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APPENDIX I

Definition 1. Let Ω be a domain in R^n and $\bar{R}$ an extended real number system. A function $u: \Omega \to \bar{R}$ is said to be subharmonic in Ω if

(i) u is upper semi-continuous in Ω.

(ii) If $x_0 \in \Omega$ and $r > 0$ satisfies $\bar{B}(x_0,r) \subset \Omega$, then

$$u(x_0) \leq s_m^{-1} r^{-m+1} \int_{\partial B(x_0,r)} u(x)d\sigma(x),$$

where $d\sigma(x)$ denotes the surface area element on $\partial B(x_0,r)$ and $s_m = 2\pi^{m/2}/\Gamma(m/2)$.

A function $u: \Omega \to \bar{R}$ is said to be superharmonic in Ω if $-u$ is subharmonic in Ω.

Definition 2. If u is subharmonic (superharmonic) in a domain $\Omega \subset R^m$, h is harmonic in Ω and $u \leq h$ $(u \geq h)$ on Ω, then h is called a harmonic majorant (harmonic minorant) of u. The function v is called a least harmonic majorant (greatest harmonic minorant) of u if v is a harmonic majorant (harmonic minorant) of u and $v \leq h$ $(v \geq h)$ whenever h is a harmonic majorant (harmonic minorant) of u.

APPENDIX II

In this section, we shall state some relevant definitions and basic properties of Banach lattices that we used without stating or proving in Chapter IV. All of these materials are drawn from Schaefer ([20], Chapter II). Moreover, we shall prove one lemma that was often referred in Chapter IV.

1. Definitions and basic properties of Banach lattices

An ordered set is a set A endowed with a binary relation, usually denoted by " $\leqslant$ ", which is supposed to be transitive ($x \leqslant y$ & $y \leqslant z \Rightarrow x \leqslant z$), reflexive ($x \leqslant x$, all $x \in A$), and anti-symmetric ($x \leqslant y$ & $y \leqslant x \Rightarrow x = y$).

Let $(A, \leqslant)$ be ordered set. We write $y \geqslant x$ to mean $x \leqslant y$ and $x < y$ to express that ($x < y$ & $x \neq y$); similary for $y > x$. A subset B of A is called majorized (minorized) if there exists $a_0 \in A$ such that $b \leqslant a_0$ for all $b \in B$ (respectively, $a_0 \leqslant b$ for all $b \in B$); a_0 is called a majorant or upper bound (respectively, a minorant or lower bound)of B in A. An order interval $[x,y]$, where $x,y \in A$, is the set of all $z \in A$ satisfying $x \leqslant z \leqslant y$; a subset $C \subset A$ contained in some ordered interval of A, is called order bounded. If B is a majorized subset of A and if there exists a majorant of B that minorizes all majorants of B (in A), such an element is unique, called the supremum or least upper bound of B, and denoted by sup B; similarly for inf B (the infimum or greatest lower bound of B). Note that these concepts depend on the set A of which B is thought to be a subset; hence, sometimes the notation $\sup_A B$, $\inf_A B$ are used.

Definition 1. An ordered set $(L, \leqslant)$ is called a lattice if for each pair $(x,y) \in L \times L$, the elements $x \vee y := \sup\{x,y\}$ and $x \wedge y := \inf\{x,y\}$ exist in L.

Definition 2. A vector space E over R, endowed with an order relation $\leqslant$, is called an ordered vector space if these axioms are satisfied:

$(LO)_1$ $\quad x \leqslant y \Rightarrow x + z \leqslant y + z \quad$ for all $x,y,z \in E$,

$(LO)_2$ $\quad x \leqslant y \Rightarrow \lambda x \leqslant \lambda y \quad$ for all $x,y \in E$ and $\lambda \in R_+$.

A vector lattice is an ordered vector space such that $x \vee y$ and $x \wedge y$ exist for all $x,y \in E$.

If $(E, \leqslant)$ is an ordered vector space, the subset $E_+ := \{x \in E \mid x \geqslant 0\}$ is called the positive cone of E; elements $x \in E_+$ are called positive.

It is readily seen from $(LO)_1$ that whenever sup A exists for a non-void subset $A \subset E$ then $\inf(-A)$ exists (and conversely); moreover for each $x \in E$ we have the following:

$$x + \sup A = \sup(x + A),$$

$$x + \inf A = \inf(x + A),$$

$$\sup A = -\inf(-A).$$

Definition 3. Let E be a vector lattice. For all $x \in E$, we define $x^+ := x \vee 0$, $x^- := (-x) \vee 0$, $|x| := x \vee (-x)$. x^+, x^- and $|x|$ are called the positive part, the negative part, and the modulus (or absolute value) of x, respectively.

Proposition 4. Let E be a vector lattice. For all $x, y, x_1, y_1 \in E$ and all $\lambda \in R$, the following relations are valid.

(1) $$x = x^+ - x^-$$

(2) $$|x| = x^+ + x^-.$$

(3) $$|x| = 0 \Leftrightarrow x = 0; \quad |\lambda x| = |\lambda||x|, \quad |x+y| \leqslant |x| + |y|.$$

(4) $$x + y = x \vee y + x \wedge y$$

(5) $$|x - y| = x \vee y - x \wedge y.$$

(6) $$|x \vee y - x_1 \vee y_1| \leqslant |x - x_1| + |y - y_1|$$

(7) $$|x \wedge y - x_1 \wedge y_1| \leqslant |x - x_1| + |y - y_1|.$$

(8) $$x \wedge y = -((-x) \vee (-y))$$

(9) $$x \vee y = -((-x) \wedge (-y)).$$

(10) $$(x + y)^+ \leqslant x^+ + y^+$$

(11) $$(x + y)^- \leqslant x^- + y^-.$$

(12) $$x \leqslant y \Leftrightarrow x^+ \leqslant y^+ \ \& \ y^- \leqslant x^-.$$

For the proof of (1)-(7) can be found in ([20], page 51) and the remaining relations are easy consequences of the preceeding ones.

Definition 5. Let E be a vector lattice. A norm $\| \ \|$ on E is called a lattice norm if $|x| \leqslant |y|$ implies $\|x\| \leqslant \|y\|$ for all $x, y \in E$. If $\| \ \|$ is a lattice norm on E. the pair $(E, \| \ \|)$ is called a <u>normed (vector) lattice</u>; if, in addition, $(E, \| \ \|)$ is norm complete it is called a <u>Banach lattice</u>.

<u>Definition</u> 6. A lattice norm $x \mapsto \|x\|$ on a vector lattice E is called an <u>L-norm</u> if it satisfies the axiom

$$\|x + y\| = \|x\| + \|y\| \quad (x,y \in E_+).$$

$(E, \| \ \|)$ is called an <u>L-normed space</u>, and an L-normed Banach lattice is called an <u>abstract L-space</u> (briefly, <u>AL-space</u>).

The dual E' of a Banach lattice E is also a Banach lattice provided that its positive cone is defined by $x' \geqslant 0$ in E' iff $x'(x) \geqslant 0$. for every $x \geqslant 0$ in E. It is easily verified that, for any $x', y' \in E'$ and every $x \geqslant 0$ in E, we have

$$(x' \vee y')(x) = \sup \{x'(u) + y'(x-u) \mid 0 \leqslant u \leqslant x\}$$

and

$$(x' \wedge y')(x) = \sup \{x'(v) + y'(x-v) \mid 0 \leqslant v \leqslant x\}.$$

<u>Proof of Lemma</u> 4.1

Let E be a Banach lattice. The dual E' of the Banach lattice E is also a Banach lattice provided that its positive cone is defined by $e' \geqslant 0$ in E' iff $e'(e) \geqslant 0$, for each $e \in E_+$. Define a mapping i from E into its second dual E" as follow:

$$i(e) = e'' \quad (e \in E),$$

where $e''(e') = e'(e)$. It is obvious that i is a positive operator.

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If we can prove that $i^{-1}: i(E) \to E$ is also a positive operator, then our lemma is proved. To see this, let $e_0 \in E$. The assumption,

$$e_0''(e') = e'(e_0) \geqslant 0 \qquad (\text{for all } e' \in E'_+)$$

of Lemma 4.1, implies that $i(e_0) = e_0''$ is a positive element of $i(E)$. By using positivity of i^{-1}, we get $i^{-1}(e_0'') = e_0 \geqslant 0$. Hence, it remains to show that i^{-1} is positive. To prove this, we begin by showing that

$$(13) \qquad i(x) \vee i(y) = i(x \vee y)$$

for all $x, y \in E$. We prove this first under the assumption that $x \wedge y = 0$. For each $u' \in E'_+$, we have

$$\begin{aligned}(i(x) \vee i(y))(u') &= \sup\{i(x)(v') + i(y)(u'-v') \mid 0 \leqslant v' \leqslant u'\} \\ &= \sup\{u'(y) + v'(x-y) \mid 0 \leqslant v' \leqslant u'\}.\end{aligned}$$

By putting $w'(z) = \sup_n \{u'(z \wedge nx)\}$ for each $z \in E'_+$, we define a bounded linear functional $w' \in E'$ (the linearity of w' is a consequence of the inequality $(a+b) \wedge c \leqslant a \wedge c + b \wedge c \leqslant (a+b) \wedge 2c$, which holds for all $a,b,c \in E_+$). The functional w' satisfies $0 \leqslant w' \leqslant u'$, $w'(x) = u'(x)$ and (since $x \wedge y = 0$) $w'(y) = 0$. It follows that $(i(x) \vee i(y))(u') \geqslant u'(y) + w'(x-y) = u'(x+y) = u'(x \vee y) = i(x \vee y)(u')$, for every positive $u' \in E'$. Hence,

$$(14) \qquad i(x) \vee i(y) \geqslant i(x \vee y).$$

Since i is positive, we have $i(x) \leqslant i(x \vee y)$ and $i(y) \leqslant i(x \vee y)$. Thus

$$(15) \qquad i(x) \vee i(y) \leqslant i(x \vee y).$$

It follows from (14) and (15) that

$$i(x) \vee i(y) = i(x \vee y).$$

Assume now that x,y are arbitrary elements of E. Put $u = x - x \wedge y$, $v = y - x \wedge y$. Then we have $u \wedge v \geqslant 0$,

$$u \wedge v + x \wedge y \leqslant x,$$

and

$$u \wedge v + x \wedge y \leqslant y.$$

Hence $u \wedge v + x \wedge y \leqslant x \wedge y$ and thus $u \wedge v = 0$. Therefore

(16) $$i(u) \vee i(v) = i(u)+i(v)-i(u) \wedge i(v) = i(u \vee v) = i(u)+i(v)-i(u \wedge v).$$

So (16) implies that $i(u) \wedge i(v) = i(u \wedge v) = i(0) = 0$. Moreover, we note that $i(x) = i(u) + i(x \wedge y)$ and $i(y) = i(v) + i(x \wedge y)$. Thus

(17) $$\begin{aligned} i(x) \wedge i(y) &= [i(u)+i(x \wedge y)] \wedge]i(v)+i(x \wedge y)] \\ &= i(u) \wedge i(v) + i(x \wedge y) = i(x \wedge y). \end{aligned}$$

It follows from (17) that

(18) $$i(x) \vee i(y) = i(x)+i(y)-i(x) \wedge i(y) = i(x)+i(y)-i(x \wedge y) = i(x \vee y).$$

This proves (13).

To see that i^{-1} is positive, we let $e_1'' \in E_+'' \cap i(E)$. Then there exists $e_1 \in E$ such that $i(e_1) = e_1''$. We note that

$$0 \leqslant e_1'' = i(e_1) = i(e_1) \vee 0 = i(e_1) \vee i(0) = i(e_1 \vee 0) = i(e_1^+),$$

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where the fourth equality follows from (13). Hence $i(e_1) = i(e_1^+)$ and this implies $e_1 = e_1^+ \geqslant 0$. This proves Lemma 4.1.

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