CHAPTER IV

SKEW RINGS OF RIGHT [LEFT] DIFFERENCES OF SEMIRINGS

In [4], some theorems for skew rings of right [left] differences of semirings were proven. In this chapter, we shall prove additional theorems for skew rings of right [left] differences of semirings.

<u>Definition 4.1</u>. Let S be a semiring and $(a,b) \in S\times S$. Then (a,b) is called a <u>unitive pair</u> if for all $x,y \in S$ there exist $u,v,u',v' \in S$ such that

$$ax+by+u = x+v,$$

$$ay+bx+u = y+v,$$

$$xa+yb+u' = x+v'$$

and

$$xb+ya+u' = y+v'.$$

If a semiring S contains a unitive pair, then S is called a <u>unitive semiring</u>.

<u>Example 4.2</u>. Let S be a nonempty set. Define + and $\cdot$ on S by $x+y = y = x\cdot y$ for all $x,y \in S$. Then $(S,+,\cdot)$ is a semiring and for any $(a,b) \in S\times S$, (a,b) is a unitive pair. Hence S is a unitive semiring.

<u>Theorem 4.3</u>. Let S be a semiring having D(S) as its skew ring of right [left] differences. Then D(S) has a multiplicative identity 1 if and only if S is a unitive semiring. Furthermore, for all $(a,b) \in S\times S$, $[(a,b)] = 1$ if and only if (a,b) is a unitive pair.

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<u>Theorem 4.3</u>. Let S be a semiring having D(S) as its skew ring of right [left] differences. Then D(S) has a multiplicative identity 1 if and only if S is a unitive semiring. Furthermore, for all $(a,b) \in S \times S$, $[(a,b)] = 1$ if and only if (a,b) is a unitive pair.

Proof. Assume that D(S) has a multiplicative identity, say $[(a,b)]$. Let $x,y \in S$. Then $[(x,y)] \in D(S)$ and $[(a,b)]\,[(x,y)] = [(x,y)] = [(x,y)]\,[(a,b)]$. Therefore $[(ax+by,ay+bx)] = [(x,y)] = [(xa+yb,xb+ya)]$. There exist $u,v,u',v' \in S$ such that $ax+by+u = x+v$, $ay+bx+u = y+v$, $xa+yb+u' = x+v'$ and $xb+ya+u' = y+v'$. Hence S is unitive.

Conversely, assume that S is a unitive semiring. Then there exists an $(a,b) \in S \times S$ such that (a,b) is a unitive pair. Let $[(x,y)] \in D(S)$. Then $x,y \in S$ and there exist $u,v,u',v' \in S$ such that $ax+by+u = x+v$, $ay+bx+u = y+v$, $xa+yb+u' = x+v'$ and $xb+ya+u' = y+v'$. Thus $(ax+by,ay+bx) \sim (x,y) \sim (xa+yb,xb+ya)$. Therefore $[(a,b)]\,[(x,y)] = [(x,y)] = [(x,y)][(a,b)]$, so $[(a,b)]$ is a multiplicative identity of D(S).

Clearly, $[(a,b)] = 1$ if and only if (a,b) is a unitive pair.

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Corollary 4.4. Let S be a semiring having D(S) as its skew ring of right or left differences. Then S is additively commutative.

Definition 4.5. A skew ring $(S,+,\cdot)$ is called a skew field if $(S \setminus \{0\},\cdot)$ is a group.

Remark. If $(S,+,\cdot)$ is a skew field, then $(S,+)$ is an abelian group.

Definition 4.6. A skew ring R is called a simple skew ring if for any ideal J of R, $J = \{0\}$ or $J = R$.

Remark. Every skew field is a simple skew ring.

Definition 4.7. Let S be a unitive semiring. Then S is called exact if for any unitive pair $(a,b) \in S \times S$ and for all distinct $x,y \in S$ there exist $u,v,z,w,z',w' \in S$ such that

$$xu+yv+z = a+w,$$

$$xv+yu+z = b+w,$$

$$ux+vy+z' = a+w'$$

and

$$uy+vx+z' = b+w'.$$

Theorem 4.8. Let S be a semiring having D(S) as its skew ring of right [left] differences. Then D(S) is a skew field if and only if S is exact.

Proof. Assume that D(S) is a skew field. Then D(S) has a multiplicative identity 1 and hence S is a unitive semiring. Let $(a,b) \in S\times S$ be a unitive pair. By Theorem 4.3, $[(a,b)] = 1$. Let $x,y \in S$ be distinct. Then $[(x,y)] \neq 0$. There exist $u,v \in S$ such that $[(x,y)][(u,v)] = [(a,b)] = [(u,v)][(x,y)]$. Therefore $(xu+yv,xv+yu) \sim (a,b) \sim (ux+vy,uy+vx)$. There exist $z,w,z',w' \in S$ such that $xu+yv+z = a+w$, $xv+yu+z = b+w$, $ux+vy+z' = a+w'$ and $uy+vx+z' = b+w'$. Hence S is exact.

Conversely, assume that S is exact. Then S is a unitive semiring. Let $(a,b) \in S\times S$ be a unitive pair. Then $[(a,b)]$ is a multiplicative identity of D(S). Let $\alpha \in D(S)\setminus\{0\}$. Choose $(x,y) \in \alpha$, so $x \neq y$. Since S is exact, there exist $u,v,z,w,z',w' \in S$ such that $xu+yv+z = a+w$, $xv+yu+z = b+w$, $ux+vy+z' = a+w'$ and $uy+vx+z' = b+w'$. Therefore $(xu+yv,xv+yu) \sim (a,b) \sim (ux+vy,uy+vx)$. Thus $[(x,y)][(u,v)] = [(a,b)] = [(u,v)][(x,y)]$ and hence D(S) is a skew field.

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Corollary 4.9. Let S be a semiring having D(S) as its skew ring of right [left] differences. If S is exact, then D(S) is a simple skew ring.

Definition 4.10. A semiring S is said to be *strongly multiplicatively cancellative* (S.M.C.) if for all $x,y,z,w \in S$

$$xz+yw = xw+yz \text{ implies that } x = y \text{ or } z = w$$

and

$$xz+yw = yz+xw \text{ implies that } x = y \text{ or } z = w.$$

Example 4.11. $\mathbb{Z}^+$ with the usual addition and multiplication is S.M.C..

In [7] the definition of a prime ring was given. We shall generalize this definition to skew rings.

Definition 4.12. A skew ring R is called a *prime skew ring* if for any ideals I,J in R, $IJ = \{0\}$ implies that $I = \{0\}$ or $J = \{0\}$.

Example 4.13. $\mathbb{Z}$ with the usual addition and multiplication is a prime skew ring.

Definition 4.14. A skew ring R is called a *strongly prime skew ring* if for any weak ideals I,J in R, $IJ = \{0\}$ implies that $I = \{0\}$ or $J = \{0\}$.

Remark. Every strongly prime skew ring is a prime skew ring.

Theorem 4.15. Let S be a semiring having D(S) as its skew ring of right [left] differences. If S is S.M.C., then D(S) is a strongly prime skew ring.

Proof. Assume that S is S.M.C.. Let I and J be weak ideals of D(S) such that $IJ = \{0\}$. Assume that $I \neq \{0\}$. We must show that $J = \{0\}$. Let $\alpha \in I$ be such that $\alpha \neq 0$ and let $\beta \in J$. Choose $(x,y) \in \alpha$ and $(z,w) \in \beta$. Then $[(x,y)][(z,w)] = \alpha\beta \in IJ = \{0\}$. Therefore $[(xz+yw,xw+yz)] = 0$. Hence $xz+yw = xw+yz$. Since S is S.M.C., $x = y$ or $z = w$. If $x = y$, then $\alpha = 0$. This is a contradiction, so $z = w$.

Therefore $\beta = 0$ and hence $J = \{0\}$.

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Corollary 4.16. Let S be a semiring having D(S) as its skew ring of right [left] differences. If S is S.M.C., then D(S) is a prime skew ring.

Theorem 4.17. Let S be a multiplicatively commutative semiring having D(S) as its skew ring of right [left] differences. If D(S) is a prime skew ring, then S is S.M.C..

Proof. Assume that D(S) is a prime skew ring. Let $\alpha \in D(S)$ and $\langle\alpha\rangle$ denote the ideal generated by α. Let

$$I_\alpha = \left\{ \sum_{i=1}^{n} (g_i+r_i\alpha-g_i)/n \in \mathbb{Z}^+,\ g_i \in D(S) \text{ and } r_i \in D(S) \cup \mathbb{Z} \text{ for all } i = 1,2, \ldots, n \right\}.$$

Claim that $I_\alpha = \langle\alpha\rangle$. Clearly, $\alpha \in I_\alpha$.

Let $\beta,\beta' \in I_\alpha$. Then $\beta = \sum_{i=1}^{n} (g_i+r_i\alpha-g_i)$ and $\beta' = \sum_{i=1}^{n} (g_i'+r_i'\alpha-g_i')$ for some $n \in \mathbb{Z}^+$, $g_i,g_i' \in D(S)$ and $r_i,r_i' \in D(S) \cup \mathbb{Z}$ for all $i = 1,2, \ldots, n$. Therefore $\beta-\beta' = (g_1+r_1\alpha-g_1)+ \ldots +(g_n+r_n\alpha-g_n)-(g_n'+r_n'\alpha-g_n')- \ldots -(g_1'+r_1'\alpha-g_1') = (g_1+r_1\alpha-g_1)+ \ldots +(g_n+r_n\alpha-g_n)+(g_n'+(-r_n')\alpha-g_n')+ \ldots +(g_1'+(-r_1')\alpha-g_1') \in I_\alpha$.

Let $\beta \in D(S)$. To show that $\beta+I_\alpha-\beta \subseteq I_\alpha$, let $\beta' \in \beta+I_\alpha-\beta$. Then $\beta' = \beta+ \sum_{i=1}^{n} (g_i+r_i\alpha-g_i)-\beta$ for some $n \in \mathbb{Z}^+$, $g_i \in D(S)$ and $r_i \in D(S) \cup \mathbb{Z}$ for all $i = 1,2, \ldots, n$. Hence $\beta' = \beta+(g_1+r_1\alpha-g_1)-\beta+\beta+(g_2+r_2\alpha-g_2)-\beta+\beta+ \ldots -\beta+\beta+(g_n-r_n\alpha-g_n)-\beta = (\beta+g_1+r_1\alpha-g_1-\beta)+(\beta+g_2+r_2\alpha-g_2-\beta)+$

$\ldots +(\beta+g_n+r_n\alpha-g_n-\beta) \in I_\alpha$.

Let $\beta \in D(S)$ and $\beta' \in I_\alpha$. Then $\beta' = \sum_{i=1}^{n} (g_i+r_i\alpha-g_i)$ for some $n \in \mathbb{Z}^+$, $g_i \in D(S)$ and $r_i \in D(S) \cup \mathbb{Z}$ for all $i = 1,2, \ldots, n$. Therefore

$$\beta\beta' = \sum_{i=1}^{n} \beta(g_i+r_i\alpha-g_i) = \sum_{i=1}^{n} (\beta g_i+(\beta r_i)\alpha-\beta g_i) \in I_\alpha.$$

Let J be an ideal of $D(S)$ such that $\alpha \in J$. We must show that $I_\alpha \subseteq J$. Let $\beta \in I_\alpha$. Then $\beta = \sum_{i=1}^{n} (g_i+r_i\alpha-g_i)$ for some $n \in \mathbb{Z}^+$, $g_i \in D(S)$ and $r_i \in D(S) \cup \mathbb{Z}$ for all $i = 1,2, \ldots, n$. Since $\alpha \in J$, $r_i\alpha \in J$ for all $i = 1,2, \ldots, n$. Since J is an ideal of $D(S)$, $g_i+r_i\alpha-g_i \in J$ for all $i = 1,2, \ldots, n$. Since J is closed under addition, $\beta \in J$. Hence $I_\alpha \subseteq J$.

So we have the claim.

Next, claim that $\langle\alpha\rangle\langle\beta\rangle \subseteq \langle\alpha\beta\rangle$ for all $\alpha,\beta \in D(S)$. Let $\alpha,\beta \in D(S)$. Then $\langle\alpha\rangle\langle\beta\rangle = \{uv \mid u \in \langle\alpha\rangle \text{ and } v \in \langle\beta\rangle\}$ =

$\{ (\sum_{i=1}^{l} (g_i+r_i\alpha-g_i))(\sum_{j=1}^{m} (g_j'+r_j'\beta-g_j')) \mid l,m \in \mathbb{Z}^+, g_i,g_j' \in D(S)$ and $r_i,r_j' \in D(S) \cup \mathbb{Z}$ for all $i = 1,2, \ldots, l$ and all $j = 1,2, \ldots, m\}$ =

$\{ \sum_{i=1}^{l} \sum_{j=1}^{m} ((g_i+r_i\alpha-g_i)(g_j'+r_j'\beta-g_j')) \mid l,m \in \mathbb{Z}^+, g_i,g_j' \in D(S)$ and $r_i,r_j' \in D(S) \cup \mathbb{Z}$ for all $i = 1,2, \ldots, l$ and all $j = 1,2, \ldots, m\}$ =

$\{ \sum_{i=1}^{l} \sum_{j=1}^{m} (g_ig_j'+g_ir_j'\beta-g_ig_j'+r_i\alpha g_j'+r_i\alpha r_j'\beta-r_i\alpha g_j'-g_ig_j'-g_ir_j'\beta+g_ig_j') \mid l,m \in \mathbb{Z}^+$, $g_i,g_j' \in D(S)$ and $r_i,r_j' \in D(S) \cup \mathbb{Z}$ for all $i = 1,2, \ldots, l$ and all $j = 1,2, \ldots, m\} = \{ \sum_{i=1}^{l} \sum_{j=1}^{m} r_ir_j'\alpha\beta \mid l,m \in \mathbb{Z}^+$ and $r_i,r_j' \in D(S) \cup \mathbb{Z}$

for all $i = 1,2, \ldots, l$ and all $j = 1,2, \ldots, m\} \subseteq \langle\alpha\beta\rangle$, so we have the claim.

Now, let $x,y,z,w \in S$ be such that $xz+yw = xw+yz$. Then $[(xz+yw,xw+yz)] = 0$. Therefore $\langle[(x,y)][(z,w)]\rangle = \{0\}$. Hence $\langle[(x,y)]\rangle\langle[(z,w)]\rangle \subseteq \langle[(x,y)][(z,w)]\rangle = \{0\}$ which implies that $\langle[(x,y)]\rangle\langle[(z,w)]\rangle = \{0\}$. Since D(S) is a prime skew ring, $\langle[(x,y)]\rangle = \{0\}$ or $\langle[(z,w)]\rangle = \{0\}$. Thus $x = y$ or $z = w$. Similarly, we can show that if $xz+yw = yz+xw$, then $x = y$ or $z = w$. Hence S is S.M.C..

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Corollary 4.18. Let S be a multiplicatively commutative semiring having D(S) as its skew ring of right [left] differences. If D(S) is a strongly prime skew ring, then S is S.M.C..

Proposition 4.19. Let S be a unitive semiring having D(S) as its skew ring of right [left] differences and $(a,b),(c,d) \in S \times S$. If (a,b) and (c,d) are unitive pairs, then the following statements hold:

(1) There exist $z,w \in S$ such that $a+z = c+w$ and $b+z = d+w$.

(2) $(ac+bd,ad+bc)$ is a unitive pair.

Proof. (1) By Theorem 4.3, $[(a,b)] = 1 = [(c,d)]$ where 1 is a multiplicative identity of D(S). Therefore there exist $z,w \in S$ such that $a+z = c+w$ and $b+z = d+w$, so done.

(2) Since $[(a,b)] = 1 = [(c,d)]$, $[(a,b)][(c,d)] = 1$. Therefore $[(ac+bd,ad+bc)] = 1$ and hence $(ac+bd,ad+bc)$ is a unitive pair.

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Proposition 4.20. Let S be a semiring with a multiplicative identity 1 having D(S) as its skew ring of right [left] differences. Then $[(1+1,1)]$ is a multiplicative identity of D(S) if and only if S is additively commutative.

<u>Proof</u>. Assume that $[(1+1,1)]$ is a multiplicative identity of $D(S)$. Let $x,y \in S$. Then $[(x,y)][(1+1,1)] = [(x,y)]$. Therefore $(x+x+y,x+y+y) \sim (x,y)$. There exist $u,v \in S$ such that $x+x+y+u = x+v$ and $x+y+y+u = y+v$. Since S is A.C., $x+y = y+x$.

Conversely, assume that S is additively commutative. Let $[(x,y)] \in D(S)$. Then $[(x,y)][(1+1,1)] = [(x+x+y,x+y+y)] = [(x,y)] = [(1+1,1)][(x,y)]$. Hence $[(1+1,1)]$ is a multiplicative identity of $D(S)$.

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<u>Proposition 4.21</u>. Let S be a semiring having D(S) as its skew ring of right [left] differences, $i : S \to D(S)$ the right [left] difference embedding and $K \subseteq S$ nonempty. Then $i(K)-i(K) = \{[(x,y)]/x,y \in K\}$.

<u>Proof</u>. Let $x,y \in K$. Claim that $[(x+x,x)]+[(y,y+y)] = [(x,y)]$. There exist $u,v \in S$ such that $x+u = y+v$. Thus $[(x+x,x)]+[(y,y+y)] = [(x+x+u,y+y+v)]$. Since $x+u = y+v, y+x+u+v = y+y+v+v$. Therefore $(x+x+u,y+y+v) \sim (x,y)$, so we have the claim.

$$\begin{aligned} \text{Now, } i(K)-i(K) &= \{a-b/a,b \in i(K)\} \\ &= \{i(x)-i(y)/x,y \in K\} \\ &= \{[(x+x,x)]-[(y+y,y)]/x,y \in K\} \\ &= \{[(x+x,x)]+[(y,y+y)]/x,y \in K\} \\ &= \{[(x,y)]/x,y \in K\}. \end{aligned}$$

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<u>Definition 4.22</u>. Let S be a commutative semiring and $K \subseteq S$ nonempty. Then K is called a <u>quasi-ideal</u> <u>in</u> <u>S</u> if

(1) for all $x,y,z,w \in K$ there exist $u,v \in K$ such that $x+w+v = y+z+u$

and (2) for all $x,y \in K$ and all $z,w \in S$ there exist $u,v \in K$ such that $xz+yw+v = xw+yz+u$.

Example 4.23. Let S be a commutative semiring and $x \in S$. Then S and $\{x\}$ are quasi-ideals in S.

Proposition 4.24. Let S be a commutative semiring and $K_1, K_2 \subseteq S$ nonempty. If K_1 and K_2 are quasi-ideals in S, then K_1+K_2 is a quasi-ideal in S.

Proof. Let $x,y,z,w \in K_1+K_2$. Then $x = x_1+x_2$, $y = y_1+y_2$, $z = z_1+z_2$ and $w = w_1+w_2$ for some $x_1,y_1,z_1,w_1 \in K_1$ and some $x_2,y_2,z_2,w_2 \in K_2$. There exist $u_1,v_1 \in K_1$ such that $x_1+w_1+v_1 = y_1+z_1+u_1$. Also, there exist $u_2,v_2 \in K_2$ such that $x_2+w_2+v_2 = y_2+z_2+u_2$. Let $u = u_1+u_2$ and $v = v_1+v_2$. Then $u,v \in K_1+K_2$ and $x+w+v = (x_1+x_2)+(w_1+w_2)+(v_1+v_2) = (x_1+w_1+v_1)+(x_2+w_2+v_2) = (y_1+z_1+u_1)+(y_2+z_2+u_2) = (y_1+y_2)+(z_1+z_2)+(u_1+u_2) = y+z+u$.

Let $x,y \in K_1+K_2$ and $z,w \in S$. Then $x = x_1+x_2$ and $y = y_1+y_2$ for some $x_1,y_1 \in K_1$ and some $x_2,y_2 \in K_2$. There exist $u_1,v_1 \in K_1$ such that $x_1z+y_1w+v_1 = x_1w+y_1z+u_1$. Also, there exist $u_2,v_2 \in K_2$ such that $x_2z+y_2w+v_2 = x_2w+y_2z+u_2$. Let $u = u_1+u_2$ and $v = v_1+v_2$. Then $u,v \in K_1+K_2$ and $xz+yw+v = (x_1+x_2)z+(y_1+y_2)w+(v_1+v_2) = x_1z+x_2z+y_1w+y_2w+v_1+v_2 = (x_1z+y_1w+v_1)+(x_2z+y_2w+v_2) = (x_1w+y_1z+u_1)+(x_2w+y_2z+u_2) = (x_1+x_2)w+(y_1+y_2)z+(u_1+u_2) = xw+yz+u$.

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Corollary 4.25. Let S be a commutative semiring and $K \subseteq S$ nonempty. If K is a quasi-ideal in S, then K+K is a quasi-ideal in S.

Proposition 4.26. Let S_1 and S_2 be commutative semirings and $K_1 \subseteq S_1$, $K_2 \subseteq S_2$ nonempty. If K_1 and K_2 are quasi-ideals in S_1 and S_2, respectively, then $K_1 \times K_2$ is a quasi-ideal in $S_1 \times S_2$.

Proof. Clearly, $S_1 \times S_2$ is a commutative semiring. Let $x,y,z,w \in K_1 \times K_2$. Then $x = (x_1,x_2)$, $y = (y_1,y_2)$, $z = (z_1,z_2)$ and $w = (w_1,w_2)$ for some $x_1,y_1,z_1,w_1 \in K_1$ and some $x_2,y_2,z_2,w_2 \in K_2$. There exist $u_1,v_1 \in K_1$ and $u_2,v_2 \in K_2$ such that $x_1+w_1+v_1 = y_1+z_1+u_1$ and $x_2+w_2+v_2 = y_2+z_2+u_2$. Let $u = (u_1,u_2)$ and $v = (v_1,v_2)$. Then $u,v \in K_1 \times K_2$ and $x+w+v = (x_1,x_2)+(w_1,w_2)+(v_1,v_2) = (x_1+w_1+v_1,x_2+w_2+v_2) = (y_1+z_1+u_1,y_2+z_2+u_2) = (y_1,y_2)+(z_1,z_2)+(u_1,u_2) = y+z+u$.

Let $x,y \in K_1 \times K_2$ and $z,w \in S_1 \times S_2$. Then $x = (x_1,x_2)$, $y = (y_1,y_2)$, $z = (z_1,z_2)$ and $w = (w_1,w_2)$ for some $x_1,y_1 \in K_1$, $x_2,y_2 \in K_2$, $z_1,w_1 \in S_1$ and some $z_2,w_2 \in S_2$. There exist $u_1,v_1 \in K_1$ and $u_2,v_2 \in K_2$ such that $x_1z_1+y_1w_1+v_1 = x_1w_1+y_1z_1+u_1$ and $x_2z_2+y_2w_2+v_2 = x_2w_2+y_2z_2+u_2$. Let $u = (u_1,u_2)$ and $v = (v_1,v_2)$. Then $u,v \in K_1 \times K_2$ and $xz+yw+v = (x_1,x_2)(z_1,z_2)+(y_1,y_2)(w_1,w_2)+(v_1,v_2) = (x_1z_1+y_1w_1+v_1,x_2z_2+y_2w_2+v_2) = (x_1w_1+y_1z_1+u_1,x_2w_2+y_2z_2+u_2) = (x_1,x_2)(w_1,w_2)+(y_1,y_2)(z_1,z_2)+(u_1,u_2) = xw+yz+u$.
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Corollary 4.27. Let S be a commutative semiring and $K \subseteq S$ nonempty. If K is a quasi-ideal in S, then KxK is a quasi-ideal in SxS.

Proposition 4.28. Let S be a commutative semiring and $K \subseteq S$ nonempty. If K is a semiring-ideal of S, then K is a quasi-ideal in S.

Proof. Let $x,y,z,w \in K$. Let $u = x+w$ and $v = y+z$. Then $u,v \in K$ and $x+w+v = x+w+(y+z) = y+z+(x+w) = y+z+u$.

Let $x,y \in K$ and $z,w \in S$. Let $u = xz+yw$ and $v = xw+yz$. Then $u,v \in K$ and $xz+yw+v = xz+yw+(xw+yz) = xw+yz+(xz+yw) = xw+yz+u$. Thus K is a quasi-ideal in S.

#

The converse of this proposition is not always true as the next example shows.

Example 4.29. Z^+ with the usual addition and multiplication is a commutative semiring and $\{1\}$ is a quasi-ideal in Z^+. But $1+1 = 2 \notin \{1\}$, so $\{1\}$ is not a semiring-ideal of Z^+.

Proposition 4.30. Let S be a commutative semiring having D(S) as its ring of differences, $i : S \to D(S)$ the difference embedding and $J \subseteq D(S)$ nonempty. If J is an ideal of D(S), then $i^{-1}(J)$ is a semiring-ideal of S.

Proof. Let $x,y \in i^{-1}(J)$. Then $i(x),i(y) \in J$. Hence $i(x+y) = i(x)+i(y) \in J$. Thus $x+y \in i^{-1}(J)$.

Let $x \in i^{-1}(J)$ and $s \in S$. Then $i(x) \in J$ and $i(s) \in D(S)$. Therefore $i(xs) = i(x)i(s) \in J$ and hence $xs \in i^{-1}(J)$.

#

Corollary 4.31. Let S be a commutative semiring having D(S) as its ring of differences, $i : S \to D(S)$ the difference embedding and J an ideal of D(S). Then the following statements hold:

(1) $i^{-1}(J)$ is a quasi-ideal in S.

(2) $i^{-1}(J)+i^{-1}(J)$ is a quasi-ideal in S+S.

(3) $i^{-1}(J)\times i^{-1}(J)$ is a quasi-ideal in $S\times S$.

Theorem 4.32. Let S be a commutative semiring having D(S) as its ring of differences, $i : S \to D(S)$ the difference embedding and $K \subseteq S$ nonempty. Then $i(K)-i(K)$ is an ideal of D(S) if and only if K is a quasi-ideal in S.

Proof. Assume that $i(K)-i(K)$ is an ideal of D(S). Let $x,y,z,w \in K$. Then $[(x+w,y+z)] = [(x,y)]+[(w,z)] = [(x,y)]-[(z,w)] \in i(K)-i(K)$. Hence there exist $u,v \in K$ such that $[(x+w,y+z)] = [(u,v)]$ which implies that $x+w+v = y+z+u$. Let $x,y \in K$ and $z,w \in S$. Then $[(x,y)] \in i(K)-i(K)$ and $[(z,w)] \in D(S)$. Thus $[(x,y)][(z,w)] \in i(K)-i(K)$. Therefore $[(xz+yw,xw+yz)] \in i(K)-i(K)$. There exist $u,v \in K$ such that $[(xz+yw,xw+yz)] = [(u,v)]$. Therefore $xz+yw+v = xw+yz+u$. Hence K is a quasi-ideal in S.

Conversely, assume that K is a quasi-ideal in S. Let $x,y \in i(K)-i(K)$. Then $x = [(x_1,x_2)]$ and $y = [(y_1,y_2)]$ where $x_1,x_2,y_1,y_2 \in K$. There exist $u,v \in K$ such that $x_1+y_2+v = x_2+y_1+u$. Thus $x-y = [(x_1+y_2,x_2+y_1)] = [(u,v)] \in i(K)-i(K)$.
Let $[(x_1,x_2)] \in i(K)-i(K)$ and $[(r_1,r_2)] \in D(S)$. Since $x_1,x_2 \in K$ and $r_1,r_2 \in S$, there exist $u,v \in K$ such that $x_1r_1+x_2r_2+v = x_1r_2+x_2r_1+u$. Thus $[(x_1r_1+x_2r_2,x_1r_2+x_2r_1)] = [(u,v)]$, so $[(x_1,x_2)][(r_1,r_2)] \in i(K)-i(K)$. Hence $i(K)-i(K)$ is an ideal of D(S).

#

Corollary 4.33. Let S be a commutative semiring having D(S) as its ring of differences and $i : S \to D(S)$ the difference embedding. Then:

(1) If J is an ideal of D(S), then $i(i^{-1}(J))-i(i^{-1}(J))$ is an ideal of D(S).

(2) If K is a quasi-ideal in S, then $i^{-1}(i(K)-i(K))$ is a quasi-ideal in S.

In [6] the concept of a C-set for a distributive ratio seminear-ring was given. For a ratio semiring (which is commutative) this definition is equivalent to the one given below :

<u>Definition 4.34</u>. Let D be a commutative ratio semiring and $C \subseteq D$ nonempty. Then C is called a <u>C-set</u> <u>in</u> <u>D</u> if

(1) for all $x,y \in C$, $xy^{-1} \in C$

and (2) for all $x \in C$ and all $d \in D$, $(x+d)(1+d)^{-1} \in C$.

<u>Example 4.35</u>. Let D be a commutative ratio semiring and 1 the identity of $(D,\cdot)$. Then D and $\{1\}$ are C-sets in D.

<u>Definition 4.36</u>. Let S be a commutative semiring and $K \subseteq S$ nonempty. Then K is called a <u>quasi-C-set</u> <u>in</u> <u>S</u> if

(1) for all $a,b,c,d \in K$ there exist $e,f \in K$ such that $adf = bce$

and (2) for all $a,b \in K$ and all $c,d \in S$ there exist $e,f \in K$ such that $ad^2f+bcdf = bd^2e+bcde$.

<u>Example 4.37</u>. Let S be a commutative semiring and $x \in S$. Then S and $\{x\}$ are quasi-C-sets in S.

<u>Proposition 4.38</u>. Let S_1 and S_2 be commutative semirings. Let K_1 and K_2 be quasi-C-sets in S_1 and S_2, respectively. Then $K_1 \times K_2$ is a quasi-C-set in $S_1 \times S_2$.

<u>Proof</u>. Clearly, $S_1 \times S_2$ is a commutative semiring. Let $x,y,z,w \in K_1 \times K_2$. Then $x = (x_1,x_2)$, $y = (y_1,y_2)$, $z = (z_1,z_2)$ and

$w = (w_1, w_2)$ for some $x_1, y_1, z_1, w_1 \in K_1$ and some $x_2, y_2, z_2, w_2 \in K_2$. There exist $u_1, v_1 \in K_1$ and $u_2, v_2 \in K_2$ such that $x_1w_1v_1 = y_1z_1u_1$ and $x_2w_2v_2 = y_2z_2u_2$. Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$. Then $u, v \in K_1 \times K_2$ and $xwv = (x_1, x_2)(w_1, w_2)(v_1, v_2) = (x_1w_1v_1, x_2w_2v_2) = (y_1z_1u_1, y_2z_2u_2) = (y_1, y_2)(z_1, z_2)(u_1, u_2) = yzu$.

Let $x, y \in K_1 \times K_2$ and $z, w \in S_1 \times S_2$. Then $x = (x_1, x_2)$, $y = (y_1, y_2)$, $z = (z_1, z_2)$ and $w = (w_1, w_2)$ for some $x_1, y_1 \in K_1$, $x_2, y_2 \in K_2$, $z_1, w_1 \in S_1$ and some $z_2, w_2 \in S_2$. There exist $u_1, v_1 \in K_1$ and $u_2, v_2 \in K_2$ such that $x_1w_1^2v_1 + y_1z_1w_1v_1 = y_1w_1^2u_1 + y_1z_1w_1u_1$ and $x_2w_2^2v_2 + y_2z_2w_2v_2 = y_2w_2^2u_2 + y_2z_2w_2u_2$. Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$. Then $u, v \in K_1 \times K_2$ and $xw^2v + yzwv = (x_1, x_2)(w_1, w_2)(w_1, w_2)(v_1, v_2) + (y_1, y_2)(z_1, z_2)(w_1, w_2)(v_1, v_2) = (x_1w_1^2v_1, x_2w_2^2v_2) + (y_1z_1w_1v_1, y_2z_2w_2v_2) = (x_1w_1^2v_1 + y_1z_1w_1v_1, x_2w_2^2v_2 + y_2z_2w_2v_2) = (y_1w_1^2u_1 + y_1z_1w_1u_1, y_2w_2^2u_2 + y_2z_2w_2u_2) = (y_1w_1^2u_1, y_2w_2^2u_2) + (y_1z_1w_1u_1, y_2z_2w_2u_2) = (y_1, y_2)(w_1, w_2)(w_1, w_2)(u_1, u_2) + (y_1, y_2)(z_1, z_2)(w_1, w_2)(u_1, u_2) = yw^2u + yzwu$. #

Corollary 4.39. Let S be a commutative semiring and K_1, K_2 quasi-C-sets in S. Then $K_1 \times K_2$ is a quasi-C-set in $S \times S$.

Proposition 4.40. Let S be a commutative semiring and K a semiring-ideal of S. Then K is a quasi-C-set in S.

Proof. Let $a, b, c, d \in K$. Let $f = bc$ and $e = ad$. Then $f, e \in K$ and $adf = bce$.

Let $a, b \in K$ and $c, d \in S$. Let $e = ad^2 + bcd$ and $f = bd^2 + bcd$.

Then $e,f \in K$ and $ad^2f+bcdf = (ad^2+bcd)f = (ad^2+bcd)(bd^2+bcd) = e(bd^2+bcd) = bd^2e+bcde$.

#

<u>Proposition 4.41</u>. Let S be a commutative semiring having Q(S) as its commutative ratio semiring of quotients, $i : S \to Q(S)$ its quotient embedding and $K \subseteq S$ nonempty. Then $i(K)\cdot i(K)^{-1} = \{[(a,b)]/a,b \in K\}$.

<u>Proof</u>. $i(K)\cdot i(K)^{-1} = \{xy^{-1}/x,y \in i(K)\}$

$= \{i(a)(i(b))^{-1}/a,b \in K\}$

$= \{[(a^2,a)]([(b^2,b)])^{-1}/a,b \in K\}$

$= \{[(a^2,a)][(b,b^2)]/a,b \in K\}$

$= \{[(a^2b,ab^2)]/a,b \in K\}$

$= \{[(a,b)]/a,b \in K\}$.

#

<u>Theorem 4.42</u>. Let S be a commutative semiring having Q(S) as its commutative ratio semiring of quotients, $i : S \to Q(S)$ the quotient embedding and $K \subseteq S$ nonempty. Then $i(K)\cdot i(K)^{-1}$ is a C-set in Q(S) if and only if K is a quasi-C-set in S.

<u>Proof</u>. Assume that $i(K)\cdot i(K)^{-1}$ is a C-set in Q(S). Let $a,b,c,d \in K$. Then $[(a,b)],[(c,d)] \in i(K)\cdot i(K)^{-1}$. Therefore $[(a,b)][(d,c)] \in i(K)\cdot i(K)^{-1}$. There exist $e,f \in K$ such that $[(a,b)][(d,c)] = [(e,f)]$. Thus $[(ad,bc)] = [(e,f)]$, so $adf = bce$. Let $a,b \in K$ and $c,d \in S$. Then $[(a,b)] \in i(K)\cdot i(K)^{-1}$ and $[(c,d)] \in Q(S)$. Thus $[((ad+bc)d,bd(d+c))] = ([(ad+bc,bd)])([(d+c,d)])^{-1} =$

$([(a,b)]+[(c,d)])([(a,a)]+[(c,d)])^{-1} \in i(K)\cdot i(K)^{-1}$. There exist $e,f \in K$ such that $[(ad+bc)d,bd(d+c)] = [(e,f)]$. Therefore $ad^2f+bcdf = bd^2e+bcde$ and hence K is a quasi-C-set in S.

Conversely, assume that K is a quasi-C-set in S. Let $x,y \in i(K)\cdot i(K)^{-1}$. Then $x = [(a,b)]$ and $y = [(c,d)]$ for some $a,b,c,d \in K$. There exist $e,f \in K$ such that $adf = bce$. Thus $xy^{-1} = [(a,b)][(c,d)]^{-1} = [(a,b)][(d,c)] = [(ad,bc)] = [(e,f)] \in i(K)\cdot i(K)^{-1}$. Let $x \in i(K)\cdot i(K)^{-1}$ and $y \in Q(S)$. Then $x = [(a,b)]$ and $y = [(c,d)]$ for some $a,b \in K$ and some $c,d \in S$. There exist $e,f \in K$ such that $ad^2f+bcdf = bd^2e+bcde$. Thus $(ad^2+bcd,bd^2+bcd) \sim (e,f)$. Therefore $(x+y)(1+y)^{-1} = [(ad+bc,bd)]([(c,c)]+[(c,d)])^{-1} = [(ad+bc,bd)]([(d+c,d)])^{-1} = [(ad+bc,bd)][(d,d+c)] = [(ad^2+bcd,bd^2+bcd)] = [(e,f)] \in i(K)\cdot i(K)^{-1}$ and hence $i(K)\cdot i(K)^{-1}$ is a C-set in Q(S).

#

<u>Corollary 4.43</u>. Let S be a commutative semiring having Q(S) as its commutative ratio semiring of quotients, $i : S \to Q(S)$ the quotient embedding and $K \subseteq S$ nonempty. If K is a semiring-ideal of S, then $i(K)\cdot i(K)^{-1}$ is a C-set in Q(S).

<u>Proof</u>. It follows immediately from Proposition 4.40 and Theorem 4.42.

#

<u>Definition 4.44</u>. Let S be a commutative semiring with a multiplicative identity 1 and $A \subseteq S$ nonempty. Then A is called a <u>c-set</u> <u>in</u> <u>S</u> if

(1) for all $x,y \in A$, $xy \in A$

and (2) for all $x \in A$ and all $s \in S$, $x+s \in A(1+s)$.

Remark. Let S be a commutative ratio semiring and $A \subseteq S$ a C-set in S. Then A is a c-set in S.

Example 4.45. $\mathbb{R}_0^+$ with the usual addition and multiplication is a commutative semiring with a multiplicative identity 1.

Claim that $[1,\infty)$ is a c-set in $\mathbb{R}_0^+$, let $x,y \in [1,\infty)$. Then $x \geqslant 1$ and $y \geqslant 1$. Therefore $xy \geqslant 1$, that is, $xy \in [1,\infty)$.

Let $x \in [1,\infty)$ and $s \in \mathbb{R}_0^+$. Then $x+s \geqslant 1+s$, so $\frac{x+s}{1+s} \geqslant 1$.

Hence $x+s = \frac{x+s}{1+s} \cdot (1+s) \in [1,\infty)(1+s)$, so we have the claim.

Remark. $\mathbb{Z}^+$ with the usual addition and multiplication is not a c-set in $\mathbb{Z}^+$ because $3+2 = 5 \notin \mathbb{Z}^+(1+2)$.

Proposition 4.46. Let S be a commutative semiring with a multiplicative identity having Q(S) as its commutative ratio semiring of quotients, $i : S \to Q(S)$ the quotient embedding and $C \subseteq S$ nonempty. If $i(C)$ is a C-set in Q(S), then C is a c-set in S.

Proof. Let $x,y \in S$. Then $i(x),i(y) \in i(C)$. Therefore $i(xy) = i(x)i(y) \in i(C)$ and hence $xy \in i^{-1}(i(C)) = C$.

Let $x \in C$ and $s \in S$. Then $i(x) \in i(C)$ and $i(s) \in Q(S)$. Therefore $(i(x)+i(s))(1+i(s))^{-1} \in i(C)$ which implies that $i(x)+i(s) \in i(C)(1+i(s))$. Thus $i(x+s) \in i(C(1+s))$. Hence $x+s \in i^{-1}(i(C(1+s))) = C(1+s)$.

#

Proposition 4.47. Let S_1 and S_2 be commutative semirings with a multiplicative identity. Let C_1 and C_2 be c-sets in S_1 and S_2, respectively. Then $C_1 \times C_2$ is a c-set in $S_1 \times S_2$.

Proof. Let $x,y \in C_1 \times C_2$. Then $x = (x_1,x_2)$ and $y = (y_1,y_2)$ for some $x_1,y_1 \in C_1$ and some $x_2,y_2 \in C_2$. Therefore $x_1y_1 \in C_1$ and $x_2y_2 \in C_2$. Thus $xy = (x_1y_1,x_2y_2) \in C_1 \times C_2$.

Let $x = (x_1,x_2) \in C_1 \times C_2$ and $s = (s_1,s_2) \in S_1 \times S_2$. Then $x_1+s_1 \in C_1(1+s_1)$ and $x_2+s_2 \in C_2(1+s_2)$. Therefore $x_1+s_1 = c_1(1+s_1)$ and $x_2+s_2 = c_2(1+s_2)$ for some $c_1 \in C_1$ and some $c_2 \in C_2$. Thus $(x_1+s_1,x_2+s_2) = (c_1(1+s_1),c_2(1+s_2))$. Hence $x+s = (x_1,x_2)+(s_1,s_2) = (c_1,c_2)((1,1)+(s_1,s_2)) \in (C_1 \times C_2)(1+s)$. #

Corollary 4.48. Let S be a commutative semiring with a multiplicative identity and A a c-set in S. Then $A \times A$ is a c-set in $S \times S$.

Proposition 4.49. Let S be a commutative M.C. semiring with a multiplicative identity and C_1,C_2 c-sets in S such that $C_1 \cap C_2 \neq \phi$. Then $C_1 \cap C_2$ is a c-set in S.

Proof. Let $x,y \in C_1 \cap C_2$. Since $x,y \in C_1$, $xy \in C_1$. Also, $xy \in C_2$. Thus $xy \in C_1 \cap C_2$.

Let $x \in C_1 \cap C_2$ and $s \in S$. Then $x+s \in C_1(1+s)$ and $x+s \in C_2(1+s)$. Therefore $c_1(1+s) = x+s = c_2(1+s)$ for some $c_1 \in C_1$ and some $c_2 \in C_2$. Since S is M.C., $c_1 = c_2$. Hence $x+s \in (C_1 \cap C_2)(1+s)$. #

Proposition 4.50. Let S be a commutative semiring with a multiplicative identity 1. Then {1} is a c-set in S. Furthermore, if S is both A.C. and M.C. and there exists an element $x \varepsilon S$ such that {x} is a c-set in S, then $x = 1$.

Proof. Let $a,b \varepsilon \{1\}$. Then $a = b = 1$. Therefore $ab \varepsilon \{1\}$. Let $a \varepsilon \{1\}$ and $s \varepsilon S$. Then $a = 1$ and $a+s = 1+s = 1(1+s) \varepsilon \{1\}(1+s)$. Hence {1} is a c-set in S.

Furthermore, assume that S is A.C. and M.C. and there exists an $x \varepsilon S$ such that {x} is a c-set in S. Then $x+x \varepsilon \{x\}(1+x)$. Thus $x+x = x(1+x) = x1+xx$. Since S is A.C., $x = xx$. Since S is M.C., $x = 1$.

#