CHAPTER III

TRANSFORMATION SEMIGROUPS ADMITTING HYPERRING STRUCTURE

In [9], standard transformation semigroups admitting ring structure were characterized. The purpose of this chapter is to characterize such standard transformation semigroups admitting hyperring structure. Also, we characterize some other transformation semigroups admitting hyperring structure.

First, we recall the notations of the following transformation semigroups. For a set X, let

P_X = the partial transformation semigroup on X,

T_X = the full transformation semigroup on X,

I_X = the 1-1 partial transformation semigroup on X (the symmetric inverse semigroup on X),

G_X = the symmetric group on X,

CP_X = the transformation semigroup of all constant partial transformations of X (including 0),

CT_X = the transformation semigroup of all constant transformations of X,

U_X = the transformation semigroup of all almost identical partial transformations of X,

V_X = the transformation semigroup of all almost identical transformations of X,

W_X = the transformation semigroup of all almost identical 1-1 partial transformations of X,

M_X = the transformation semigroup of all 1-1 transformations of X,

E_X = the transformation semigroup of all onto transformations of X,

AM_X = the transformation semigroup of all almost 1-1 transformations of X and

AE_X = the transformation semigroup of all almost onto transformations of X.

For convenience in giving the proofs of some theorems in this chapter, the following notations are required : Let X be a set. For a nonempty subset A of X and $x \in X$, let A_x denote the constant partial transformation of X with domain A and range $\{x\}$ and for $a, b \in X$, let (a,b) be the element of G_X defined by

$$x(a,b) = \begin{cases} b & \text{if } x = a, \\ a & \text{if } x = b, \\ x & \text{if } x \in X \setminus \{a,b\}. \end{cases}$$

Observe that

$$CP_X = \{A_x \mid \phi \neq A \subseteq X,\ x \in X\} \cup \{0\} \text{ and}$$

$$CT_X = \begin{cases} \{X_x \mid x \in X\} & \text{if } X \neq \phi, \\ \{0\} & \text{if } X = \phi. \end{cases}$$

The following results obtained from [9] show that each of the transformation semigroups P_X, T_X, I_X, G_X, CP_X, U_X, V_X, W_X, M_X and E_X is almost never the multiplicative structure of any ring :

(I) If X is a set and S is any one of P_X, T_X, I_X, CP_X, U_X, V_X or W_X, then S admits a ring structure if and only if $|X| \leq 1$.

(II) If X is a set and S is any one of G_X, M_X or E_X, then S admits a ring structure if and only if $|X| \leq 2$.

Although hyperrings are a generalization of rings, the following two theorems show that (I) also holds for the case of admitting hyperring structure.

Theorem 3.1. If X is a set and S is any one of P_X, I_X, CP_X, U_X or W_X, then S admits a hyperring structure if and only if $|X| \leq 1$.

Proof : Let X be a set and $S = P_X$, I_X, CP_X, U_X or W_X.

Assume that S admits a hyperring structure under an addition +. To prove that $|X| \leq 1$, suppose this is false. Then $|X| > 1$. Thus there exist a, b ε X such that $a \neq b$. Therefore $\{a\}_a, \{a\}_b, \{b\}_a \in S$ and so $\phi \neq \{a\}_a + \{a\}_b \subseteq S$. Let $\alpha \in \{a\}_a + \{a\}_b$. Then $\alpha\{a\}_a \in (\{a\}_a + \{a\}_b)\{a\}_a$. But $(\{a\}_a + \{a\}_b)\{a\}_a =$ $\{a\}_a\{a\}_a + \{a\}_b\{a\}_a = \{a\}_a + 0 = \{\{a\}_a\}$, so we have that $\alpha\{a\}_a = \{a\}_a$. This implies that $a \in \Delta\alpha$ and $a\alpha = a$ and hence $\{a\}_a\alpha = \{a\}_a$. From $\alpha \in \{a\}_a + \{a\}_b$, we have that $\{a\}_a\alpha \in \{a\}_a(\{a\}_a + \{a\}_b)$. Since $\{a\}_a(\{a\}_a + \{a\}_b) = \{a\}_a\{a\}_a + \{a\}_a\{a\}_b = \{a\}_a + \{a\}_b$, it follows that $\{a\}_a \in \{a\}_a + \{a\}_b$. Hence $\{a\}_a\{b\}_a \in (\{a\}_a + \{a\}_b)\{b\}_a$ $= \{a\}_a\{b\}_a + \{a\}_b\{b\}_a = 0 + \{a\}_a = \{\{a\}_a\}$ which implies that $\{a\}_a\{b\}_a = \{a\}_a$. This is a contradiction since $\{a\}_a\{b\}_a = 0$. This proves that $|X| \leq 1$.

The converse is obvious since $S = \{0\}$ if $|X| = 0$ and if $|X| = 1$, then $S = \{0, 1_X\}$ which is (isomorphic to) $\mathbb{Z}_2$ under multiplication.

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Theorem 3.2. If X is a set and S is any one of T_X, V_X, AM_X or AE_X, then S admits a hyperring structure if and only if $|X| \leq 1$.

Proof : Let X be a set and $S = T_X, V_X, AM_X$, or AE_X.

Assume that S admits a hyperring structure under an addition +. To prove that $|X| \leq 1$, assume that $|X| \neq 0$. Let a, b ε X. Define $\alpha, \beta : X \to X$ by

$$x\alpha = \begin{cases} a & \text{if } x \in \{a,b\}, \\ x & \text{otherwise} \end{cases}$$

and

$$x\beta = \begin{cases} b & \text{if } x \in \{a,b\}, \\ x & \text{otherwise.} \end{cases}$$

Then $\alpha, \beta \in S$. Since $a\alpha^2 = (a\alpha)\alpha = a\alpha$, $b\alpha^2 = (b\alpha)\alpha = a\alpha = b\alpha$ and $x\alpha^2 = (x\alpha)\alpha = x\alpha$ for $x \in X \setminus \{a,b\}$, we have that $\alpha^2 = \alpha$. Since $a(\beta\alpha) = (a\beta)\alpha = b\alpha = a\alpha$, $b(\beta\alpha) = (b\beta)\alpha = b\alpha$ and $x(\beta\alpha) = (x\beta)\alpha = x\alpha$ for $x \in X \setminus \{a,b\}$, we have that $\beta\alpha = \alpha$. Let γ be the inverse of α in the hypergroup $(S^o,+)$. Then $0 \in \alpha + \gamma$, so $0 \in (\alpha + \gamma)\alpha = \alpha^2 + \gamma\alpha = \alpha + \gamma\alpha$. Since γ is the unique inverse of α in $(S^o,+)$, it follows that $\gamma = \gamma\alpha$. Then $0 \in \beta\alpha + \gamma\alpha$ since $0 \in \alpha + \gamma$, $\alpha = \beta\alpha$ and $\gamma = \gamma\alpha$. But $\beta\alpha + \gamma\alpha = (\beta + \gamma)\alpha$ and $\alpha \neq 0$, so we have $0 \in \beta + \gamma$. Therefore $\alpha = \beta$ since α is the unique inverse of γ in $(S^o,+)$. By defining α and β, we get a = b. This proves that $|X| = 1$.

The converse is obvious since $|S| = 1$ if $|X| \leq 1$.

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The following theorem is obtained from the fact that any nontrivial right zero semigroup does not admit a hyperring structure.

Theorem 3.3. For a set X, CT_X admits a hyperring structure if and only if $|X| \leq 1$.

Proof : Since for a, b ε X, $X_aX_b = X_b$, it follows that CT_X is a right zero semigroup. By Corollary 2.7(2), CT_X admits a hyperring structure if and only if $|CT_X| = 1$. But $|CT_X| = 1$ if and only if $|X| \leq 1$, so the theorem is proved.

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Since every group admits a hyperring structure (see the remark of Chapter I, page 24), the following theorem is obtained.

Theorem 3.4. For a set X, G_X admits a hyperring structure.

We know that if X is a finite set, then $M_X = E_X = G_X$. Therefore it follows from Theorem 3.4 that for any set X and for $S = M_X$ or E_X, if X is finite, then S admits a hyperring structure. It is natural to ask whether this converse is true. The following theorem tells us that this converse is true. A basic fact which will be used to prove the theorem is the following : For a set X, the center of the group G_X is trivial if $|X| \geq 3$.

Theorem 3.5. If X is a set and S is any one of M_X or E_X, then S admits a hyperring structure if and only if X is finite.

Proof : Let X be a set and $S = M_X$ or E_X.

Assume that S admits a hyperring structure under an addition +. To show that X is finite, suppose on the contrary that X is infinite. Since $1_X \in S$, there exists an inverse α of 1_X in the hypergroup $(S^o,+)$ such that $0 \in 1_X + \alpha$. Let $\beta \in S$. Then $0 \in (1_X + \alpha)\beta = \beta + \alpha\beta$ and $0 \in \beta(1_X + \alpha) = \beta + \beta\alpha$. Therefore $\alpha\beta$ and $\beta\alpha$ are both inverses of β in $(S^o,+)$. It then follows that $\alpha\beta = \beta\alpha$. This proves that $\alpha\beta = \beta\alpha$ for all $\beta \in S$. Since $G_X \subseteq S$, $(x,x\alpha) \in S$ for all $x \in X$. Then $\alpha(x,x\alpha) = (x,x\alpha)\alpha$ for all $x \in X$. For $x \in X$, $x(\alpha(x,x\alpha)) = x$ and $x((x,x\alpha)\alpha) = x\alpha^2$. It follows that $\alpha^2 = 1_X$ which implies that $\alpha \in G_X$. Therefore α is an element in the center of G_X. Since X is infinite, $\alpha = 1_X$. Thus $0 \in 1_X + 1_X$ and hence $0 \in \beta + \beta$ for all $\beta \in S$.

Let $a, b \in X$ be such that $a \neq b$. Since X is infinite, $|X| = |X \setminus \{a,b\}|$, so there exist a 1-1 map γ from X onto $X \setminus \{a,b\}$ and a map λ from $X \setminus \{a,b\}$ onto X. Extend λ to $\mu : X \to X$ by defining $a\mu = b\mu = a$. Then $\gamma \in M_X$, $\gamma(a,b) = \gamma$, $\mu \in E_X$ and $(a,b)\mu = \mu$.

<u>Case 1</u> : $S = M_X$. Then $0 \in \gamma + \gamma$. Since $\gamma + \gamma = \gamma(a,b) + \gamma 1_X = \gamma((a,b) + 1_X)$, we have that $0 \in (a,b) + 1_X$. Since 1_X is the unique inverse of 1_X in $(M_X^o,+)$, we get $(a,b) = 1_X$ which is a contradiction since $a \neq b$.

<u>Case 2</u> : $S = E_X$. Then $0 \in \mu + \mu$. Since $\mu + \mu = (a,b)\mu + 1_X\mu = ((a,b) + 1_X)\mu$, it follows that $0 \in (a,b) + 1_X$. This implies that $(a,b) = 1_X$ which is a contradiction.

Hence X is finite.

The converse follows from the fact mentioned above. #