CHAPTER II

THE GINZBURG-LANDAU THEORY

Derivation of the Ginzburg-Landau Equations

In 1950, that is, even before the microscopic (BCS) theory was developed, Ginzburg and Landau proposed a theory currently known as the Ginzburg-Landau theory (Ginzburg and Landau, 1950), which describes the properties of superconductors near T_C. The first assumption of the Ginzburg-Landau theory is that the behavior of the superconducting electrons may be described by an *effective wave function*, Ψ (Rose-Innes and Rhoderick, 1978), which has the significance that $|\Psi|^2$ is equal to the density of *superelectrons*. It is then assumed that the free energy of the superconducting state differs from the normal state by an amount which can be written as a power series in $|\Psi|^2$. We shall deal here with the simple derivation given in the fundamental work of Ginzburg and Landau (1950).

We start with the definition of the "effective wave function" Ψ, which is commonly called the *order parameter*. The order parameter is a complex quantity, and as we have mentioned above, near the critical temperature the order parameter Ψ(r) is small and the free energy density f may be expanded as a series in Ψ. In the absence of an external field in a bulk superconductor Ψ is independent of the coordinates. Since Ψ is a complex quantity and f is real, the expansion is carried out in powers of $|\Psi|^2$. Thus, we have

$$f_s = f_n + a\,|\Psi|^2 + \frac{b}{2}|\Psi|^4 + \ \dots \tag{2.1}$$

where f_n is the corresponding free energy density for the normal state. The total free energy is

$$F_s = \int d\mathbf{r} f_s(\mathbf{r})$$

We can perform a variational calculation with respect to Ψ^*, that is

$$\frac{\partial f_s}{\partial \Psi^*} = 0$$

and if we neglect terms in f_s of order $|\Psi|^6$ and higher,

$$a\Psi + b|\Psi|^2\Psi = 0 \qquad (2.2)$$

There are two solutions:

$$\Psi = 0 \qquad \text{(normal state)} \qquad (2.3)$$

$$|\Psi|^2 = -\frac{a}{b} \qquad \text{(superconducting state)} \qquad (2.4)$$

The first solution clearly represents the normal state because f_s then equals f_n. The second solution is acceptable only if a/b is negative; it represents the superconducting state with a corresponding free-energy density

$$f_s = f_n - \frac{1}{2}\frac{a^2}{b} \qquad (2.5)$$

It must be the case that $a < 0$ and $b > 0$, so that $f_s < f_n$ (we must have $|\Psi|^2 > 0$). Following Ginzburg and Landau, we assume that b is independent of temperature, while

$$a(T) = (T - T_c)a'$$

is negative for $T<T_c$, vanishing linearly at T_c.

Presence of Magnetic Fields

Suppose now that an external magnetic field is applied to the superconductor. In this case, both the field in the superconductor and Ψ depend on the coordinates (Abrikosov, 1988). For this case some flux may penetrate the sample. The energy of the field per unit volume is $h^2/8\pi$ and we can write the internal magnetic field $\boldsymbol{h}(\boldsymbol{r})$ in terms of an electromagnetic vector potential, $\boldsymbol{A}(\boldsymbol{r})$, as

$$\boldsymbol{h} = \nabla \times \boldsymbol{A} \tag{2.6}$$

In general, for a particle of charge q in a magnetic field the Hamiltonian operator (Landau and Lifshitz, 1977) is

$$\widehat{H} = \frac{1}{2m}\left(-i\hbar\nabla - \frac{q}{c}A\right)^2 \tag{2.7}$$

We will assume that the variation of Ψ^* in space occurs slowly. This permits one to consider only the fluctuation correction $|\nabla\Psi|^2$ to the free energy, ignoring higher-order derivatives. However, the order parameter, Ψ, can be interpreted as the wave function of the Cooper pairs (Cooper, 1956), so we set q = 2e in eq.(2.7), where $-e$ is the negative electronic charge (Zimmerman and Mercerau, 1965). This is also known as the effective charge, e*. A is the vector potential. We recall that the theory must be invariant under a gauge transformation of the vector potential of the form $A \rightarrow A + c\nabla\phi$, where c is a constant and ϕ is a function of position. In the expression $\left(-i\hbar\nabla + \frac{2e}{c}A\right)\Psi$, a change in A can be compensated for by a change in the phase of the function Ψ, so ϕ in the transformation above can be identified as that phase. We have taken into account that the pair has a charge $e^* = 2e$

In order to include a new term in the free energy, we write it as the kinetic energy of a particle of mass m* (m* = 2m)

$$\frac{1}{2m^*}\left|\left(-i\hbar\nabla + \frac{e^*A}{c}\right)\Psi\right|^2 \tag{2.8}$$

The free energy density of the entire superconductor is now

$$f_s = f_n + a\left|\Psi\right|^2 + \frac{b}{2}\left|\Psi\right|^4 + \frac{1}{2m^*}\left|\left(-i\hbar\nabla + \frac{e^*A}{c}\right)\Psi\right|^2 + \frac{h^2}{8\pi} \tag{2.9}$$

In an experimental situation we have an external magnetic field H which can be controlled. But the internal field $\boldsymbol{h}$ cannot be controlled. It is convenient to make a Legendre transformation(Duzer and Turner, 1981) to a new (Gibbs) free energy density given by

$$g_s = f_n - \frac{\boldsymbol{h}\cdot\boldsymbol{H}}{4\pi} + \frac{H^2}{8\pi} \tag{2.10}$$

So eq.(2.9) will be changed to a new form as

$$g_s = f_n + a\,|\Psi|^2 + \frac{b}{2}|\Psi|^4 + \frac{1}{2m^*}\left|\left(-i\hbar\nabla + \frac{e^*A}{c}\right)\Psi\right|^2 + \frac{(h-H)^2}{8\pi} \tag{2.11}$$

This free energy density is a function of the external magnetic field. We must now find the order parameter $\Psi(r)$ and the internal magnetic field $h(r)$ by the variational method.

<u>The First Ginzburg-Landau Equation</u> : *variation with respect to* Ψ^*

Eq.(2.11) of the previous section can be integrated over the volume of the sample to give the total Gibbs free energy. First consider the term

$$\frac{1}{2m^*}\int dr\left|\left(-i\hbar\nabla + \frac{e^*A}{c}\right)\Psi\right|^2 = \frac{1}{2m^*}\int dr\left[\hbar^2\nabla\Psi^*\cdot\nabla\Psi + i\hbar\frac{e^*A}{c}\cdot\Psi\nabla\Psi^* - i\hbar\frac{e^*A}{c}\cdot\Psi^*\nabla\Psi + \frac{e^{*2}A^2}{c^2}|\Psi|^2\right] \tag{2.12}$$

From the divergence theorem

$$\int_v dr\,\nabla\cdot P = \int_s P\cdot dS \tag{2.13}$$

and we can obtain that

$$\frac{1}{2m^*}\int \mathrm{dr}\left|\left(-i\hbar\nabla + \frac{e^*A}{c}\right)\Psi\right|^2 = \frac{1}{2m^*}\int \mathrm{dr}\,\Psi^*\left(-i\hbar\nabla + \frac{e^*A}{c}\right)^2\Psi$$

$$+ \frac{i\hbar}{2m^*}\int_S \Psi^*\left(-i\hbar\nabla + \frac{e^*A}{c}\right)\Psi \cdot \mathrm{d}\mathbf{S}$$

$$(2.14)$$

We choose the surface condition (Lifshitz and Piteavskii, 1980),

$$\hat{n}\cdot\left(-i\hbar\nabla + \frac{e^*}{c}A\right)\Psi = 0 \qquad (2.15)$$

for all points on the surface of the superconductor, where $\hat{n}$ is the normal vector. The free energy is then

$$G_s = \int \mathrm{dr}\; g_s$$

$$= \int \mathrm{dr}\left[f_n + a\,|\Psi|^2 + \frac{b}{2}|\Psi|^4 + \Psi^*\frac{1}{2m^*}\left(-i\hbar\nabla + \frac{e^*A}{c}\right)^2\Psi + \frac{(h-H)^2}{8\pi}\right]$$

Therefore, variation with respect to Ψ^* gives,

$$\frac{1}{2m^*}\left(-i\hbar\nabla + \frac{e^*A}{c}\right)^2\Psi + a\,\Psi + b|\Psi|^2\Psi = 0 \qquad (2.16)$$

which is the first of the Ginzburg-Landau equations.

<u>The Second Ginzburg-Landau Equation</u> : *variation with respect to* $\boldsymbol{A}$

Again we start with the equation (2.11), that is

$$g_s \;=\; f_n + a\,|\Psi|^2 + \frac{b}{2}|\Psi|^4 + \frac{1}{2m^*}\left|\left(-i\hbar\nabla + \frac{e^*\boldsymbol{A}}{c}\right)\Psi\right|^2 + \frac{(\boldsymbol{h}-\boldsymbol{H})^2}{8\pi} \tag{2.11}$$

and vary $G_s = \int g_s\,d\mathbf{r}$ with respect to $\boldsymbol{A}$ to get

$$\int d\mathbf{r}\left(\frac{\partial g_s}{\partial A_i} - \sum_j \frac{\partial}{\partial r_j}\left[\frac{\partial g_s}{\partial(\partial A_i/\partial r_j)}\right]\right)\delta A_i \;=\; 0 \tag{2.17}$$

Since the free energy density is a function of $\boldsymbol{A}$ and its spatial derivatives (remember that $\boldsymbol{h} = \nabla \times \boldsymbol{A}$), we require that (Arfken , 1985)

$$\frac{\partial g_s}{\partial A_i} - \sum_j \frac{\partial}{\partial r_j}\left[\frac{\partial g_s}{\partial(\partial A_i/\partial r_j)}\right] \;=\; 0 \tag{2.18}$$

at every point inside the superconductor. Firstly,

$$\frac{\partial g_s}{\partial A_i} \;=\; \frac{1}{2m^*}\left(i\hbar\frac{e^*}{c}\Psi\frac{\partial \Psi^*}{\partial r_i} - i\hbar\frac{e^*}{c}\Psi^*\frac{\partial \Psi}{\partial r_i} + \frac{2e^{*2}}{c^2}A_i|\Psi|^2 \right) \tag{2.19}$$

Secondly,

$$h \;=\; \nabla \times A \;=\; \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

so that

$$\frac{h^2}{8\pi} - \frac{h \cdot H}{4\pi} = \frac{1}{8\pi}\left[\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)^2 + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)^2 + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)^2\right]$$

$$- \frac{1}{4\pi}\left[\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)H_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)H_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)H_z\right]$$

(2.20)

We will now consider the term $\dfrac{\partial g_s}{\partial(\partial A_i/\partial r_j)}$; when i, j = 1, 2, 3 (or x, y, z).

Then for i = 1 (or x) we get

$$\frac{\partial g_s}{\partial(\partial A_x/\partial x)} = 0$$

$$\frac{\partial g_s}{\partial(\partial A_x/\partial y)} = -\frac{1}{4\pi}\left[\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - H_z\right] = -\frac{1}{4\pi}[\,h_z - H_z\,]$$

$$\frac{\partial g_s}{\partial(\partial A_x/\partial z)} = \frac{1}{4\pi}\left[\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) - H_y\right] = \frac{1}{4\pi}[\,h_y - H_y\,]$$

(2.21)

Now

$$\frac{\partial}{\partial x}\left[\frac{\partial g_s}{\partial(\partial A_i/\partial x)}\right] + \frac{\partial}{\partial y}\left[\frac{\partial g_s}{\partial(\partial A_i/\partial y)}\right] + \frac{\partial}{\partial z}\left[\frac{\partial g_s}{\partial(\partial A_i/\partial z)}\right]$$

$$= -\frac{1}{4\pi}\frac{\partial}{\partial y}[h_z - H_z] + \frac{1}{4\pi}\frac{\partial}{\partial z}[h_y - H_y]$$

$$= -\frac{1}{4\pi}[\nabla\times(\boldsymbol{h} - \boldsymbol{H})]_x \quad (2.22)$$

The other two components are treated similarly , giving

$$\sum_j \frac{\partial}{\partial r_j}\left[\frac{\partial g_s}{\partial(\partial A_i/\partial r_j)}\right] = -\frac{1}{4\pi}[\nabla\times(\boldsymbol{h} - \boldsymbol{H})]_i \quad (2.23)$$

Writing this in vector form, eq.(2.18) becomes

$$\frac{1}{2m^*}\left(i\hbar\frac{e^*}{c}\Psi\frac{\partial\Psi^*}{\partial r_i} - i\hbar\frac{e^*}{c}\Psi^*\frac{\partial\Psi}{\partial r_i} + \frac{2e^{*2}}{c^2}A\,|\Psi|^2\right) = -\frac{1}{4\pi}[\nabla\times(\boldsymbol{h} - \boldsymbol{H})]_i$$

(2.24)

We will consider the external magnetic field to be uniform. In particular we have $\nabla\times\boldsymbol{H} = 0$. From Maxwell's equation (Jackson, 1962), the current is given by

$$\boldsymbol{J} = \frac{c}{4\pi}\nabla\times\boldsymbol{h} \quad (2.25)$$

provided there is no time-varying electric field. In our case we identify this as the supercurrent and write

$$J \;=\; -\frac{\hbar e^*}{2m^* i}\left(\Psi^*\nabla\Psi \;-\; \Psi\nabla\Psi^*\right) \;-\; \frac{e^{*2}}{m^* c} A\,|\Psi|^2 \qquad (2.26)$$

This is the second of the Ginzburg - Landau equations.

<u>Ginzburg-Landau Coherence Length</u>

All superconductors have their own characteristic lengths called the *coherence length* and *penetration depth* (the details will be described in the next section). The relationship between these two lengths will help to classify the type of superconducting material . Firstly, we will describe the coherence length.

Since the order parameter is complex, we can write $\Psi \;=\; |\Psi|\, e^{i\phi}$, where ϕ is the phase of the order parameter. Then

$$\Psi^*\nabla\Psi \;-\; \Psi\nabla\Psi^* \;=\; 2i\,|\Psi|^2\nabla\phi \qquad (2.27)$$

and the supercurrent is

$$J \;=\; \frac{c}{4\pi}\nabla\times \boldsymbol{h} \;=\; -\frac{e^{*2}}{m^* c}\left(\frac{\hbar c}{e^*}\nabla\phi \;+\; A\right)|\Psi|^2 \qquad (2.28)$$

When the Meissner effect occurs we have $\boldsymbol{h} = 0$ (zero internal field) and $|\Psi| \neq 0$ (superconducting phase). We must therefore have

$$A \;=\; -\frac{\hbar c}{e^*}\nabla\phi \qquad (2.29)$$

Note that $\nabla\times A = 0$ since $\nabla\times\nabla\phi = 0$. The first Ginzburg-Landau equation eq.(2.16), is

$$\frac{1}{2m^*}\left(-i\hbar\nabla + \frac{e^*A}{c}\right)^2\Psi + a\Psi + b|\Psi|^2\Psi = 0$$

For the case of $h = 0$ and $|\Psi| \neq 0$, we can substitute eq.(2.29) to get

$$\frac{\hbar^2}{2m^*}\left(i\nabla + \nabla\phi\right)^2\Psi + a\Psi + b|\Psi|^2\Psi = 0 \qquad (2.30)$$

But

$$\left(i\nabla + \nabla\phi\right)^2\Psi = \left(-\nabla^2 + i\nabla^2\phi + i\nabla\phi\cdot\nabla + (\nabla\phi)^2\right)|\Psi|e^{i\phi}$$

$$= -e^{i\phi}\nabla^2|\Psi| \qquad (2.31)$$

So

$$-\frac{\hbar^2}{2m^*}\nabla^2|\Psi| + a|\Psi| + b|\Psi|^3 = 0$$

In this case ϕ is irrelevant and we consider only $|\Psi|$, a real parameter.

If $|\Psi|$ is uniform everywhere, then

$$|\Psi|^2 = -\frac{a}{b} = \frac{|a|}{b} \sim T_c - T$$

In general, we can write $|\Psi| = \sqrt{\frac{|a|}{b}}\, f$, where f is a trial function. Then

$$-\frac{\hbar^2}{2m^*}\left(\frac{|a|}{b}\right)^{\frac{1}{2}}\nabla^2 f + |a|\left(\frac{|a|}{b}\right)^{\frac{1}{2}} f + b\left(\frac{|a|}{b}\right)^{\frac{3}{2}} f^3 = 0 \qquad (2.32)$$

or

$$-\frac{\hbar^2}{2m^*|a|}\nabla^2 f \;-\; f \;+\; f^3 = 0 \qquad (2.33)$$

Considering the superconductor sample to have a flat surface in the y-z plane perpendicular to the x-axis, so that f only depends on x then eq.(2.33) is

$$-\frac{\hbar^2}{2m^*|a|}\frac{d^2 f}{dx^2} \;-\; f \;+\; f^3 = 0 \qquad (2.34)$$

The Ginzburg - Landau coherence length is defined by

$$\xi = \left(\frac{\hbar^2}{2m^*|a|}\right)^{\frac{1}{2}} \sim \frac{1}{\sqrt{T_c - T}} \qquad (2.35)$$

so eq.(2.34) becomes

$$-\xi^2 \frac{d^2 f}{dx^2} \;-\; f \;+\; f^3 = 0 \qquad (2.36)$$

For a single boundary condition of f(x=0) = 0, the exact solution (Fetter and Walecka, 1971) is $f(x) = tanh\left[x/(\sqrt{2}\,\xi)\right]$. This solution is appropriate for a normal metal-superconductor interface. The spatial variation in *f(x)* is basically confined to lie within a distance ξ of the surface. For $x >> \xi$, *f(x)* = 1 as shown in fig. 2.1. Close to the boundary f(x) does not show the behavior expected from the surface condition (2.15). This is a defect of the Ginzburg-Landau theory. Figure 2.1 is drawn according to physical expectations.

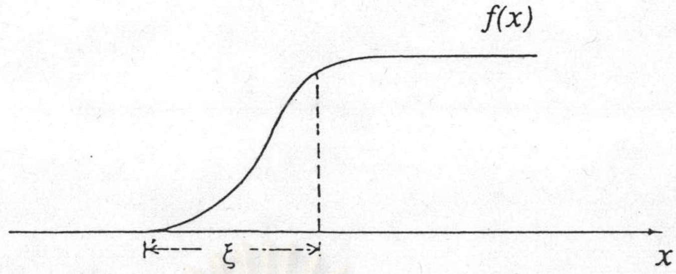

Fig. 2.1 The Ginzburg-Landau coherence length ξ (Tinkham, 1975)

Penetration Depth

Consider eq.(2.26) for the case of a uniform order parameter. The supercurrent is then

$$J = \frac{c}{4\pi}\nabla\times h = -\frac{e^{*2}}{m^{*}c}|\Psi|^{2}A \qquad (2.37)$$

The first Ginzburg-Landau equation, (2.16), gives

$$\frac{1}{2m^{*}}\left(-i\hbar\frac{e^{*}}{c}\nabla\cdot A + \frac{e^{*2}A^{2}}{c^{2}}\right)\Psi + a\Psi + b|\Psi|^{2}\Psi = 0 \qquad (2.38)$$

We can always choose a gauge so that $\nabla\cdot A = 0$ without changing the value of $h = \nabla\times A$. Also, if we keep only the terms linear in A, then

$$|\Psi|^{2} = \frac{|a|}{b} \qquad (2.39)$$

Now

$$\frac{c}{4\pi}\nabla\times\boldsymbol{h} \;=\; -\,\frac{e^{*2}}{m^{*}c}\frac{|a|}{b}\boldsymbol{A} \qquad (2.40)$$

and

$$\frac{c}{4\pi}\nabla\times\nabla\times\boldsymbol{h} \;=\; -\,\frac{e^{*2}}{m^{*}c}\frac{|a|}{b}\boldsymbol{h} \qquad (2.41)$$

Consider again our sample with a surface perpendicular to the x-axis. Let there be an external magnetic field $\boldsymbol{H} = H_0\hat{z}$ parallel to the surface. The internal magnetic field will be of the form

$$\boldsymbol{h} \;=\; h(x)\hat{z}$$

so that

$$\nabla\times h \;=\; -\,\frac{\mathrm{d}h(x)}{\mathrm{d}x}\,\hat{y}$$

and

$$\nabla\times\nabla\times\boldsymbol{h} \;=\; -\,\frac{\mathrm{d}^2h(x)}{\mathrm{d}x^2}\,\hat{z}$$

Now,

$$\frac{c}{4\pi}\,\frac{\mathrm{d}^2h(x)}{\mathrm{d}x^2} \;=\; \frac{e^{*2}}{m^{*}c}\frac{|a|}{b}\,h(x) \qquad (2.42)$$

so that the penetration depth is defined by

$$\lambda = \sqrt{\frac{m^* c^2 b}{4\pi e^{*2}|a|}} \sim \frac{1}{\sqrt{T_c - T}} \qquad (2.43)$$

and eq.(2.42) becomes

$$\frac{d^2 h(x)}{dx^2} - \frac{1}{\lambda^2} h(x) = 0 \qquad (2.44)$$

We try the solution (Kittel, 1986).

$$h(x) = H_0 e^{-x/\lambda} \qquad (2.45)$$

and find that it satisfies the above equation. The penetration of the magnetic field is confined to a surface layer of thickness λ, which is shown below in fig. 2.2

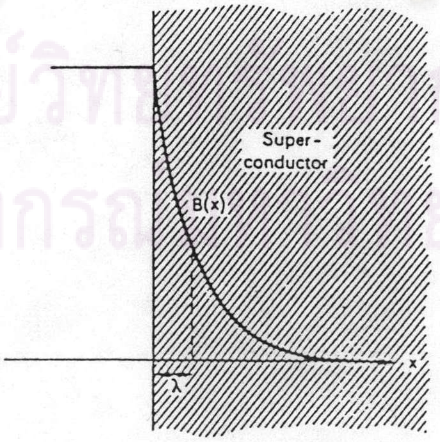

Fig. 2.2 The penetration depth of a superconducting material (Rose-Innes and Rhoderick, 1978)