CHAPTER II

HYPERRINGS

There are many standard well-known theorems in ring theory. Hyperrings are a generalization of rings. The main purpose of this chapter is to give some theorems concerning hyperrings which generalize some standard theorems in ring theory. Included in this chapter, some general properties of hypergroups and hyperrings are also given.

The following two propositions give some general properties of hypergroups.

A hypergroup need not have an identity and an element of hypergroup with identity need not have an inverse (see Example 3 on page 11 and Example 4 on page 12). The first proposition shows that a semihypergroup becomes a hypergroup if it has an identity and each of its elements has an inverse. This proposition will be used to prove some theorems about hyperrings in this chapter.

<u>Proposition 2.1.</u> If a semihypergroup H has an identity and every element of H has an inverse in H, then H is a hypergroup.

<u>Proof</u> : To prove that H is a hypergroup, let $x \in H$. Then x has an inverse y in H. Thus there exists an identity e in H such that $e \in (xoy) \cap (yox)$ where o is the hyperoperation of H. To show that $Hox = H = xoH$, let $h \in H$. Then $h \in hoe$. Since $e \in yox$, we have that $hoe \subseteq ho(yox)$, and therefore $hoe \subseteq (hoy)ox \subseteq Hox$. Hence $h \in Hox$.

This proves that $H = Hox$. It can be shown similarly that $H = xoH$. This proves that $Hox = H = xoH$ for all $x \in H$. Hence H is a hypergroup.

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It is well-known in group theory that a nonempty subset S of a group G is a subgroup of G if and only if $xy^{-1} \in S$ for all $x, y \in S$. Example 5 on page 13 shows that a subhypergroup of a canonical hypergroup need not be canonical and Example 6 on page 16 shows that a canonical subhypergroup of a canonical hypergroup H need not contain the scalar identity of H. The following proposition characterizes a canonical subhypergroup of a canonical hypergroup H containing the scalar identity of H and it generalizes the fact mentioned above for the abelian ease. However, its proof shows that the property of commutativity can be omitted.

Proposition 2.2. Let (H,o) be a canonical hypergroup with scalar identity e and S a nonempty subset of H. Then S is a canonical subhypergroup of H containing e if and only if $xoy' \subseteq S$ for all $x, y \in S$.

Proof : Assume that S is a canonical subhypergroup of H containing e. Then e is the scalar identity of S which is the unique identity of S. Let $x, y \in S$. Then $y' \in S$ since S is canonical and $e \in S$ which implies that $xoy' \subseteq S$.

Conversely, assume that $xoy' \subseteq S$ for all $x, y \in S$. Then for $x \in S$, $e \in xox' \subseteq S$ which implies that $x' \in eox' \subseteq S$. Hence for $x, y \in S$, $xoy = xo(y')' \subseteq S$. Therefore S is a semihypergroup containing the scalar identity e and each of its elements has a unique inverse in S. By Proposition 2.1, S is a hypergroup. Since H is commutative and reversible, we have that S is commutative and reversible. Hence S is a canonical subhypergroup of H containing e.

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We have an immediate consequence of Proposition 2.2 as follows :

Corollary 2.3. Let A be a hyperring and B a nonempty subset of A. Then B is a subhyperring of A containing 0 if and only if for $x, y \in B$, $x + y' \subseteq B$ and $xy \in B$.

We give a significant sufficient condition such that a hyperring becomes a ring.

Proposition 2.4. If A is a hyperring in which $x + x' = \{0\}$ for all $x \in A$, then A is a ring.

Proof : To show that A is a ring, it suffices to show that $|x + y| = 1$ for all $x, y \in A$. Let $x, y \in A$. Let $z \in x + y$. Then

$$z + y' \subseteq (x + y) + y' = x + (y + y') = x + 0 = \{x\}.$$

Thus $z + y' = \{x\}$. Therefore

$$x + y = (z + y') + y = z + (y' + y) = z + 0 = \{z\}.$$

This proves that for all $x, y \in A$, $|x + y| = 1$. Hence A is a ring. #

The following proposition gives a generalization of the following well-known theorem of rings : "Every Boolean ring is commutative".

Proposition 2.5. If A is a hyperring in which $(x + y)^2 = x + y$ for all $x, y \in A$, then A is a commutative hyperring.

Proof : For $x \in A$, $\{x^2\} = (x + 0)^2 = x + 0 = \{x\}$, so $x^2 = x$. Then for $x \in A$, $x' = x'x' = xx = x$.

Let $x, y \in A$. Then

$$xy + xyx = (xy + xyx)^2$$
$$= (xy)^2 + xyxyx + xyxxy + (xyx)^2$$
$$= xy + xyx + xy + xyx$$
$$= xy + xy + xyx + xyx$$
$$= xy + (xy)' + xyx + (xyx)'$$

and

$$yx + xyx = (yx + xyx)^2$$
$$= (yx)^2 + yxxyx + xyxyx + (xyx)^2$$
$$= yx + yx + xyx + xyx$$
$$= yx + (yx)' + xyx + (xyx)'.$$

Since 0 is an element of $xy + (xy)'$, $xyx + (xyx)'$ and $yx + (yx)'$, it follows that $0 \in xy + xyx$ and $0 \in yx + xyx$. By uniqueness of an additive inverse of xyx, we have that $xy = (xyx)' = yx$.

Hence, A is a commutative hyperring.

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It follows from Corollary 1.5 of [9] that for any ring R, if $R \setminus \{0\}$ is a left [right] group under multiplication, then $R \setminus \{0\}$ is a group. The following proposition shows that this is also true for the case of hyperrings.

Proposition 2.6. Let A be a hyperring. If $A \setminus \{0\}$ is a left [right] group under multiplication, then $A \setminus \{0\}$ is a group under multiplication.

Proof : Since $A \setminus \{0\}$ is a left group under multiplication, we have that $A \setminus \{0\}$ is a union of multiplicative subgroups of $A \setminus \{0\}$ and $ef = e$ for all $e, f \in A \setminus \{0\}$ such that $e^2 = e$ and $f^2 = f$.

Let $e, f \in A \setminus \{0\}$ be such that $e^2 = e$ and $f^2 = f$. Then

$$0 \in e + e' = ef + (ee)' = ef + ee' = e(f + e').$$

If $0 \notin f + e'$, then $f + e' \subseteq A \setminus \{0\}$, so $e(f + e') \subseteq A \setminus \{0\}$ since $A \setminus \{0\}$ is closed under multiplication, and hence $0 \in A \setminus \{0\}$ which is a contradiction. This proves that $0 \in f + e'$. By uniqueness of an additive inverse of f, we have that $f' = e'$ and hence $e = f$. Therefore $A \setminus \{0\}$ has a unique element e such that $e^2 = e$. Since $A \setminus \{0\}$ is a union of multiplicative subgroups of $A \setminus \{0\}$, we have that $A \setminus \{0\}$ is a group under multiplication.

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From Proposition 2.6, we obtain the following corollary.

Corollary 2.7. (1) A left [right] group admits a hyperring structure if and only if it is a group.

(2) A left [right] zero semigroup S admits a hyperring structure if and only if $|S| = 1$.

Proof : (1) follows directly from Proposition 2.6 and the fact that every group admits a hyperring structure.

(2) follows from (1) and the fact that a left [right] zero semigroup is a left [right] group and a left [right] zero semigroup is a union of trivial subgroups.

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It has been known from [9] that a ring whose multiplicative structure is a Kronecker semigroup contains at most two elements. This is also true for the case of hyperrings.

Proposition 2.8. Let A be a hyperring. If the multiplicative structure of A is a Kronecker semigroup, then $|A| \leq 2$.

Proof : Since the multiplicative structure of A is a Kronecker semigroup, it follows that for x, y $\in$ A,

$$xy = yx = \begin{cases} x & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases} \quad \text{..........} (*)$$

Suppose that $|A| > 2$. Let x and y be two distinct nonzero elements of A. Then $xx = x$, $yy = y$ and $xy = yx = 0$. Therefore $x(x + y) = xx + xy = x + 0 = \{x\}$. It follows from (*) that $x + y = \{x\}$. Symmetrically, $y + x = \{y\}$. Since $x + y = y + x$, we have that $x = y$ which is a contradiction. Hence $|A| \leq 2$.

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The next proposition gives some properties of hyperideals of hyperrings. Such properties are well-known for the case of rings.

Proposition 2.9. Let A be a hyperring. Then the following statements hold.

(1) If I and J are hyperideals of A, then I + J is a hyperideal of A.

(2) If $\{I_\alpha\}_{\alpha\in\Lambda}$ is a collection of hyperideals of A, then $\bigcap_{\alpha\in\Lambda} I_\alpha$ is a hyperideal of A.

Proof : To prove (1), let I and J be hyperideals of A. Let $x, y \in I + J$. Then there exist elements $a, b \in I$ and $c, d \in J$ such that $x \in a + c$ and $y \in b + d$. From Chapter I, page 8, $b' \in I$ and $d' \in J$, and hence $a + b' \in I$ and $c + d' \in J$. Since the additive structure of A is reversible and $y \in b + d$, we have that $y' \in b' + d'$. Therefore $x + y' \subseteq (a + c) + (b' + d') = (a + b') + (c + d') \subseteq I + J$. By Proposition 2.2, I + J is a canonical subhypergroup of A under addition. Since $AI \subseteq I$, $AJ \subseteq J$, $IA \subseteq I$ and $JA \subseteq J$, it follows that $A(I + J) = AI + AJ \subseteq I + J$ and $(I + J)A = IA + JA \subseteq I + J$. Hence I + J is a hyperideal of A.

To prove (2), let $\{I_\alpha\}_{\alpha\varepsilon\Lambda}$ be a collection of hyperideals of A. From Chapter I, page 8, $0 \varepsilon I_\alpha$ for all $\alpha \varepsilon \Lambda$. Then $0 \varepsilon \bigcap_{\alpha\varepsilon\Lambda} I_\alpha$. Let $x, y \varepsilon \bigcap_{\alpha\varepsilon\Lambda} I_\alpha$. Then $x, y \varepsilon I_\alpha$ for all $\alpha \varepsilon \Lambda$. Since such I_α is a hyperideal of A, we have that $y' \varepsilon I_\alpha$ for all $\alpha \varepsilon \Lambda$. Thus $x + y' \subseteq I_\alpha$ for all $\alpha \varepsilon \Lambda$ which implies that $x + y' \subseteq \bigcap_{\alpha\varepsilon\Lambda} I_\alpha$. Hence by Proposition 2.2, $\bigcap_{\alpha\varepsilon\Lambda} I_\alpha$ is a canonical subhypergroup of A under addition. Since for every $\beta \varepsilon \Lambda$, $\bigcap_{\alpha\varepsilon\Lambda} I_\alpha \subseteq I_\beta$, $AI_\beta \subseteq I_\beta$ and $I_\beta A \subseteq I_\beta$, we have that $A(\bigcap_{\alpha\varepsilon\Lambda} I_\alpha) \subseteq AI_\beta \subseteq I_\beta$ and $(\bigcap_{\alpha\varepsilon\Lambda} I_\alpha)A \subseteq I_\beta A \subseteq I_\beta$ for all $\beta \varepsilon \Lambda$. Hence $A(\bigcap_{\alpha\varepsilon\Lambda} I_\alpha) \subseteq \bigcap_{\alpha\varepsilon\Lambda} I_\alpha$ and $(\bigcap_{\alpha\varepsilon\Lambda} I_\alpha)A \subseteq \bigcap_{\alpha\varepsilon\Lambda} I_\alpha$. This proves that $\bigcap_{\alpha\varepsilon\Lambda} I_\alpha$ is a hyperideal of A.

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We know that the image of a ring homomorphism is a subring of the codomain of that ring homomorphism. The next proposition shows that the image of a hyperring homomorphism is also a subhyperring of the codomain of that hyperring homomorphism but it need not contain the zero of the codomain which is shown by Example 6 on page 16.

<u>Proposition 2.10</u>. If φ is a homomorphism of a hyperring A into a hyperring B, then $\varphi(A)$ is a subhyperring of B having $\varphi(0)$ as its zero and for each $x \varepsilon A$, $\varphi(x')$ is the unique inverse of $\varphi(x)$ in $\varphi(A)$.

<u>Proof</u> : Since φ is a homomorphism, it is clear that $\varphi(x)\varphi(0) = \varphi(0)\varphi(x) = \varphi(0)$ for all $x \varepsilon A$. To prove the proposition, it suffices to prove that the following statements hold.

(1) For $x, y \in A$, $\varphi(x) + \varphi(y) \subseteq \varphi(A)$ and $\varphi(x)\varphi(y) \in \varphi(A)$.

(2) For $x \in A$, $\varphi(0) + \varphi(x) = \{\varphi(x)\}$.

(3) For $x \in A$, $\varphi(0) \in \varphi(x) + \varphi(x')$ and if $\varphi(0) \in \varphi(x) + \varphi(y)$, then $\varphi(y) = \varphi(x')$.

(4) For $x, y, z \in A$, $\varphi(x) \in \varphi(y) + \varphi(z)$ implies that $\varphi(z) \in \varphi(y') + \varphi(x)$.

(1) follows from the facts that φ is a homomorphism, $A + A \subseteq A$ and $AA \subseteq A$. Since $0 + x = \{x\}$ for all $x \in A$, we have that $\varphi(0) + \varphi(x) = \varphi(0 + x) = \{\varphi(x)\}$ for all $x \in A$. Then (2) holds.

To prove (3), let $x \in A$. Since $0 \in x + x'$, $\varphi(0) \in \varphi(x + x') = \varphi(x) + \varphi(x')$. Let $y \in A$ be such that $\varphi(0) \in \varphi(x) + \varphi(y)$. Then $\varphi(0) \in \varphi(x + y)$. Thus there exists an element $a \in x + y$ such that $\varphi(0) = \varphi(a)$. Since A is reversible under addition and $a \in x + y$, it follows that $y \in x' + a$ which implies that $\varphi(y) \in \varphi(x' + a) = \varphi(x') + \varphi(a) = \varphi(x') + \varphi(0) = \varphi(x' + 0) = \{\varphi(x')\}$. Hence $\varphi(y) = \varphi(x')$.

To prove (4), let $x, y, z \in A$ be such that $\varphi(x) \in \varphi(y) + \varphi(z)$. Then $\varphi(x) \in \varphi(y + z)$. Thus there exists an element $u \in y + z$ such that $\varphi(x) = \varphi(u)$. Since A is reversible under addition and $u \in y + z$, we have that $z \in y' + u$. Then $\varphi(z) \in \varphi(y' + u) = \varphi(y') + \varphi(u)$. Since $\varphi(u) = \varphi(x)$, $\varphi(z) \in \varphi(y') + \varphi(x)$.

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The kernel of a hyperring homomorphism may be empty. It is shown by Example 6, page 16. General properties of a hyperring homomorphism with nonempty kernel are obtained in the same way as those of a ring homomorphism as follows :

<u>Proposition 2.11</u>. If φ is a homomorphism of a hyperring A into a hyperring B, then the following statements are equivalent.

(1) $\ker \varphi \neq \phi$.

(2) $0 \in \ker \varphi$.

(3) $\varphi(0) = 0$.

(4) $0 \in \varphi(A)$.

(5) $(\varphi(x))' = \varphi(x')$ for all $x \in A$.

(6) $\ker \varphi$ is a hyperideal of A.

Proof : Assume that (1) holds. To prove (2), let $x \in \ker \varphi$. Then $\varphi(x) = 0$. Thus $\{\varphi(0)\} = \varphi(0) + 0 = \varphi(0) + \varphi(x) = \varphi(0 + x) = \{\varphi(x)\}$, so $\varphi(0) = \varphi(x)$. Hence $\varphi(0) = 0$. Therefore $0 \in \ker \varphi$.

It is clear that (2) implies (3), (3) implies (4) and (4) implies (1). Now, we have that (1), (2), (3) and (4) are equivalent.

If $\varphi(0) = 0$, then for $x \in A$, $0 = \varphi(0) \in \varphi(x + x') = \varphi(x) + \varphi(x')$ which implies that $(\varphi(x))' = \varphi(x')$. If $(\varphi(x))' = \varphi(x')$ for all $x \in A$, then $(\varphi(0))' = \varphi(0') = \varphi(0)$ and therefore $0 \in \varphi(0) + (\varphi(0))' = \varphi(0) + \varphi(0) = \varphi(0 + 0) = \{\varphi(0)\}$ which implies that $\varphi(0) = 0$. This proves that (3) $\Longleftrightarrow$ (5) holds. Now, we prove that (1), (2), (3), (4) and (5) are equivalent.

Assume that (1) holds. Then (5) holds. To prove (6), let $x, y \in \ker \varphi$. Then $\varphi(x) = 0 = \varphi(y)$. By (5), $\varphi(y') = (\varphi(y))' = 0' = 0$. Hence $\varphi(x + y') = \varphi(x) + \varphi(y') = 0 + 0 = \{0\}$. This implies that $x + y' \subseteq \ker \varphi$. By Proposition 2.2, $\ker \varphi$ is a canonical subhypergroup of A under addition. Since $\varphi(A(\ker \varphi)) = \varphi(A)\varphi(\ker \varphi) = \varphi(A)\{0\} = \{0\}$ and $\varphi((\ker \varphi)A) = \varphi(\ker \varphi)\varphi(A) = \{0\}\varphi(A) = \{0\}$, we have that $A(\ker \varphi) \subseteq \ker \varphi$ and $(\ker \varphi)A \subseteq \ker \varphi$. Therefore $\ker \varphi$ is a hyperideal of A, that is, (6) holds. It is trivial that (6) implies (1).

Hence the proposition is proved. #

The following two corollaries are immediate consequences of Proposition 2.10 and Proposition 2.11.

Corollary 2.12. If φ is a homomorphism of a hyperring A into a hyperring B such that ker $\varphi \neq \phi$, then $\varphi(A)$ is a subhyperring of B containing 0 and for each $x \in A$, $\varphi(x') = (\varphi(x))' \in \varphi(A)$.

Corollary 2.13. If φ is a homomorphism of a hyperring A onto a hyperring B, then ker φ is a hyperideal of A and for each $x \in A$, $\varphi(x') = (\varphi(x))' \in \varphi(A)$.

The next corollary which follows from Proposition 2.11 generalizes a standard result in ring theory.

Corollary 2.14. If φ is a homomorphism of a hyperring A into a hyperring B such that ker $\varphi \neq \phi$, then φ is 1-1 if and only if ker $\varphi = \{0\}$.

Proof : By Proposition 2.11, we have that $\varphi(0) = 0$ and $(\varphi(x))' = \varphi(x')$ for all $x \in A$ since ker $\varphi \neq \phi$.

Assume that φ is 1-1. Let $x \in$ ker φ. Then $\varphi(x) = 0 = \varphi(0)$. Since φ is 1-1, $x = 0$. Hence ker $\varphi = \{0\}$.

Conversely, assume that ker $\varphi = \{0\}$. Let $x, y \in A$ be such that $\varphi(x) = \varphi(y)$. Then $0 \in \varphi(x) + (\varphi(x))' = \varphi(x) + (\varphi(y))' = \varphi(x) + \varphi(y') = \varphi(x + y')$, so there exists an element $z \in x + y'$ such that $\varphi(z) = 0$. Thus $z \in$ ker φ. Since ker $\varphi = \{0\}$, $z = 0$ which implies that $0 \in x + y'$. Then $x' = y'$, so $x = y$. Hence φ is 1-1.

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Using Corollary 2.14, we get

Corollary 2.15. Let φ be a homomorphism of a hyperring A into a hyperfield F and $|A| > 1$. Then the following statements hold :

(1) φ is 1-1 if and only if $\ker \varphi = \{0\}$.

(2) If A is a hyperfield and φ is 1-1, then $\varphi(1) = 1$ and $\varphi(x^{-1}) = (\varphi(x))^{-1}$ for all $x \in A$, $x \neq 0$.

Proof : (1) Assume that φ is 1-1. Since $\varphi(0) = \varphi(00) = (\varphi(0))^2$ and F is a hyperfield, $\varphi(0) = 0$ or $\varphi(0) = 1$. If $\varphi(0) = 1$, then for $x \in A$, $\varphi(x) = \varphi(x)1 = \varphi(x)\varphi(0) = \varphi(x0) = \varphi(0) = 1$ which is a contradiction since φ is 1-1. Thus $\varphi(0) = 0$. By Corollary 2.14, $\ker\varphi = \{0\}$.

The converse of (1) follows from Corollary 2.14.

(2) Assume that A is a hyperfield and φ is 1-1. By (1), $\ker \varphi = \{0\}$ and $\varphi(1) \neq \varphi(0) = 0$. Therefore $\varphi(1) = \varphi(1)1 =$ $= \varphi(1)(\varphi(1)(\varphi(1))^{-1}) = (\varphi(1)\varphi(1))(\varphi(1))^{-1} = \varphi(1)(\varphi(1))^{-1} = 1$. Since φ is 1-1, $\varphi(x) \neq 0$ for all $x \in A \setminus \{0\}$. Hence for $x \in A \setminus \{0\}$, $\varphi(x^{-1}) = \varphi(x^{-1})\varphi(x)(\varphi(x))^{-1} = \varphi(x^{-1}x)(\varphi(x))^{-1} = \varphi(1)(\varphi(x))^{-1}$ $= 1(\varphi(x))^{-1} = (\varphi(x))^{-1}$.

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The following four theorems generalize isomorphism theorems in ring theory.

Theorem 2.16. If φ is a homomorphism of a hyperring A onto a hyperring B, then $A/\ker \varphi \cong B$.

Proof : By Corollary 2.13, $\ker \varphi$ is a hyperideal of A and $\varphi(x') = (\varphi(x))'$ for all $x \in A$. Define $\Phi : A/\ker \varphi \to B$ by

$$\Phi(x + \ker \varphi) = \varphi(x)$$

for all $x \in A$. To show that Φ is well-defined, let $x, y \in A$ be such that $x + \ker \varphi = y + \ker \varphi$. Then $y \in x + \ker \varphi$, so there exists an element $z \in \ker \varphi$ such that $y \in x + z$. Thus $\varphi(z) = 0$ and therefore $\varphi(y) \in \varphi(x + z) = \varphi(x) + \varphi(z) = \varphi(x) + 0 = \{\varphi(x)\}$. Hence $\varphi(x) = \varphi(y)$.

To show that Φ is 1-1, let $x, y \in A$ be such that $\Phi(x + \ker \varphi) = \Phi(y + \ker \varphi)$. Then $\varphi(x) = \varphi(y)$. Thus $0 \in \varphi(x) + (\varphi(x))' = \varphi(y) + \varphi(x') = \varphi(y + x')$, so there exists an element $u \in y + x'$ such that $\varphi(u) = 0$. Then $u \in \ker \varphi$. Since $u \in y + x'$, by reversibility of A under addition, we have that $y \in u + x$. Then $y \in x + u \subseteq x + \ker \varphi$ which implies that $x + \ker \varphi = y + \ker \varphi$. Hence Φ is 1-1.

Next, to show that Φ is a homomorphism, let $x, y \in A$. Then

$$\begin{aligned}\Phi((x + \ker \varphi) + (y + \ker \varphi)) &= \Phi(\{z + \ker \varphi \mid z \in x + y\}) \\ &= \{\Phi(z + \ker \varphi) \mid z \in x + y\} \\ &= \{\varphi(z) \mid z \in x + y\} \\ &= \varphi(x + y) \\ &= \varphi(x) + \varphi(y) \\ &= \Phi(x + \ker \varphi) + \Phi(y + \ker \varphi)\end{aligned}$$

and

$$\begin{aligned}\Phi((x + \ker \varphi)(y + \ker \varphi)) &= \Phi(xy + \ker \varphi) \\ &= \varphi(xy) \\ &= \varphi(x)\varphi(y) \\ &= \Phi(x + \ker \varphi)\Phi(y + \ker \varphi).\end{aligned}$$

Since φ is onto, it follows that Φ is onto. Now, we have that Φ is a 1-1 homomorphism of A onto B. Hence $A/\ker \varphi \cong B$.

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<u>Lemma 2.17</u>. Let A and B be hyperrings, $\varphi : A \to B$ an onto homomorphism and I a hyperideal of A. Then $\varphi(I)$ is a hyperideal of B.

<u>Proof</u> : Since I is a hyperideal of A, we have that I is a subhyperring of A (see Chapter I, page 8). By Proposition 2.10, $\varphi(I)$ is a subhyperring of B. In particular, $\varphi(I)$ is a subhypergroup of B under addition. Since φ is onto, $B = \varphi(A)$. We have that $AI \subseteq I$

and $IA \subseteq I$ since I is a hyperideal of A. These imply that

$B\varphi(I) = \varphi(A)\varphi(I) = \varphi(AI) \subseteq \varphi(I)$ and $\varphi(I)B = \varphi(I)\varphi(A) = \varphi(IA) \subseteq \varphi(I)$.

Hence $\varphi(I)$ is a hyperideal of B.

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Theorem 2.18. Let A and B be hyperrings, $\varphi : A \to B$ an onto homomorphism and I a hyperideal of A such that $\ker \varphi \subseteq I$. Then $A/I \cong B/\varphi(I)$.

Proof : By Lemma 2.17, $\varphi(I)$ is a hyperideal of B. Define $\Phi : A \to B/\varphi(I)$ by

$$\Phi(x) = \varphi(x) + \varphi(I)$$

for all $x \in A$. Since φ is onto, Φ is onto. To show that Φ is a homomorphism, let $x, y \in A$. Then

$$\begin{aligned}\Phi(x + y) &= \{\Phi(k) \mid k \in x + y\} \\ &= \{\varphi(k) + \varphi(I) \mid k \in x + y\} \\ &= \{z + \varphi(I) \mid z \in \varphi(x + y)\} \\ &= \{z + \varphi(I) \mid z \in \varphi(x) + \varphi(y)\} \\ &= (\varphi(x) + \varphi(I)) + (\varphi(y) + \varphi(I)) \\ &= \Phi(x) + \Phi(y)\end{aligned}$$

and

$$\begin{aligned}\Phi(xy) &= \varphi(xy) + \varphi(I) \\ &= \varphi(x)\varphi(y) + \varphi(I) \\ &= (\varphi(x) + \varphi(I))(\varphi(y) + \varphi(I)) \\ &= \Phi(x)\Phi(y).\end{aligned}$$

By Theorem 2.16, $A/\ker \Phi \cong B/\varphi(I)$.

By Corollary 2.13, $\varphi(x') = (\varphi(x))'$ for all $x \in A$ since φ is onto. Next, we claim that $\ker \Phi = I$. To prove this, let $x \in \ker \Phi$. Then $\Phi(x) = \varphi(I)$. But $\Phi(x) = \varphi(x) + \varphi(I)$, so $\varphi(x) \in \varphi(I)$. Then there exists an element $y \in I$ such that $\varphi(x) = \varphi(y)$. This implies that $0 \in \varphi(y) + (\varphi(y))' = \varphi(x) + \varphi(y') = \varphi(x + y')$. Thus there exists an element $z \in x + y'$ such that $\varphi(z) = 0$, so $z \in \ker \varphi$. Since A is reversible under addition and $z \in x + y'$, we have that $x \in z + y$. Therefore $x \in \ker \varphi + I \subseteq I + I \subseteq I$ since $\ker \varphi \subseteq I$. Hence we get $\ker \Phi \subseteq I$. If $x \in I$, then $\Phi(x) = \varphi(x) + \varphi(I) = \varphi(I)$ which implies that $x \in \ker \Phi$. This proves that $I \subseteq \ker \Phi$. Hence we have the claim. From $A/\ker \Phi \cong B/\varphi(I)$ and $\ker \Phi = I$, we obtain that $A/I \cong B/\varphi(I)$ as required.

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<u>Lemma 2.19</u>. If I and J are hyperideals of a hyperring A such that $I \subseteq J$, then J/I is a hyperideal of A/I.

<u>Proof</u> : It follows directly from Lemma 2.17 and the fact that the map $x \to x + I$ is a homomorphism of A onto A/I.

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<u>Lemma 2.20</u>. If I and J are hyperideals of a hyperring A such that $I \subseteq J$, then the map $\varphi : A/I \to A/J$ defined by $\varphi(x + I) = x + J$ $(x \in A)$ is an onto homomorphism.

<u>Proof</u> : First, to show that φ is well-defined, let $x, y \in A$ be such that $x + I = y + I$. Then $x \in y + I$. Since $I \subseteq J$, $y + I \subseteq y + J$. Therefore $x \in y + J$ which implies that $x + J = y + J$ (see Chapter I, page 8). The map φ is clearly onto.

To show that φ is a homomorphism, let $x, y \in A$. Then

$$\begin{aligned} \varphi((x + I) + (y + I)) &= \varphi(\{z + I \mid z \in x + y\}) \\ &= \{\varphi(z + I) \mid z \in x + y\} \\ &= \{z + J \mid z \in x + y\} \\ &= (x + J) + (y + J) \\ &= \varphi(x + I) + \varphi(y + I) \end{aligned}$$

and

$$\begin{aligned}\varphi((x + I)(y + I)) &= \varphi(xy + I)\\ &= xy + J\\ &= (x + J)(y + J)\\ &= \varphi(x + I)\varphi(y + I).\end{aligned}$$

#

Theorem 2.21. If I and J are hyperideals of a hyperring A such that $I \subseteq J$, then $(A/I)/(J/I) \cong A/J$.

Proof : By Lemma 2.19, J/I is a hyperideal of A/I. By Lemma 2.20, the map $\varphi : A/I \to A/J$ defined by $\varphi(x + I) = x + J$ $(x \in A)$ is an onto homomorphism. By Theorem 2.16, we have that $(A/I)/\ker\varphi \cong A/J$. To prove that $\ker\varphi = J/I$, let $x + I \in \ker\varphi$ $(x \in A)$. Then $\varphi(x + I) = J$, so $x + J = J$ which implies that $x \in J$. Thus $x + I \in J/I$. Hence $\ker\varphi \subseteq J/I$. Conversely, let $y \in J$. Then $y + J = J$ and therefore $\varphi(y + I) = y + J = J$ which implies that $y + I \in \ker\varphi$. Hence we have that $\ker\varphi = J/I$. Since $(A/I)/\ker\varphi \cong A/J$, we obtain that $(A/I)/(J/I) \cong A/J$.

#

Theorem 2.22. If I and J are hyperideals of a hyperring A, then $I/(I \cap J) \cong (I + J)/J$.

Proof : By Proposition 2.9, $I \cap J$ and $I + J$ are hyperideals of A. Since $I \cap J \subseteq I$, $I \cap J$ is a hyperideal of I. We have that J is a hyperideal of $I + J$ since J is a hyperideal of A and $J \subseteq I + J$. Define $\varphi : I \to (I + J)/J$ by

$$\varphi(x) = x + J$$

for all $x \in I$. If $x \in I$ and $y \in J$, then $(x + y) + J = x + (y + J)$ $= x + J = \varphi(x)$. This proves that φ is onto. Since the map $x \to x + J$

is a homomorphism of A onto A/J (see Chapter I, page 8), it follows that φ is a homomorphism. By Theorem 2.16, we obtain that $I/\ker\varphi \cong (I + J)/J$. Next, to show that $\ker\varphi = I \cap J$, let $x \in \ker\varphi$. Then $x \in I$ and $\varphi(x) = J$. But $\varphi(x) = x + J$, so $x + J = J$ which implies that $x \in J$. Therefore $x \in I \cap J$. Hence $\ker\varphi \subseteq I \cap J$. If $x \in I \cap J$, then $J = x + J = \varphi(x)$, so $x \in \ker\varphi$. Thus $I \cap J \subseteq \ker\varphi$. Now, we have that $\ker\varphi = I \cap J$. From $I/\ker\varphi \cong (I + J)/J$, it follows that $I/(I \cap J) \cong (I + J)/J$.
#

It is well-known in ring theory that if I is an ideal of a ring R, then there is a bijection from the set of all subrings of R containing I onto the set of all subrings of R/I such that the bijection takes the set of all ideals of R containing I onto the set of all ideals of R/I. The next theorem generalizes this result in terms of hyperrings.

Theorem 2.23. If I is a hyperideal of a hyperring A, then there exists a bijection from the set of all subhyperrings of A containing I onto the set of all subhyperrings of A/I having I as their zero such that the bijection takes the set of all hyperideals of A containing I onto the set of all hyperideals of A/I.

Proof : Let $\mathcal{A}$ be the set of all subhyperrings of A containing I and let $\mathcal{B}$ be the set of all subhyperrings of A/I having I as their zero. Define $\Phi : \mathcal{A} \to \mathcal{B}$ by

$$\Phi(S) = S/I$$

for all $S \in \mathcal{A}$.

To show that Φ is 1-1, let $S_1, S_2 \in \mathcal{A}$ be such that $\Phi(S_1) = \Phi(S_2)$.

Then $S_1/I = S_2/I$. Let $x \in S_1$. Then $x + I \in S_1/I$. Since $S_1/I = S_2/I$, $x + I \in S_2/I$, so there exists an element $y \in S_2$ such that $x + I = y + I$. Thus $x \in y + I \subseteq S_2 + S_2 \subseteq S_2$. Hence $S_1 \subseteq S_2$. By a similar proof, we have that $S_2 \subseteq S_1$. Hence $S_1 = S_2$.

Next, to show that Φ is onto, let $S^* \in \mathcal{B}$. Set

$$S = \{x \in A \mid x + I \in S^*\}.$$

Since $I \in S^*$ and $x + I = I$ for all $x \in I$, it follows that $I \subseteq S$. Let $x, y \in S$. Then $x + I, y + I \in S^*$. Since S^* contains I as its zero, by Corollary 2.3, $(x + I) + (y' + I) = \{z + I \mid z \in x + y'\} \subseteq S^*$ and $(x + I)(y + I) = xy + I \in S^*$. These imply that $x + y' \subseteq S$ and $xy \in S$. By Corollary 2.3, S is a subhyperring of A containing 0. It is clear that $S/I = S^*$, that is, $\Phi(S) = S^*$.

If J is a hyperideal of A containing I, then by Lemma 2.19, $\Phi(J) = J/I$ is a hyperideal of A/I.

Let J^* be a hyperideal of A/I. Since Φ is onto, there exists a subhyperring J of A containing I such that $\Phi(J) = J^*$. Then $J/I = J^*$. Let $x \in A$ and $y \in J$. Then $x + I \in A/I$ and $y + I \in J/I$, so $xy + I = (x + I)(y + I) \in (A/I)(J/I) \subseteq J/I$ since $J/I = J^*$ is a hyperideal of A/I. Thus there exists an element $z \in J$ such that $xy + I = z + I$ which implies that $xy \in z + I$. Since $z \in J$ and $I \subseteq J$, we have that $xy \in J + J \subseteq J$. Similarly, $yx \in J$. Hence J is a hyperideal of A and $\Phi(J) = J^*$.

#

It is known that every field has exactly two ideals and any commutative ring with identity which has exactly two ideals is a field. This is true for the case of hyperrings which is shown by the next theorem. In order to prove the theorem, we need the following lemma and the fact that if S is a semigroup with zero 0 and identity $1 \neq 0$

such that for $x \in S \setminus \{0\}$, $xy = yx = 1$ for some $y \in S$, then $S \setminus \{0\}$ forms a group under the operation of S.

Lemma 2.24. If A is a hyperring, then for each $x \in A$, Ax [xA] is a left [right] hyperideal of A.

Proof : Let $x \in A$. Then for $a, b \in A$, $ax + (bx)' = ax + b'x = (a + b')x \subseteq Ax$. By Proposition 2.2, Ax is a canonical subhypergroup of A under addition. Also, we have that $A(Ax) = (AA)x \subseteq Ax$. Hence Ax is a left hyperideal of A.

#

Theorem 2.25. Let A be a commutative hyperring with identity and $|A| > 1$. Then A is a hyperfield if and only if $\{0\}$ and A are the only hyperideals of A.

Proof : First, assume that A is a hyperfield. A and $\{0\}$ are hyperideals of A. Let I be a hyperideal of A such that $I \neq \{0\}$. Then there exists an element $x \in I$ such that $x \neq 0$. Since A is a hyperfield, the multiplicative inverse of x, x^{-1} exists in A. Then $1 = xx^{-1} \in I$ which implies that $I = A$.

For the converse, assume that $\{0\}$ and A are the only hyperideals of A. To show that A is a hyperfield, as mentioned above, it suffices to show that for each $x \in A$, $x \neq 0$, there exists an element $y \in A$ such that $xy = 1$. To prove this, let $x \in A$ and $x \neq 0$. By Lemma 2.24 and the commutativity of A, xA is a hyperideal of A. Then by assumption, we have that $xA = \{0\}$ or $xA = A$. Since $0 \neq x = x1 \in xA$, $xA \neq \{0\}$, so $xA = A$. But $1 \in A$, it follows that there exists an element $y \in A$ such that $xy = 1$.

#

The following five theorems and one corollary about hyperrings concerning maximal hyperideals and prime hyperideals generalize standard theorems of rings relating to maximal ideals and prime ideals.

Theorem 2.26. Let A be a commutative hyperring with identity and M a hyperideal of A. Then M is a maximal hyperideal of A if and only if A/M is a hyperfield.

Proof : We note that $1 + M$ is the identity of the quotient hyperfield A/M and the nonzero elements of A/M are $x + M$ for $x \in A \setminus M$. Since A is commutative, A/M is commutative.

Assume that M is a maximal hyperideal of A. Then A/M is a commutative hyperring with identity $1 + M$ and $1 + M \neq M$. To show that A/M is a hyperfield, by the fact on page 43, it suffices to show that for $x \in A \setminus M$, there exists an element $y \in A$ such that $1 \in xy + M$ (which is equivalent to $(x + M)(y + M) = 1 + M$). Let $x \in A \setminus M$. By Lemma 2.24 and the commutativity of A, we have that xA is a hyperideal of A. It then follows from Proposition 2.9, $xA + M$ is a hyperideal of A. Since $1 \in A$ and $0 \in M$, we get $x \in xA + M$. But $M \subseteq xA + M$ and $x \notin M$, by assumption, we have that $xA + M = A$. Since $1 \in A$, it follows that there exists an element $y \in A$ such that $1 \in xy + M$.

Conversely, assume that A/M is a hyperfield. Then $A \neq M$. Let K be a hyperideal of A such that $M \subseteq K$ and $M \neq K$. Thus there exists an element $x \in K \setminus M$. Since A/M is a hyperfield, the multiplicative inverse of $x + M$ exists in A/M. Then there exists an element $y \in A$ such that $1 + M = (x + M)(y + M)$, so $1 + M = xy + M$ which implies that $1 \in xy + M$. Since K is a hyperideal of A and $x \in K$, $xy \in K$. Then $1 \in K + M \subseteq K + K \subseteq K$ since $M \subseteq K$. Hence $K = A$. This proves that M is a maximal hyperideal of A.

#

Theorem 2.27. Let A be a commutative hyperring and P a hyperideal of A. Then P is a prime hyperideal of A if and only if A/P is a hyperintegral domain.

Proof : Assume that P is a prime hyperideal of A. Let $x, y \in A$. be such that $(x + P)(y + P) = P$. Then $xy + P = P$, so $xy \in P$. Since P is a prime hyperideal of A, $x \in P$ or $y \in P$. Thus $x + P = P$ or $y + P = P$. This proves that A/P is a hyperintegral domain.

Conversely, assume that A/P is a hyperintegral domain. Let $x, y \in A$ be such that $xy \in P$. Then $(x + P)(y + P) = xy + P = P$. Since A/P is a hyperintegral domain, $x + P = P$ or $y + P = P$. Therefore $x \in P$ or $y \in P$. Hence P is a prime hyperideal of A.

#

As a consequence of Theorem 2.26 and Theorem 2.27, we have the following corollary.

Corollary 2.28. In a commutative hyperring A with identity, every maximal hyperideal of A is a prime hyperideal of A.

In the ring of integers, {0} is a proper prime ideal but not a maximal ideal. This shows that the converse of Corollary 2.28 is not true. The converse of Corollary 2.28 is true in Boolean hyperrings as follows :

Theorem 2.29. Let I be a hyperideal of a Boolean hyperring A with identity. If I is a proper prime hyperideal of A, then I is a maximal hyperideal of A.

Proof : First, we note that for $x \in A$, $x' = x$ since $x' = x'x' = xx = x$. Hence $0 \in x + x$ for all $x \in A$.

To show that I is a maximal hyperideal of A, let K be a hyperideal of A such that $I \subseteq K$ and $I \neq K$. Let $x \in K \setminus I$. Since $0 \in x + x = x1 + xx = x(1 + x)$, there exists an element $y \in 1 + x$ such that $xy = 0$, so $xy \in I$. Since I is a prime hyperideal of A and $x \notin I$, $y \in I$ which implies that $y \in K$ since $I \subseteq K$. From $y \in 1 + x$, by reversibility of A under addition, we have that $1 \in y + x'$. But $y + x' = y + x \subseteq K$, so $1 \in K$. Hence $K = A$.

#

Theorem 2.30. In a hyperring A with identity, each proper hyperideal of A is contained in a maximal hyperideal of A.

Proof : Let I be a proper hyperideal of A and let

$$\mathscr{C} = \{J \mid I \subseteq J \text{ and } J \text{ is a proper hyperideal of } A\}.$$

Since $I \in \mathscr{C}$, $\mathscr{C} \neq \phi$. Partially order $\mathscr{C}$ by inclusion. Let $\mathscr{M}$ be a chain in $\mathscr{C}$. Let $K = \bigcup_{J \in \mathscr{M}} J$. Clearly, $I \subseteq K$. Let $x, y \in K$. Then there exist $J_1, J_2 \in \mathscr{M}$ such that $x \in J_1$ and $y \in J_2$. Without loss of generality, assume that $J_1 \subseteq J_2$. Thus $x, y \in J_2$, so $x + y' \subseteq J_2 \subseteq K$. By Proposition 2.2, K is a canonical subhypergroup of A under addition. It is clear that $KA \subseteq K$ and $AK \subseteq K$. Then K is a hyperideal of A. Since for each $J \in \mathscr{C}$, $1 \notin J$, it follows that $1 \notin K$. Therefore $K \in \mathscr{C}$ and K is an upper bound of $\mathscr{M}$. By Zorn's Lemma, $\mathscr{C}$ has a maximal element, say M. Then $I \subseteq M$. Clearly, M is a maximal hyperideal of A containing I.

#

Theorem 2.31. Let A be a commutative hyperring with identity having exactly one maximal hyperideal M. If $x \in A$ is such that $x^2 = x$, then x is either 0 or 1.

Proof : Let $x \varepsilon A$ be such that $x^2 = x$. Then $0 \varepsilon x + x'$ $= x + (xx)' = x + xx' = x(1 + x')$, so there exists an element $y \varepsilon 1 + x'$ such that $xy = 0$. By reversibility of A under addition, $1 \varepsilon y + x$. By Lemma 2.24 and the commutativity of A, xA and yA are hyperideals of A. If xA and yA are proper hyperideals of A, by Theorem 2.30 and M is the only one maximal hyperideal of A, we have that $xA \subseteq M$ and $yA \subseteq M$ which imply that $1 \varepsilon y + x \subseteq yA + xA \subseteq M + M \subseteq M$ which is a contradiction since $M \neq A$. Then $xA = A$ or $yA = A$. If $xA = A$, then $1 = xz$ for some $z \varepsilon A$, so $x = x1 = xxz = xz = 1$. If $yA = A$, then $1 = yw$ for some $w \varepsilon A$ which implies that $x = x1 = xyw = 0$ since $xy = 0$.

Hence the theorem is proved.

#

It is immediately seen in any commutative ring R that R is an integral domain if and only if R is multiplicatively cancellative. We need some verification to show that this is also true in hyperrings.

Proposition 2.32. Let A be a commutative hyperring. Then A is a hyperintegral domain if and only if A is multiplicatively cancellative, that is, for $x, y, z \varepsilon A$, $xy = xz$ and $x \neq 0$ imply $y = z$.

Proof : Assume that A is a hyperintegral domain. Let $x, y, z \varepsilon A$ be such that $xy = xz$ and $x \neq 0$. Then $0 \varepsilon xy + (xy)' =$ $= xy + (xz)' = xy + xz' = x(y + z')$, so there exists an element $u \varepsilon y + z'$ such that $xu = 0$. Since A is a hyperintegral domain and $x \neq 0$, we have that $u = 0$. Thus $0 \varepsilon y + z'$. Hence $y = z$.

For the converse, assume that A is multiplicatively cancellative. Let $x, y \varepsilon A$ be such that $xy = 0$ and $x \neq 0$. Then $xy = x0$ and $x \neq 0$. Since A is multiplicatively cancellative, $y = 0$. This proves that A is a hyperintegral domain.

#

As in ring theory, we can prove by Proposition 2.32 that every finite hyperintegral domain containing more than one element is a hyperfield.

Theorem 2.33. A finite hyperintegral domain of order greater than 1 is a hyperfield.

Proof : Let A be a finite hyperintegral domain such that $|A| > 1$. Let $A^* = A \setminus \{0\} = \{x_1,\ldots,x_n\}$, $x_i \neq x_j$ if $i \neq j$. Since in A, $xy = 0$ implies $x = 0$ or $y = 0$, it follows that A^* is closed under multiplication. To show that A^* is a group under multiplication, it is equivalent to show that $xA^* = A^* = A^*x$ for all $x \in A^*$. Let $x \in A^*$. Then $xA^* = \{xx_1,\ldots,xx_n\}$. If $i, j \in \{1,2,\ldots,n\}$ are such that $xx_i = xx_j$, then by Proposition 2.32, $x_i = x_j$. Therefore $|xA^*| = n$ But $xA^* \subseteq A^*$ and $|xA^*| = |A^*| = n$, so we have that $xA^* = A^*$. Since A is commutative, $A^*x = xA^* = A^*$. Therefore A^* is a group under multiplication. Hence A is a hyperfield.

#

An integral domain has the feature that we can enlarge it to its field of quotients. The proof of the following theorem shows that we can perform a similar construction for any hyperintegral domain.

Theorem 2.34. Every hyperintegral domain can be embedded in a hyperfield.

Proof : Let A be a hyperintegral domain. If $A = \{0\}$, then A can be embedded in every hyperfield. Assume that $|A| > 1$. Define the relation $\sim$ on $A \times (A \setminus \{0\})$ by

$$(a,b) \sim (c,d) \Longleftrightarrow ad = bc$$

for all $(a,b), (c,d) \in A \times (A \setminus \{0\})$. Clearly $\sim$ is reflexive on $A \times (A \setminus \{0\})$. Since A is commutative, $\sim$ is symmetric. To show that $\sim$ is transitive, let $(a,b), (c,d), (e,f) \in A \times (A \setminus \{0\})$ be such that $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$. Then $ad = bc$ and $cf = de$, so $adf = bcf$ and $bcf = bde$. Thus $adf = bde$. Since A is a hyperintegral domain and $d \neq 0$, we have by Proposition 2.32 that $af = be$ which implies that $(a,b) \sim (e,f)$.

Therefore $\sim$ is transitive. Hence $\sim$ is an equivalence relation on $A \times (A \setminus \{0\})$.

It is easy to see that for $a \in A$, $b, c \in A \setminus \{0\}$,

(1) $(a,b) \sim (0,c) \Longleftrightarrow a = 0$,

(2) $(a,b) \sim (c,c) \Longleftrightarrow a = b$ and

(3) $(a,b) \sim (ax,bx)$ for all $x \in A \setminus \{0\}$.

Let $F = (A \times (A \setminus \{0\}))/\sim = \{[(a,b)] \mid (a,b) \in A \times (A \setminus \{0\})\}$ where $[(a,b)]$ is the equivalence class of $\sim$ containing (a,b). Define the hyperoperation $\oplus$ and the operation $\odot$ on F by

$$[(a,b)] \oplus [(c,d)] = \{[(x,bd)] \mid x \in ad + bc\}$$

and

$$[(a,b)] \odot [(c,d)] = [(ac,bd)]$$

for all $(a,b), (c,d) \in A \times (A \setminus \{0\})$. Claim that $(F,\oplus,\odot)$ is a hyperfield and A can be embedded in $(F,\oplus,\odot)$.

To show that $\oplus$ and $\odot$ are well-defined, let $(a,b), (a^*,b^*)$, $(c,d), (c^*,d^*) \in A \times (A \setminus \{0\})$ be such that $[(a,b)] = [(a^*,b^*)]$ and $[(c,d)] = [(c^*,d^*)]$. Then

$$[(a,b)] \oplus [(c,d)] = \{[(x,bd)] \mid x \in ad + bc\},$$

$$[(a^*,b^*)] \oplus [(c^*,d^*)] = \{[(x,b^*d^*)] \mid x \in a^*d^* + b^*c^*\},$$

$$[(a,b)] \odot [(c,d)] = [(ac,bd)] \text{ and}$$

$$[(a^*,b^*)]\odot[(c^*,d^*)] \;=\; [(a^*c^*,b^*d^*)].$$

Since $[(a,b)] = [(a^*,b^*)]$ and $[(c,d)] = [(c^*,d^*)]$, it follows that $(a,b) \sim (a^*,b^*)$ and $(c,d) \sim (c^*,d^*)$ which imply that $ab^* = ba^*$ and $cd^* = dc^*$. Then $(ad + bc)b^*d^* = ab^*dd^* + bcd^*b^* = ba^*dd^* + bdc^*b^* = (a^*d^* + b^*c^*)bd$. To show that $[(a,b)] \oplus [(c,d)] = [(a^*,b^*)] \oplus [(c^*,d^*)]$, let $y \varepsilon ad + bc$, that is, $[(y,bd)] \varepsilon [(a,b)] \oplus [(c,d)]$. Then $yb^*d^* \varepsilon (ad + bc)b^*d^*$. But $(ad + bc)b^*d^* = (a^*d^* + b^*c^*)bd$, so we have that $yb^*d^* \varepsilon (a^*d^* + b^*c^*)bd$. This implies that there exists an element $z \varepsilon a^*d^* + b^*c^*$ such that $yb^*d^* = zbd$. Then $(y,bd) \sim (z,b^*d^*)$ and hence $[(y,bd)] = [(z,b^*d^*)]$. Since $z \varepsilon a^*d^* + b^*c^*$, $[(z,b^*d^*)] \;\varepsilon\; [(a^*,b^*)] \oplus [(c^*,d^*)]$. It then follows that $[(y,bd)] \varepsilon [(a^*,b^*)] \oplus [(c^*,d^*)]$. This proves that $[(a,b)] \oplus [(c,d)] \subseteq [(a^*,b^*)] \oplus [(c^*,d^*)]$. The inclusion $[(a^*,b^*)] \oplus [(c^*,d^*)] \subseteq [(a,b)] \oplus [(c,d)]$ can be proved similarly. Hence $[(a,b)] \oplus [(c,d)] = [(a^*,b^*)] \oplus [(c^*,d^*)]$. From $ab^* = ba^*$ and $cd^* = dc^*$, we have that $ab^*cd^* = ba^*dc^*$ which implies that $(ac,bd) \sim (a^*c^*,b^*d^*)$. Then $[(ac,bd)] = [(a^*c^*,b^*d^*)]$ and hence $[(a,b)]\odot[(c,d)] = [(a^*,b^*)]\odot[(c^*,d^*)]$.

Since A is commutative, the hyperoperation $\oplus$ and the operation $\odot$ are clearly commutative on F. To show that $\oplus$ and $\odot$ are associative on F, let $[(a,b)], [(c,d)], [(e,f)] \varepsilon F$. Then

$$
\begin{aligned}
([(a,b)] \oplus [(c,d)]) \oplus [(e,f)] &= \{[(x,bd)] \mid x \varepsilon ad + bc\} \oplus [(e,f)] \\
&= \bigcup_{y \varepsilon ad+bc} ([(y,bd)] \oplus [(e,f)]) \\
&= \bigcup_{y \varepsilon ad+bc} \{[(x,bdf)] \mid x \varepsilon yf + bde\} \\
&= \{[(x,bdf)] \mid x \varepsilon (ad + bc)f + bde\} \\
&= \{[(x,bdf)] \mid x \varepsilon adf + b(cf + de)\}
\end{aligned}
$$

$$= \bigcup_{y \varepsilon cf+de} \{[(x,bdf)] \mid x \varepsilon adf + by\}$$

$$= \bigcup_{y \varepsilon cf+de} ([(a,b)] \oplus [(y,df)])$$

$$= [(a,b)] \oplus \{[(x,df)] \mid x \varepsilon cf + de\}$$

$$= [(a,b)] \oplus ([(c,d)] \oplus [(e,f)]) \text{ and}$$

$$([(a,b)]\odot[(c,d)])\odot[(e,f)] = [(ac,bd)]\odot[(e,f)]$$

$$= [((ac)e,(bd)f)]$$

$$= [(a(ce),b(df))]$$

$$= [(a,b)]\odot[(ce,df)]$$

$$= [(a,b)]\odot([(c,d)]\odot[(e,f)]).$$

If $u \varepsilon A \setminus \{0\}$ and $(a,b) \varepsilon A \times (A \setminus \{0\})$, then

$$[(a,b)] \oplus [(0,u)] = \{[(x,bu)] \mid x \varepsilon au + b0\}$$

$$= \{[(x,bu)] \mid x \varepsilon \{au\}\}$$

$$= \{[(au,bu)]\}$$

$$= \{[(a,b)]\} \text{ (from (3)).}$$

This proves that for $u \varepsilon A \setminus \{0\}$, $[(0,u)]$ is the scalar identity of the semihypergroup $(F,\oplus)$.

To show that for $[(a,b)] \varepsilon F$, $[(a',b)]$ is the unique inverse of $[(a,b)]$ in $(F,\oplus)$, let $[(a,b)] \varepsilon F$. Then

$$[(a,b)] \oplus [(a',b)] = \{[(x,b^2)] \mid x \varepsilon ab + ba'\}$$

$$= \{[(x,b^2)] \mid x \varepsilon ab + (ab)'\}.$$

Since $0 \varepsilon ab + (ab)'$, $[(0,b^2)] \varepsilon [(a,b)] \oplus [(a',b)]$. But $[(0,b^2)]$ is the scalar identity of $(F,\oplus)$, thus $[(a',b)]$ is an inverse of $[(a,b)]$ in $(F,\oplus)$. Let $[(c,d)]$ be an inverse of $[(a,b)]$ in $(F,\oplus)$. Since $[(0,b)]$ is the scalar identity of $(F,\oplus)$, $[(0,b)]$ is the unique identity of $(F,\oplus)$. Then $[(0,b)] \varepsilon [(a,b)] \oplus [(c,d)]$. But $[(a,b)] \oplus [(c,d)] = \{[(x,bd)] \mid x \varepsilon ad + bc\}$, so there exists an element $y \varepsilon ad + bc$ such that $[(0,b)] = [(y,bd)]$. Thus $(0,b) \sim (y,bd)$, so $by = 0$.

Since A is a hyperintegral domain and $b \neq 0$, $y = 0$, so $0 \in ad + bc$. This implies that $bc = (ad)' = a'd$, thus $(a',b) \sim (c,d)$. Hence $[(a',b)] = [(c,d)]$.

To show that $(F,\oplus)$ is reversible, let $[(a,b)]$, $[(c,d)]$, $[(e,f)] \in F$ be such that $[(a,b)] \in [(c,d)] \oplus [(e,f)]$. From the previous proof, $[(c',d)]$ is the unique inverse of $[(c,d)]$ in $(F,\oplus)$. Since $[(c,d)] \oplus [(e,f)] = \{[(x,df)] \mid x \in cf + de\}$, $[(a,b)] = [(y,df)]$ for some $y \in cf + de$. Then $(a,b) \sim (y,df)$, so $adf = by$. Since $(A,+)$ is reversible and $y \in cf + de$, it follows that $de \in (cf)' + y$. Then $ed \in c'f + y$, and so $edb \in (c'f + y)b = c'fb + yb = c'fb + adf$ $= f(c'b + ad)$. Therefore $edb = fz$ for some $z \in c'b + ad$. Then $(e,f) \sim (z,db)$, so $[(e,f)] = [(z,db)]$. But $[(c',d)] \oplus [(a,b)]$ $= \{[(x,db)] \mid x \in c'b + ad\}$, we have that $[(e,f)] \in [(c',d)] \oplus [(a,b)]$.

Now, we have that $(F,\oplus)$ is a canonical hypergroup with scalar identity $[(0,u)]$ where $u \in A \setminus \{0\}$, for each $[(a,b)] \in F$, $[(a',b)]$ is the unique inverse of $[(a,b)]$ in $(F,\oplus)$ and $(F,\odot)$ is a commutative semigroup.

If $u \in A \setminus \{0\}$ and $[(a,b)] \in F$, the $[(a,b)]\odot[(0,u)] = [(0,bu)]$ $= [(0,u)]$ (from (1)). This shows that the scalar identity of the hypergroup $(F,\oplus)$ is the zero of the semigroup $(F,\odot)$. For $[(a,b)]$, $[(c,d)]$, $[(e,f)] \in F$, we have that

$$
\begin{aligned}
[(a,b)]\odot([(c,d)] \oplus [(e,f)]) &= [(a,b)]\odot\{[(x,df)] \mid x \in cf + de\} \\
&= \{[(a,b)]\odot[(x,df)] \mid x \in cf + de\} \\
&= \{[(ax,bdf)] \mid x \in cf + de\} \\
&= \{[(axb,bdfb)] \mid x \in cf + de\} \quad \text{(from (3))} \\
&= \{[(abx,bdbf)] \mid x \in cf + de\} \\
&= \{[(y,bdbf)] \mid y \in ab(cf + de)\}
\end{aligned}
$$

$$= \{[(y,bdbf)] \mid y \in acbf + bdae\}$$
$$= [(ac,bd)] \oplus [(ae,bf)]$$
$$= ([(a,b)]\odot[(c,d)]) \oplus ([(a,b)]\odot[(e,f)]).$$

Therefore $(F,\oplus,\odot)$ is a hyperring. It follows from (3) that for $x \in A \setminus \{0\}$, $[(a,b)]\odot[(x,x)] = [(a,b)]$ for all $[(a,b)] \in F$ and if $[(a,b)] \in F$ and $a \neq 0$, then $[(a,b)]\odot[(b,a)] = [(ab,ba)] = [(ab,ab)]$. This proves that $[(x,x)]$ where $x \in A \setminus \{0\}$ is the identity of the semigroup $(F,\odot)$ and every nonzero element $[(a,b)]$ of F has $[(b,a)]$ as a multiplicative inverse. Hence we proved that $(F,\oplus,\odot)$ is a hyperfield, as required. Also, we have that for $x \in A \setminus \{0\}$, $[(0,x)]$ is the zero and $[(x,x)]$ is the identity of $(F,\oplus,\odot)$ and for nonzero element $[(a,b)]$ of $(F,\oplus,\odot)$ has $[(b,a)]$ as its multiplicative inverse.

Next, we shall show that A can be embedded in F. Let $k \in A \setminus \{0\}$ and define $\varphi : A \to F$ by

$$\varphi(x) = [(xk,k)]$$

for all $x \in A$. To show that φ is a homomorphism, let $a, b \in A$. Then

$$\begin{aligned}
\varphi(a + b) &= \{\varphi(x) \mid x \in a + b\} \\
&= \{[(xk,k)] \mid x \in a + b\} \\
&= \{[(xk^2,k^2)] \mid x \in a + b\} \quad \text{(from (3))} \\
&= \{[(y,k^2)] \mid y \in (a + b)k^2\} \\
&= \{[(y,k^2)] \mid y \in ak^2 + bk^2\} \\
&= [(ak,k)] \oplus [(bk,k)] \\
&= \varphi(a) \oplus \varphi(b)
\end{aligned}$$

and

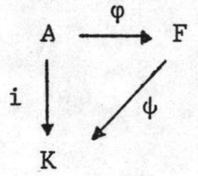
$$\begin{aligned}
\varphi(ab) &= [(abk,k)] \\
&= [(abk^2,k^2)] \quad \text{(from (3))} \\
&= [(ak,k)]\odot[(bk,k)] \\
&= \varphi(a)\odot\varphi(b).
\end{aligned}$$

To prove that φ is 1-1, let a, b ε A be such that $\varphi(a) = \varphi(b)$. Then $[(ak,k)] = [(bk,k)]$. Thus $ak^2 = bk^2$. Since A is a hyperintegral domain and $k \neq 0$, we have by Proposition 2.32 that a = b. Hence φ is 1-1.

Therefore, the theorem is completely proved.

#

The hyperfield $(F,\oplus,\odot)$ constructed from the hyperintegral domain A with $|A| > 1$ in the proof of Theorem 2.34 is in fact the smallest hyperfield containing A, that is, for any hyperfield K and any 1-1 homomorphism $i : A \to K$, there exists a unique 1-1 homomorphism $\psi : F \to K$ such that $i = \psi o \varphi$. To prove this, let K be a hyperfield and $i : A \to K$ a 1-1 homomorphism. Since i is 1-1 homomorphism,



we have from Corollary 2.15 that $i(0) = 0$ which implies that $i(x) \neq 0$ for all $x \; \varepsilon \; A \setminus \{0\}$. Define $\psi : F \to K$ by

$$\psi([(a,b)]) = i(a)(i(b))^{-1}$$

for all $[(a,b)] \; \varepsilon \; F$. Let $[(a,b)], [(c,d)] \; \varepsilon \; F$ be such that $[(a,b)] = [(c,d)]$. Then ad = bc. Therefore

$$\begin{aligned}
\psi[(a,b)] &= i(a)(i(b))^{-1} \\
&= i(a)(i(b))^{-1}i(d)(i(d))^{-1} \\
&= i(ad)(i(b))^{-1}(i(d))^{-1} \\
&= i(bc)(i(b))^{-1}(i(d))^{-1} \\
&= i(b)i(c)(i(b))^{-1}(i(d))^{-1} \\
&= i(c)(i(d))^{-1} \\
&= \psi[(c,d)].
\end{aligned}$$

Hence ψ is well-defined.

To show that ψ is 1-1, let $[(a,b)], [(c,d)] \; \varepsilon \; F$ be such that $\psi([(a,b)] = \psi([(c,d)])$. Then $i(a)(i(b))^{-1} = i(c)(i(d))^{-1}$, so

$i(a)i(d) = i(b)i(c)$. Thus $i(ad) = i(a)i(d) = i(b)i(c) = i(bc)$. Since i is 1-1, $ad = bc$. Then $(a,b) \sim (c,d)$. Hence $[(a,b)] = [(c,d)]$.

To show that ψ is a homomorphism, let $[(a,b)], [(c,d)] \varepsilon F$. Then

$$\begin{aligned}
\psi([(a,b)] \oplus [(c,d)]) &= \psi(\{[(x,bd)] \mid x \varepsilon ad + bc\}) \\
&= \{\psi([(x,bd)]) \mid x \varepsilon ad + bc\} \\
&= \{i(x)(i(bd))^{-1} \mid x \varepsilon ad + bc\} \\
&= i(ad + bc)(i(bd))^{-1} \\
&= (i(ad) + i(bc))(i(b)i(d))^{-1} \\
&= i(ad)(i(b)i(d))^{-1} + i(bc)(i(b)i(d))^{-1} \\
&= i(a)i(d)(i(b))^{-1}(i(d))^{-1} \\
&\quad + i(b)i(c)(i(b))^{-1}(i(d))^{-1} \\
&= i(a)(i(b))^{-1} + i(c)(i(d))^{-1} \\
&= \psi([(a,b)]) + \psi[(c,d)])
\end{aligned}$$

and

$$\begin{aligned}
\psi([(a,b)] \odot [(c,d)]) &= \psi([(ac,bd)]) \\
&= i(ac)(i(bd))^{-1} \\
&= i(a)i(c)(i(b))^{-1}(i(d))^{-1} \\
&= i(a)(i(b))^{-1}i(c)(i(d))^{-1} \\
&= \psi([(a,b)])\psi([(c,d)]).
\end{aligned}$$

If $x \varepsilon A$, then $(\psi o \varphi)(x) = \psi(\varphi(x)) = \psi([(xk,k)]) = i(xk)(i(k))^{-1} = i(x)i(k)(i(k))^{-1} = i(x)$ where $k \varepsilon A \setminus \{0\}$. Hence $\psi o \varphi = i$.

Let $\phi : F \to K$ be a 1-1 homomorphism such that $i = \phi o \varphi$. Let $[(a,b)] \varepsilon F$. Let $k \varepsilon A \setminus \{0\}$. Then

$$\begin{aligned}
\psi([(a,b)]) &= i(a)(i(b))^{-1} \\
&= ((\phi o \varphi)(a))((\phi o \varphi)(b))^{-1} \\
&= (\phi(\varphi(a)))(\phi(\varphi(b)))^{-1}
\end{aligned}$$

$$= (\phi(\varphi(a)))(\phi((\varphi(b))^{-1}) \quad \text{(by Corollary 2.15)}$$

$$= \phi(\varphi(a) \odot (\varphi(b))^{-1})$$

$$= \phi([(ak,k)] \odot [(bk,k)]^{-1})$$

$$= \phi([(ak,k)] \odot [(k,bk)])$$

$$= \phi([(ak^2,bk^2)])$$

$$= \phi([(a,b)]) \quad \text{(from (3))}.$$

Hence $\psi = \phi$.

The hyperfield $(F,\oplus,\odot)$ will be called the hyperfield of quotients of A.

#