$. The proof that the topology coming the seminorm is the quotient topology is the same as the proof given in Theorem 3.72. #

Remark 3.74 From Theorems 3.71 and 3.73, it follows that X/S is a normed space over $\mathbb{H}$ if and only if X is a seminormed space over $\mathbb{H}$ and S is a closed subspace.

Finite dimensional spaces over $\mathbb{H}$

Theorem 3.75 Let X be an n-dimensional separated TVS($\mathbb{H}$), $n < \infty$. Then X is linearly homeomorphic with $\mathbb{H}^n$.

Proof : We shall prove it by induction on n.

Case 1 $n = 0$. Then $X = \{0\}$, so the result is true.

Case 2 $n \in \mathbb{N}$. Claim that every linear functional on X is continuous. Let f be a linear functional on X. Since $n = \dim X = \dim(\mathrm{Im} f) + \dim \ker f = 1 + \dim \ker f$, $\dim \ker f = n-1$. Hence ker f is an $(n-1)$ dimensional subspace of X. By the induction hypothesis, ker f is linearly homeomorphic to $\mathbb{H}^{n-1}$. Since $\mathbb{H}^{n-1}$ is complete, ker f is complete ; hence ker f is closed. By Theorem 3.41, f is continuous so we have the claim. Let $\{b_1, b_2, \ldots, b_n\}$ be a basis of X.

Define $u : \mathbb{H}^n \to X$ by $u(a) = \sum_{i=1}^{n} a_i b_i$ where $a = (a_1, a_2, \ldots, a_n) \in \mathbb{H}^n$. It is clear that u is a linear bijection. For $i \in \{1,2,\ldots,n\}$, let $P_i(a) = a_i$. and $u_i(a) = a_i b_i$ where $a = (a_1, a_2, \ldots, a_n) \in \mathbb{H}^n$. Then $u_i = P_i b_i$ for all $i \in \{1,2,\ldots,n\}$. Since P_i is continuous for all i $b_i = 0$ for all i and X is separated, u_i is continuous for each i so $u = \sum_{i=1}^{n} u_i$ is also continuous. Claim that $P_i \circ U^{-1} \in X.^{\#}$ Let $x, y \in X$. Since u is surjective, there exist $a, b \in \mathbb{H}^n$ such $x = u(a)$ and $y = u(b)$ so $x + y = u(a+b)$ therefore $P_i \circ u^{-1}(x+y) = P_i(u^{-1}(x+y)) = P_i(a+b)$ $= a_i + b_i = P_i(u^{-1}(x)) + P_i(u^{-1}(y)) = P_i \circ u^{-1}(x) + P_i \circ u^{-1}(y)$. Let $\alpha \in \mathbb{H}$ and $x \in X$. Then $x = u(a)$ for some $a \in \mathbb{H}^n$ so $u(\alpha a) = \alpha u(a) = \alpha x$. Hence $P_i \circ u^{-1}(\alpha x) = P_i(u^{-1}(\alpha x)) = P_i(\alpha a) = \alpha a_i = \alpha P_i(a) = \alpha P_i(u^{-1}(x))$ $= \alpha P_i \circ u^{-1}(x)$ so $P_i \circ u^{-1}$ is a linear functional on X therefore we have the claim. By the claim, $P_i \circ u^{-1}$ is continuous on X. By Theorem 3.69, u^{-1} is continuous so u is a linear homeomorphism from $\mathbb{H}^n$ onto X. #

<u>Theorem 3.76</u> The sum of a closed and a finite-dimensional subspace of a TVS($\mathbb{H}$) must be closed.

<u>Proof</u> : Let X be a TVS($\mathbb{H}$). Let A be closed and B a finite-dimensional subspace of X. We must show that $A + B$ is closed in X. Let $q : X \to X/_A$ be the quotient map. Then $\ker q = A$ is closed. By Theorem 3.71, $X/_A$ is separated. We must show that q(B) is closed. Since B is finite dimensional, by Theorem 3.75, B is linearly homeomorhic to $\mathbb{H}^n$ for some $n \in \mathbb{N}$ so B is complete. Let $y \in \overline{q(B)}$.

Then there exists a net $(b_\delta)_{\delta \in D}$ in B such that $q(b_\delta) \to y$. Since $X/_A$ is separated, the limit is unique. Hence $b_\delta \to b$ for some $b \in B$ so $y = q(b) \in q(B)$ is closed. Since q is continuous, $q^{-1}(q(B))$ is closed. Claim that $A + B = q^{-1}(q(B))$. Let $z \in A + B$. Then $z = a + b$ for some $a \in A$ and $b \in B$ so $q(z) = q(a+b) = q(a) + q(b) = q(b)$. So $z \in q^{-1}(q(b)) \subseteq q^{-1}(q(B))$. Conversely, let $z \in q^{-1}(q(B))$. Then $q(z) \in q(B)$. Hence $q(z) = q(b)$ for some $b \in B$ therefore $q(z) = q(a) + q(b)$ for some $a \in A$ so $q(z) = q(a+b)$. Hence $z \in A + B$ so we have the claim. Since $q^{-1}(q(B))$ is closed, $A + B$ is closed in X . #

<u>Lemma 3.77</u> Let X and Y be TVS($\mathbb{H}$)'s. Let $f : X \to Y$ be a linear map. If f takes some neighberhood U of 0 into a bounded set then f is continuous. If Y is locally bounded and f is continuous then f takes some neighberhood of 0 into a bounded set.

<u>Proof</u> : Suppose that f(U) is bounded for some $U \in N(X)$. We must show that f is continuous. Let $V \in N(Y)$. Since f(U) is bounded, there exists a $t \neq 0$ such that $tf(U) \subseteq V$ therefore $f^{-1}(V) \supseteq f^{-1}(f(tU)) \supseteq tU$. Since $tU \in N(X)$, $f^{-1}(V) \in N(X)$; hence f is continuous at 0. Since f is linear and continuous at 0, f is continuous everywhere. Let Y be a locally bounded set. Let $V \in N(Y)$ be a bounded set. We must show that $f(f^{-1}(V))$ is bounded and $f^{-1}(V) \in N(X)$. Since f is continuous and $V \in N(Y)$, $f^{-1}(V) \in N(X)$. Let $W \in N(Y)$. Since V is bounded, there exists an $\varepsilon > 0$ such that $tV \subseteq W$ wherever $|t| < \varepsilon$. Let $t \in \mathbb{H}$ be such that $|t| < \varepsilon$. Then $tf(f^{-1}(V)) \subseteq tV \subseteq W$ therefore $f(f^{-1}(V))$ is bounded and $f^{-1}(V) \in N(X)$. #

Definition 3.78 Let X be a TVS($\mathbb{H}$). $S \subseteq X$ is called totally bounded (or precompact) if for each neighborhood U of 0, there is a finite set F such that $S \subseteq F + U$.

Lemma 3.79 Let X be a TVS($\mathbb{H}$) and $S \subseteq X$. Then the following hold.

(a) If S is compact then S is totally bounded.

(b) If S is totally bounded then S is bounded.

Proof : (a) Suppose that S is compact. Let U be an open neighborhood of 0. Then $\{ s + U | s \in S\}$ is an open cover of S. Since S is compact, there exist $s_1, s_2, \ldots, s_n \in S$ such that $S \subseteq \bigcup_{i=1}^{n} \{s_i + U\}$ i.e. $S \subseteq \bigcup_{i=1}^{n} \{s_i\} + U$. Let $F = \{s_1, s_2, \ldots, s_n\}$. Then $S \subseteq F + U$ so S is totally bounded.

(b) Suppose that S is totally bounded. Let U be a balanced neighborhood of 0. Let V be a balanced neighborhood of 0 such that $V + V \subseteq U$. Since S is totally bounded, there exists a finite $F \subseteq X$ such that $S \subseteq F + V$. Since F is bounded, there exists an $n_0 \geq 1$ such that $F \subseteq n_0 V$ so $S \subseteq F + V \subseteq n_0 V + V \subseteq n_0 V + n_0 V \subseteq n_0 (V + V) \subseteq n_0 U$. Let $t \in \mathbb{H}$ be such that $|t| < \frac{1}{n_0}$. Then $tS \subseteq t(n_0 U) \subseteq \frac{1}{n_0}(n_0 U) = U$ so S is bounded. #

Theorem 3.80 Let X be a separated TVS($\mathbb{H}$) which has a totally bounded neighborhood U of 0. Then X is finite dimensional.

Proof : Let $U \in N(X)$ be a totally bounded set. Then there exists a finite set $F' \subseteq X$ such that $U \subseteq F' + \frac{1}{2} U$. Let $F = \langle F' \rangle$. Then F is a finite dimensional subspace of X so $U \subseteq F' + \frac{1}{2}U \subseteq F + \frac{1}{2}U \subseteq F + \frac{1}{2}(F + \frac{1}{2}U) = \frac{3}{2}F + \frac{1}{4}U \subseteq F + \frac{1}{4}U$.

Continuing in this way we get that $U \subseteq F + 2^{-n}U$ for all $n \in \mathbb{N}$. Set $B = \{2^{-n}U \mid n \in \mathbb{N}\}$. Let $V \in N(X)$. Since U is totally bounded, by Lemma 3.79, U is bounded ; hence there exists an $\varepsilon > 0$ such that $tU \subseteq V$ whenever $|t| < \varepsilon$. Choose $n' \in \mathbb{N}$ such that $2^{-n'} < \varepsilon$. Then B is a local base of neighberhoods of 0. Claim that $\bar{F} = \bigcap\{F + W \mid W \in B\}$. Let $f \in \bar{F}$ and let $W \in B$. Choose a balanced set $U' \in N(X)$ such that $U' \subseteq W$ then $f + U'$ is a neighberhood of f so $(f + U') \cap F \neq \emptyset$. Now $f \in F - U' = F + U'$ so $f \in F + U' \subseteq F + W$ therefore $f \in \bigcap\{F + W \mid W \in B\}$. Conversely, let $f \notin \bar{F}$. Then there exists a $U'' \in N(X)$ such that $(f + U'') \cap F = \emptyset$. Since B is alocal neighberhood base of 0, $U'' \supseteq 2^{-n}W$ for some $n \in \mathbb{N}$ therefore $(f + 2^{-n}W) \cap F \subseteq (f + U'') \cap F = \emptyset$; hence $(f + 2^{-n}W) \cap F = \emptyset$. Hence $f \notin F + 2^{-n}W$ so $f \notin \bigcap\{F + W \mid W \in B\}$ thus we have the claim. Hence $U \subseteq \bigcap_{n \in \mathbb{N}}\{F + 2^{-n}W\} = \bar{F} = F$. Since F is a finite dimensional subspace of X and $F \supseteq U$ for all $U \in N(X)$, $F = X$. #

<u>Theorem 3.81</u> Let X, Y be TVS($\mathbb{H}$)'s and let $S \subseteq X$ be a totally bounded set. Let $f : X \to Y$ be a linear continuous mapping. Then $f(S)$ is totally bounded.

<u>Proof</u> : Let $U \in N(Y)$. Since f is continuous, $f^{-1}(U) \in N(X)$ so $S \subseteq F + f^{-1}(U)$ for some finite set F. Hence $f(S) \subseteq f(F + f^{-1}(U)) \subseteq f(F) + f(f^{-1}(U)) \subseteq f(F) + U$. Since F is finite, $f(F)$ is finite therefore $f(S)$ is totally bounded. #

<u>Definition 3.82</u> Let X be a TVS($\mathbb{H}$), $S \subseteq X$, $A \subseteq X$. We say that S is <u>small of order A</u> if snd only if $S - S \subseteq A$ where $S - S = \{s - t \mid s, t \in S\}$.

<u>Theorem 3.83</u> Let X be a TVS($\mathbb{H}$). Then $S \subseteq X$ is totally bounded if and only if for each $U \in N(X)$, S is a finite union of sets which are small of order U.

<u>Proof</u> : ($\Longrightarrow$) Let $U \in N(X)$. Choose $V \in N(X)$ such that $V - V \subseteq U$. Since S is totally bounded, there exists a finite set $F = \{f_1, f_2, \ldots, f_n\}$ $\subseteq F + V = \bigcup_{k=1}^{n} \{f_k + V\}$. Let $k \in \{1,2,\ldots,n\}$. Then $(f_k + V) - (f_k + V)$ $= V - V \subseteq U$ so $f_k + V$ is small of order U for each k. Let $S(f_k) =$ $(f_k + V) \cap S$. Then $S = \bigcup_{k=1}^{n} S(f_k)$ and $S(f_k) \subseteq f_k + V$ for each $k \in \{1,2,\ldots,n\}$ so S is a finite union of sets which are small of order U.

($\Longleftarrow$) Let $U \in N(X)$. By assumption, there exist $S_1, S_2, \ldots, S_n \subseteq X$ which are small of order U such that $S = \bigcup_{k=1}^{n} S_k$. Suppose that $S_k \neq \emptyset$ for all $k \in \{1,2,\ldots,n\}$. For each $k \in \{1,2,\ldots,n\}$, let $x_k \in S_k$ and $F = \{x_1, x_2, \ldots, x_n\}$. Then $S_k - x_k \subseteq S_k - S_k \subseteq U$ for each k ; hence $S_k \subseteq x_k + U$ for all $k \in \{1,2,\ldots,n\}$. As a result, $S = \bigcup_{k=1}^{n} S_k \subseteq \bigcup_{k=1}^{n} \{x_k + U\} \subseteq F + U$ so S is totally bounced. #

<u>Theorem 3.84</u> Let X be a vector space over $\mathbb{H}$, Y a TVS($\mathbb{H}$), $f : X \to Y$ a linear map and $S \subseteq X$. Then S is σf - totally bounded if and only if f(S) is toyally bounded.

<u>Proof</u> : ($\Longrightarrow$) We consider σf to be the smallest topology on X making f continuous. Since S is σf totally bounded, by Theorem 3.81, f(S) is totally bounded.

($\Leftarrow$) Let $U \in N(X, \sigma f)$. Then there exists a $V \in N(Y)$ such that $U \supseteq f(V)$. Since $f(S)$ is totally bounded, there exist $A_1, A_2, \ldots, A_n \subseteq Y$ such that A_k is small of order V for each $k \in \{1,2,\ldots,n\}$ and $f(S) = \bigcup_{k=1}^{n} A_k$. Assume that $A_k \neq \emptyset$ for each k. Set $S_k = f^{-1}(A_k)$, $k = 1,2,\ldots,n$. Claim that S_k is small of order U for each k. Let $k \in \{1,2,\ldots,n\}$. Let $z \in S_k - S_k$. Then $z = x - y$ for some $x, y \in S_k$ so $f(z) = f(x-y) = f(x) - f(y) \in A_k - A_k \subseteq V$; hence $z = x-y \in f^{-1}(V) \subseteq U$ thus we have the claim. We must show that $S \subseteq \bigcup_{k=1}^{n} S_k$. Let $s \in S$. Then $f(s) \in A_{k'}$ for some $k' \in \{1,2,\ldots,n\}$ so $s \in f^{-1}(A_{k'}) = S_{k'} \subseteq \bigcup_{k=1}^{n} S_k$. Hence $S \subseteq \bigcup_{k=1}^{n} S_k$. For each k, let $S_k' = S_k \cap S$. Let $m \in S$. Then $m \in S_{k''}$ for some $k'' \in \{1,2,\ldots,n\}$ so $m \in S_{k''} \cap S \subseteq \bigcup_{k=1}^{n} (S_k \cap S) = \bigcup_{k=1}^{n} S_k'$. Hence $S \subseteq \bigcup_{k=1}^{n} S_k'$ so $S = \bigcup_{k=1}^{n} S_k'$. Since S_k' is small of order U for each k and $S_k' \subseteq S_k$, S_k is small of order U. By Theorem 3.83, S is σf - totally bounded. #

<u>Theorem 3.85</u> Let Φ be a collection of vector topologies an a vector space X over $\mathbb{H}$. Then $S \subseteq X$ is $v\Phi$ totally bounded if and only if S is T - totally bounded for each $T \in \Phi$.

<u>Proof</u> : ($\Rightarrow$) Suppose that S is $v\Phi$ totally bounded. We must show that S is T - totally bounded for each $T \in \Phi$. Let $T \in \Phi$. Let $i : (X, v\Phi) \to (X, T)$ be the inclusion map. Since $T \subseteq v\Phi$, i is con continuous and linear. By Therem 3.81, $i(S) = S$ is T - totally bounded.

($\Leftarrow$) Suppose S is T - totally bounded for each $T \in \Phi$. We must show that S is $v\Phi$ - totally bounded. Let $U \in N(X, v\Phi)$. There exist $T_1, T_2, \ldots, T_n \in \Phi$ and $V_j \in N(X, T_j)$, $j = 1,2,\ldots,n$ such that $U \supseteq \bigcap_{j=1}^{n} V_j$. For each $j = 1,2,\ldots,n$, let $S = \bigcup\{ S_{ij} \mid i = 1,2,\ldots,m_j\}$ where each S_{ij} is small of order V_j. Choose $i(j) \in \mathbb{N}$ such that $1 \leq i(j) \leq m_j$ and let $A_{i(j)} = \bigcap\{ S_{i(j)j} \mid j = 1,2,\ldots,n \}$. Claim that $A_{i(j)}$ is small of order U. Let $z \in A_{i(j)} - A_{i(j)}$. Then $z = x' - y'$ for some $x', y' \in A_{i(j)}$. Since $x', y' \in A_{i(j)}$, $x', y' \in S_{i(j)j}$ for each j ; hence $z = x' - y' \in S_{i(j)j} - S_{i(j)j} \subseteq V_j$ for each j. Thus $z = x' - y' \in \bigcap_{j=1}^{n} V_j \subseteq U$ so we have the claim. Let $x \in S$. Then $x \in \cup\{S_{ij} \mid i = 1,2,\ldots,m_j\}$, $j = 1,2,\ldots,n$ so $x \in S_{i_o j}$ for some $i_o \in \{1,2,\ldots,m_j\}$ thus $S \subseteq \bigcup_{j=1}^{n} A_{i(j)}$. Let $A'_{i(j)} = A_{i(j)} \cap S$. Then $S = \bigcup A'_{i(j)}$. Since $A'_{i(j)}$ is small of order U for each j and $A'_{i(j)} \subseteq A_{i(j)}$, $A'_{i(j)}$ is small of order U and $S = \bigcup_{j=1}^{n} A'_{i(j)}$ so S is $v\Phi$ - totally bounded. #

<u>Corollary 3.86</u> Let X be a vector space over $\mathbb{H}$ and F a set of linear maps $\{ f_\alpha \mid X \to Y_{f_\alpha}, \alpha \in I \}$ where each Y_{f_α} is a TVS($\mathbb{H}$) and each $\alpha \in I$ an index set. Then $S \subseteq (X, \sigma F)$ is totally bounded if and only if $f_\alpha(S)$ is totally bounced for each, $\alpha \in I$.

<u>Proof</u> : Since $S \subseteq (X, v\Phi)$ where $\Phi = \{\sigma f_\alpha \mid \alpha \in I\}$ is totally bounded, S is σf_α- totally bounded by Theorem 3.85. Hence, by Theorem 3.84, $f_\alpha(S)$ is totally bounded for each $\alpha \in I$.

($\Leftarrow$) Suppose that $f_\alpha(S)$ is totally bounded for each $\alpha \in I$. By Theorem 3.84, S is σf_α - totally bounded for each $\alpha \in I$. By Theorem 3.85, S is $v\Phi$ - totally bounded where $\Phi = \{\sigma f_\alpha | \alpha \in I\}$. #

<u>Corollary 3.87</u> Let $(X_\alpha)_{\alpha \in I}$ be a collection of TVS($\mathbb{H}$)'s, $S \subseteq \Pi X_\alpha$. Then S is totally bounded if and only if each of its projections is totally bounded. In particular, if $S_\alpha \subseteq X_\alpha$ is totally bounded for each $\alpha \in I$, ΠS_α is totally bounded in ΠX_α.

Proof : ($\Rightarrow$) Suppose that S is totally bounded. For each $\alpha \in I$, let $P_\alpha : \Pi X_\beta \to X_\alpha$ be the projection map. Since P_α is linear and continuous and $S \subseteq (\Pi X_\beta, \sigma F)$ where $F = \{P_\alpha | \alpha \in I\}$ is totally bounded, by Corollary3.86, $P_\alpha(S)$ is totally bounded for each $\alpha \in I$.

($\Leftarrow$) Suppose that $P_\alpha(S)$ is totally bounded for each projection map $P_\alpha : \Pi X_\beta \to X_\alpha$, $\alpha \in I$. By Corollary 3.86, $S \subseteq (\Pi X_\alpha, \sigma F)$ where $F = \{P_\alpha | \alpha \in I\}$ is totally bounded. For the rest of the proof, suppose that $S_\alpha \subseteq X_\alpha$ is totally bounded for each $\alpha \in I$. For each $\alpha \in I$, let $P_\alpha : \Pi X_\beta \to X_\alpha$ be the projection map. Then $P_\alpha(\Pi S_\beta) = S_\alpha$, for each α. Since S_α is totally bounded for each α, by the previous result, ΠS_β is totally bounded in $(\Pi X_\beta, \sigma F)$ where $F = \{P_\alpha | \alpha \in I\}$. #

<u>Compact Sets</u>

<u>Definition 3.88</u> Let X be a TVS($\mathbb{H}$). A filterbase F on X is called <u>Cauchy</u> if and only if for each $U \in N(X)$, there exists an $S \in F$ such that $S - S \subseteq U$. F is said to converge to x, denoted by $F \to x$ if and only if $F' \supseteq N_x$ where F' is the filter generated by F and N_x is the set of all neighborhoods of x.

Remark 3.89 $F \to x$ if and only if for all $U \in N(X)$, $x+U \supseteq A$ for some $A \in F$.

Lemma 3.90   Let F be a Cauchy filter base on X, a TVS($\mathbb{H}$), suppose $x \in \bar{A}$ for each $A \in F$. Then $F \to x$.

Proof : Let $U \in N(X)$. Choose a closed set $V \in N(X)$ such that $V \subseteq U$. Since F is Cauchy, there exists an $A \in F$ such that $A - A \subseteq V$. Then $A - x \subseteq \bar{A} - \bar{A} \subseteq \overline{A-A} \subseteq V \subseteq U$; hence $A \subseteq x+U$. By Remark 3.89, $F \to x$. #

Lemma 3.91   Let X be a TVS($\mathbb{H}$) and $S \subseteq X$ a complete subset. Then each Cauchy filterbase F on S converges to a point in S.

Proof : Let F be a Cauchy filter base on S. Let $A \in F$. Choose $x_A \in A$. Set $x = \{x_A | A \in F\}$. Then x is a net in S where F is directed by reverse inclusion ; that is, $A \geq B$ if and only if $A \subseteq B$ for all $A,B \in F$. We must show that x is Cauchy in S. Let $U \in N(X)$. Since F is a Cauchy filterbase on S, there exists a $B \in F$ such that $B - B \subseteq U \cap S \subseteq U$. Let $A', A'' \in F$ be such that $A' \geq B$ and $A'' \geq B$. Then $A' \subseteq B$ and $A'' \subseteq B$; so $x_{A'} - x_{A''} \in A' - A'' \subseteq B - B \subseteq U$. Since S is complete and x is Cauchy in S, $x \to s$ for some $s \in S$. Claim that $F \to s$. By Lemma 3.90, it suffices to show that $s \in \bar{A}$ for each $A \in F$. Let $A \in F$ and $U \in N(X)$. Since $x \to s$, there exists a $B \in F$ such that $B' \subseteq B$ implies that $x_{B'} \subseteq s+U$. Since F is a filter base, there exists a $C \in F$ such that $C \subseteq A \cap B$; hence $x_C \in s+U$. Since $x_C \in C \subseteq A$, $A \cap (s+U) \neq \emptyset$ so $s \in \bar{A}$. #

<u>Definition 3.92</u> F is an ultrafilter if and only if F is a maximal filter ; that is whenever F' is a filter with $F' \supseteq F$, $F' = F$.

<u>Lemma 3.93</u> Let S be a set and F an ultrafilter on S. Let $A \subseteq S$. Then either $A \in F$ or $S \setminus A \in F$.

<u>Proof</u> : The proof is standard. #

<u>Lemma 3.94</u> Let S be a set and F an ultrafilter on S. Suppose that $S = \bigcup_{k=1}^{n} S_k$ for some $n \in \mathbb{N}$. Then $S_k \in F$ for some $k \in \{1,2,\ldots,n\}$.

<u>Proof</u> : The proof is standard. #

<u>Lemma 3.95</u> Let X be a TVS($\mathbb{H}$) , S a totally bounded set in X and F an ultrafilter on S, Then F is Cauchy.

<u>Proof</u> : The proof is standard. #

<u>Lemma 3.96</u> Let S be a set and B a collection of subsets of S with the finite intersection property ; that is for any finite subset B' of B, say $B' = \{B_1, B_2, \ldots, B_n\}$, $\bigcap_{k=1}^{n} B_k \neq \emptyset$. Then there exists an ultrafilter F' on S such that $B \subseteq F'$.

<u>Proof</u> : Let P = {filter F on S such that $F \supseteq B$ }. Let B'' = the set of finite intersections of elements in B. We must show that B'' is a filterbase on S. Since B has the finite intersection property, $\emptyset \notin B''$. Let $U, V \in B''$. Then $U = \bigcap_{k=1}^{m} C_k$ and $V = \bigcap_{j=1}^{n} D_j$ where $C_k, D_j \in B$ for $k = 1,2,\ldots,m$ and $j = 1,2,\ldots,n$. Then $U \cap V = \left(\bigcap_{k=1}^{m} C_k\right) \cap \left(\bigcap_{j=1}^{n} D_j\right)$ so $U \cap V \in B''$. Hence B'' is a filter base on S. Let B''' be the filter

generated by B'' ; that is $B''' = \{ C \subseteq S \mid$ there exists a $D \in B''$ such that $D \subseteq C \}$. Then $B''' \in P$. Hence $P \neq \emptyset$. Partially order P by set inclusion. Let $\{ C_\alpha \}_{\alpha \in I}$ be a chain in P. Let $C = \bigcup_{\alpha \in I} C_\alpha$. Then $C \in P$ and C is upperbound of the chain $\{ C_\alpha \}_{\alpha \in I}$. By Zorn's lemma, P contains a maximal element, say C'. Clearly C' is an ultrafilter and $B \subseteq C'$. #

<u>Theorem 3.97</u> Let X be a TVS($\mathbb{H}$). If K is a compact subset of X then K is complete.

<u>Proof</u> : Let $x = (x_\alpha)_{\alpha \in D}$ be a Cauchy net in K. For each $\alpha \in D$, let $T_\alpha = \{ x_\delta \mid \delta \geq \alpha \}$. Since $x_\alpha \in T_\alpha$ for each $\alpha \in D$, $T_\alpha \neq \emptyset$ for all α. Claim that for any finite subset J of D say $J = \{ \alpha_1, \alpha_2, \ldots, \alpha_n \}$, $\bigcap_{j=1}^{n} T_\alpha = \emptyset$. Since D is directed, reorder $\{ \alpha_1, \alpha_2, \ldots, \alpha_n \}$ to be $\{ \beta_1, \beta_2, \ldots, \beta_n \}$ where $\beta_1 \leq \beta_2 \leq \beta_3 \leq \ldots \leq \beta_n$. Then $\bigcap_{j=1}^{n} T_{\alpha_j} = \bigcap_{j=1}^{n} T_{\beta_j} = T_{\beta_n} \neq \emptyset$. So we have the claim. Next, we must show that $\bigcap_{\alpha \in D} \bar{T}_\alpha \neq \emptyset$. Suppose not. Then $K \subseteq \bigcup_{\alpha \in D} (\bar{T}_\alpha)^c = X$. Since K is compact, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in D$ such that $K \subseteq \bigcup_{j=1}^{n} (\bar{T}_{\alpha_j})^c$. Hence $K \subseteq \left(\bigcap_{j=1}^{n} \bar{T}_{\alpha_j} \right)^c$ so $K \cap \left(\bigcap_{j=1}^{n} \bar{T}_{\alpha_j} \right) = \emptyset$, a contradiction. Let $k \in \bigcap_{\alpha \in D} \bar{T}_\alpha$. We must show that $x \to k$. Let $U \in N(X)$ and let $V \in N(X)$ be such that $V + V \subseteq U$. Since x is Cauchy in K and therefore in X, there exists an $\alpha > 0$ such that for all $\delta \geq \alpha$, $\delta' \geq \alpha$ implies that $x_\delta - x_{\delta'} \in V$. Since $k \in \bar{T}_\alpha$, $(k + V) \cap T_\alpha \neq \emptyset$. Let $x_\delta \in (k + V) \cap T_\alpha$ so $\delta \geq \alpha$. Let $\delta' \in D$ be such that $\delta' \geq \alpha$. Then $x_{\delta'} - k = (x_{\delta'} - x_\delta) + (x_\delta - k) \in V + V \subseteq U$. Hence $x \to k$ so K is complete. #

<u>Theorem 3.98</u> Let X be a TVS($\mathbb{H}$). Then $K \subseteq X$ is compact if and only if K is totally bounded amd complete.

Proof : ($\Longrightarrow$) Suppose that K is compact. By Theorem 3.97, K is complete. By Lemma 3.80, K is totally bounded.

($\Longleftarrow$) Suppose that K is complete and totally bounded. We must show that K is compact. Suppose not. Then there exists an open cover $G = \{G_\alpha\}_{\alpha \in I}$ of K such that G has no finite subcover of K. Let B be any finite subcollection of G, say $B = \{G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}\}$.

Then $K \setminus \bigcup_{j=1}^{n} \{G_{\alpha_j}\} = \bigcap_{j=1}^{n} (K \setminus G_{\alpha_j}) \neq \emptyset$. Let $C = \{K \setminus G_\alpha | \alpha \in I\}$.

Then C is a collection of closed sets in K with the finite intersection property. By Lemma 3.96, there exists an ultrafilter F on K such that $F \supseteq C$. By Lemma 3.95, since K is totally bounded and F is an ultrafiller on K, F is Cauchy. Since K is complete, by Lemma 3.91, $F \to k'$ for some $k' \in K$. Claim that $k' \in \bigcap C$. Let $\alpha \in D$ and let $U \in N(X)$. Since F is an ultrafilter on X, $k' + U \supseteq A$ for some $A \in F$. Since $K \setminus G_\alpha \in F$, $(k' + U) \cap (K \setminus G_\alpha) \neq \emptyset$; hence $k' \in Cl_X(K \setminus G_\alpha)$. Then $k' \in Cl_X(K \setminus G_\alpha) \cap K = Cl\,(K \setminus G_\alpha) = K \setminus G_\alpha$ so $k' \in \bigcap C$ and we have the claim. Then $\bigcap_{\alpha \in D} \{K \setminus G_\alpha\} \neq \emptyset$ so $K \setminus G = K \setminus \bigcup_{\alpha \in I} G_\alpha = \bigcap_{\alpha \in I} (K \setminus G_\alpha) \neq \emptyset$ which contradicts the fact that G is an open cover of K so K is compact. #

<u>Definition 3.99</u> Let X be a TVS($\mathbb{H}$) and $S \subseteq X$. H is called the <u>balanced covex hull of S</u> if and only if H is the smallest balanced convex set containing S.

<u>Lemma 3.100</u> Lem X be a TVS($\mathbb{H}$) and let A, B be balanced convex compact sets. Then the balanced convex hull H of $A \cup B$ is compact.

<u>Proof</u> : Let $D = \{ (z, w) \in \mathbb{H}^2 \mid |z| + |w| \leq 1 \}$. Define $f : D \times A \times B \to X$ by $f(z, w, a, b) = za + wb$. Claim that $f(D \times A \times B) = H$ (the balanced convex hull of $A \cup B$). Let $h \in H$. Then

$h = \sum_{i=1}^{n} t_i x_i$ where $\sum_{i=1}^{n} |t_i| \leq 1$, $x_i \in A \cup B$, say $x_1, x_2, \ldots, x_k \in A$ and $x_{k+1}, x_{k+2}, \ldots, x_n \in B$. Let $z = \sum_{j=1}^{k} |t_j|$, $w = \sum_{i=k+1}^{n} |t_i|$,

$a = \sum_{i=1}^{k} (\frac{t_i}{z}) x_i$ and $b = \sum_{i=k+1}^{n} (\frac{t_i}{w}) x_i$. Hence $a = \sum_{i=1}^{k} (\frac{t_i}{z}) x_i \in A$ and $b = \sum_{i=k+1}^{n} (\frac{t_i}{w}) x_i \in B$. Then $f(z, w, a, b) = za + wb = z (\sum_{i=1}^{k} (\frac{t_i}{z}) x_i)$ $+ w (\sum_{i=k+1}^{n} (\frac{t_i}{w}) x_i) = \sum_{i=1}^{n} t_i x_i = h$ so $f(D \times A \times B) = H$. Clearly f is continuous. Since $D \times A \times B$ is compact, $f(D \times A \times B)$ is compact ; hence H is compact. #