CHAPTER II

ALMOST MULTIPLICATIVELY CANCELLATIVE SEMINEAR-RINGS

In [2] the concept of almost multiplicatively cancellative semirings was given. By definition, a seminear-ring is a generalization of a semiring. In this chapter we generalize the concept of an almost multiplicatively cancellative semiring to a new concept, an almost multiplicatively cancellative seminear-ring.

Definition 2.1. A seminear-ring $(S,+,\cdot)$ is called an almost multiplicatively cancellative seminear-ring (A.M.C. seminear-ring) if there exists an element $a \in S$ such that $(S \smallsetminus \{a\},\cdot)$ is a cancellative semigroup. If $a \in S$ has the property that $(S \smallsetminus \{a\},\cdot)$ is a cancellative semigroup, then S is called an A.M.C. seminear-ring w.r.t. a and a is called an essential element of S.

Remark. (1) If S is an A.M.C. seminear-ring, then $|S| > 1$.

(2) Every seminear-field is an A.M.C. seminear-ring.

Proposition 2.2. Let S be an A.M.C. seminear-ring. Let $A = \{a \in S \mid (S \smallsetminus \{a\},\cdot) \text{ is a cancellative semigroup}\}$. If there exists an $a \in A$ such that a is not M.C. in S, then $|A| \leq 2$.

Proof. Assume that there exists an $a \in A$ such that a is not M.C. in S.

Case 1. a is not L.M.C. in S. Then there exist $x,y \in S$ such that $ax = ay$ and $x \neq y$. To prove that $|A| \leq 2$, suppose not. Then there exist $b, c \in A \setminus \{a\}$ such that $b \neq c$. Clearly, $x, y \in S \setminus \{a\}$ or $x, y \in S \setminus \{b\}$ or $x, y \in S \setminus \{c\}$. If $x, y \in S \setminus \{b\}$ or $x, y \in S \setminus \{c\}$, then $x = y$. This is a contradiction, so $x, y \in S \setminus \{a\}$.

<u>Subcase 1.1</u>. a is a right multiplicative zero. Then $aa = a = ba$. Since $a, b \in S \setminus \{c\}$ and $S \setminus \{c\}$ is M.C., $a = b$. This is a contradiction.

<u>Subcase 1.2</u>. a is not a right multiplicative zero. Then there exists a $u \in S$ such that $ua \neq a$. Since $(ua)x = (ua)y$, $x = y$. This is a contradiction.

<u>Case 2</u>. a is not R.M.C. in S. Using the same proof as in Case 1, we can show that $|A| \leqslant 2$.

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<u>Proposition 2.3</u>. Let S be an A.M.C. seminear-ring w.r.t. a. If $xa = ax = x$ for all $x \in S$, then either $(S \setminus \{a\}, \cdot)$ has an identity or S is M.C..

<u>Proof</u>. We shall consider two cases.

<u>Case 1</u> . There exists an $e \in S \setminus \{a\}$ such that $e^2 = e$. Let $x \in S \setminus \{a\}$. Then $xe^2 = xe$ and $e^2x = ex$. Therefore $xe = ex = x$. Hence e is an identity of $(S \setminus \{a\}, \cdot)$.

<u>Case 2</u>. For all $x \in S \setminus \{a\}$, $x^2 \neq x$. Let $x, y, z \in S$ be such that $xy = xz$. If $x = a$, then $ay = az$. Therefore $y = z$. Assume that $x \neq a$. If $y \neq a$ and $z \neq a$, then $y = z$ so done. Without loss of generality, assume that $y = a$ and $z \neq a$. Then $xa = xz$, so $x = xz$. Therefore $xz = xz^2$ and hence $z = z^2$, a contradiction. Thus S is L.M.C.. Similarly, we can show that S is R.M.C..

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Theorem 2.4. Let S be an A.M.C. seminear-ring w.r.t. a. Then exactly one of the following statements hold :

(1) $xa = ax = a$ for all $x \varepsilon S$.

(2) $a^2 = a$ and there exists a $b \varepsilon S \setminus \{a\}$ such that $ab \neq a$ or $ba \neq a$.

(3) $a^2 \neq a$ and there exists a $b \varepsilon S \setminus \{a\}$ such that $ab = a$.

(4) $ax \neq a$, $ax \neq x$ and $xa \neq x$ for all $x \varepsilon S$.

(5) $ax \neq a$ for all $x \varepsilon S$ and $a^2 = a^n$ for all $n \varepsilon \mathbf{Z}^+ \setminus \{1\}$.

Proof. Consider a^2.

Case 1. $a^2 = a$.

Subcase 1.1. a is a multiplicative zero. Then $xa = ax = a$ for all $x \varepsilon S$.

Subcase 1.2. a is not a multiplicative zero. There exists a $b \varepsilon S$ such that $ab \neq a$ or $ba \neq a$. Clearly, $b \neq a$.

Case 2. $a^2 \neq a$.

Subcase 2.1. There exists a $b \varepsilon S \setminus \{a\}$ such that $ab = a$.

Subcase 2.2. For all $x \varepsilon S \setminus \{a\}$, $ax \neq a$. Then $ax \neq a$ for all $x \varepsilon S$.

Subcase 2.2.1. For all $x \varepsilon S$, $xa \neq x$ and $ax \neq x$.

Subcase 2.2.2. There exists an $x_o \varepsilon S$ such that $x_oa = x_o$ or $ax_o = x_o$. Assume that $x_oa = x_o$. Then $x_o \neq a$, $a^3 = aa^2 \neq a$ and $x_oa^3 = x_oa^2$. Therefore $a^3 = a^2$. By induction, $a^2 = a^n$ for all $n \varepsilon \mathbf{Z}^+ \setminus \{1\}$.

Similarly, we can show that if $ax_o = x_o$, then $a^2 = a^n$ for all $n \in Z^+ \setminus \{1\}$. #

From Theorem 2.4 we see that if S is an A.M.C. seminear-ring w.r.t. a, then there are exactly five mutually exclusive possibilities for the essential element a of S :

In (1) we say that S is a Classification A seminear-ring w.r.t. a.

In (2) we say that S is a Classification B seminear-ring w.r.t. a.

In (3) we say that S is a Classification C seminear-ring w.r.t. a.

In (4) we say that S is a Classification D seminear-ring w.r.t. a.

In (5) we say that S is a Classification E seminear-ring w.r.t. a.

If S is a Classification A, B, C, D or E seminear-ring w.r.t. some element of S, then we call S a Classification A, B, C, D or E seminear-ring, respectively.

Note that S may contain essential elements of different classifications, so S may be of several classifications.

Remark. (1) If S is a Classification A seminear-ring w.r.t. a, then SxS is never a Classification A seminear-ring w.r.t. (a,a) since for $x \in S \setminus \{a\}$ we have that $(a,x), (x,a) \in (S\times S) \setminus \{(a,a)\}$ but $(a,x)(x,a) = (ax,xa) = (a,a)$.

(2) If S is a Classification A or E seminear-ring w.r.t. a, then a is not M.C..

(3) If S is a Classification C, D or E seminear-ring w.r.t. a, then a is cancellative in $(S \setminus \{a\}, \cdot)$.

(4) If S is a Classification D seminear-ring, then $|S| > 2$.

Proposition 2.5. Let S be a Classification A seminear-ring w.r.t. a. If S is also a Classification A seminear-ring w.r.t. b, then a = b.

Proof. Since $ax = a = xa$ and $bx = b = xb$ for all $x \in S$, $a = ab = b$.

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Theorem 2.6. Let S be a Classification A seminear-ring w.r.t. a. Then:

(1) $a+a = a$.

(2) S is 0-M.C..

(3) For all $x,y \in S$, $xy = a$ if and only if $x = a$ or $y = a$.

Proof. (1) $a = (a+a)a = aa+aa = a+a$.

(2) Let $x,y,z \in S$ be such that $xy = xz$ and $x \neq a$.

Case 1. $y \neq a$ and $z \neq a$. We get that $y = z$.

Case 2. $y = a$. If $z \neq a$, then $a \neq xz = xy = xa = a$, a contradiction. Hence $z = a = y$.

Case 3. $z = a$. The proof is similar to the proof of Case 2. Similarly, we can show that $yx = zx$ and $x \neq a$ imply that $y = z$. Thus S is 0-M.C..

(3) The proof of (3) is obvious.

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Proposition 2.7. Let S be a Classification A seminear-ring w.r.t. a such that $(S,\cdot)$ satisfies the right [left] Ore condition. Then either a is an additive identity or a is an additive zero or $(S,+)$ is a right zero semigroup or $(S,+)$ is a left zero semigroup.

Proof. It follows from Proposition 1.40 and Theorem 2.6(2).

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Proposition 2.8. Let $(S,+,\cdot)$ be an A.M.C. seminear-ring. If $(S,\cdot)$ is a right or a left zero semigroup, then $(S,+,\cdot)$ is a Classification B seminear-ring and $|S| = 2$.

Proof. Assume that $(S,+,\cdot)$ is an A.M.C. seminear-ring w.r.t. a such that $(S,\cdot)$ is a right zero semigroup. Then $a^2 = a$. Let $b \in S\smallsetminus\{a\}$. Then $ab = b \neq a$. Thus $(S,+,\cdot)$ is a Classification B seminear-ring w.r.t. a. To prove that $|S| = 2$, suppose not. Then there exists a $c \in S\smallsetminus\{a,b\}$. Therefore $bb = b = cb$ which implies that $b = c$, a contradiction. Hence $|S| = 2$.

Similarly, we can show that if $(S,+,\cdot)$ is an A.M.C. seminear-ring such that $(S,\cdot)$ is a left zero semigroup, then $(S,+,\cdot)$ is a Classification B seminear-ring and $|S| = 2$.

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Theorem 2.9. Let $S = \{a,b\}$ be a Classification B seminear-ring w.r.t. a. If $ab = ba = b$, then S must be isomorphic to a seminear-field with a category I special element of order 2. Otherwise, S must be isomorphic to a seminear-field with a category II special element of order 2 or a seminear-field with a category III or IV special element.

Proof. S must have one of the structures given below :

(1)

$\cdot$	a	b
a	a	b
b	a	b

and

+	a	b
a	a	a
b	a	b

or

(2)

$\cdot$	a	b
a	a	b
b	a	b

and

+	a	b
a	a	a
b	b	b

or

(3)

·	a	b
a	a	b
b	a	b

and

+	a	b
a	a	b
b	a	b

or

(4)

·	a	b
a	a	b
b	a	b

and

+	a	b
a	a	b
b	b	b

or

(5)

·	a	b
a	a	a
b	b	b

and

+	a	b
a	a	a
b	a	a

or

(6)

·	a	b
a	a	a
b	b	b

and

+	a	b
a	a	a
b	a	b

or

(7)

·	a	b
a	a	a
b	b	b

and

+	a	b
a	a	a
b	b	b

or

(8)

·	a	b
a	a	a
b	b	b

and

+	a	b
a	a	b
b	a	b

or

(9)

·	a	b
a	a	a
b	b	b

and

+	a	b
a	a	b
b	b	a

or

(10)

·	a	b
a	a	a
b	b	b

and

+	a	b
a	a	b
b	b	b

or

(11)

·	a	b
a	a	a
b	b	b

and

+	a	b
a	b	a
b	a	b

or

(12)

·	a	b
a	a	a
b	b	b

and

+	a	b
a	b	b
b	b	b

or

(13)

·	a	b
a	a	b
b	b	b

and

+	a	b
a	a	a
b	a	b

or

(14)

·	a	b
a	a	b
b	b	b

and

+	a	b
a	a	a
b	b	b

or

(15)

·	a	b
a	a	b
b	b	b

and

+	a	b
a	a	b
b	a	b

or

(16)

·	a	b
a	a	b
b	b	b

and

+	a	b
a	a	b
b	b	b

or

(17)

·	a	b
a	a	b
b	b	b

and

+	a	b
a	b	a
b	a	b

or

(18)

·	a	b
a	a	b
b	b	b

and

+	a	b
a	b	b
b	b	b

.

Let K = {a',e'} be a seminear-field with a' as a category IV special element. Then K must have one of the structures given below :

(i)

·	a'	e'
a'	a'	e'
e'	a'	e'

and

+	a'	a'
a'	a'	a'
e'	a'	e'

or

(ii)

·	a'	e'
a'	a'	e'
e'	a'	e'

and

+	a'	e'
a'	a'	a'
e'	e'	e'

or

(iii)

·	a'	e'
a'	a'	e'
e'	a'	e'

and

+	a'	e'
a'	a'	e'
e'	a'	e'

or

(iv)

·	a'	e'
a'	a'	e'
e'	a'	e'

and

+	a'	e'
a'	a'	e'
e'	e'	e'

.

Define F : S → K by F(a) = a' and F(b) = e'. Then we can show that (1) ≅ (i), (2) ≅ (ii), (3) ≅ (iii) and (4) ≅ (iv).

Let K = {a',e'} be a seminear-field with a' as a category III special element. Then K must have one of the structures given below :

(v)

·	a'	e'
a'	a'	a'
e'	e'	e'

and

+	a'	e'
a'	a'	a'
e'	a'	a'

or

(vi)

·	a'	e'
a'	a'	a'
e'	e'	e'

and

+	a'	e'
a'	a'	a'
e'	a'	e'

or

(vii)

·	a'	e'
a'	a'	a'
e'	e'	e'

and

+	a'	e'
a'	a'	a'
e'	e'	e'

or

(viii)

·	a'	e'
a'	a'	a'
e'	e'	e'

and

+	a'	e'
a'	a'	e'
e'	a'	e'

or

(ix)

·	a'	e'
a'	a'	a'
e'	e'	e'

and

+	a'	e'
a'	a'	e'
e'	e'	a'

or

(x)

·	a'	e'
a'	a'	a'
e'	e'	e'

and

+	a'	e'
a'	a'	e'
e'	e'	e'

or

(xi)

$\cdot$	a'	e'
a'	a'	a'
e'	e'	e'

and

+	a'	e'
a'	e'	a'
e'	a'	e'

or

(xii)

$\cdot$	a'	e'
a'	a'	a'
e'	e'	e'

and

+	a'	e'
a'	e'	e'
e'	e'	e'

.

Define $F : S \to K$ by $F(a) = a'$ and $F(b) = e'$. Then we can show that (5) ≅ (v), (6) ≅ (vi), ..., (12) ≅ (xii).

Let $K = \{a', e'\}$ be a seminear-field with a' as a category I special element. Then K must have one of the structures given below :

(c)

$\cdot$	a'	e'
a'	a'	a'
e'	a'	e'

and

+	a'	e'
a'	a'	e'
e'	e'	e'

or

(d)

$\cdot$	a'	e'
a'	a'	a'
e'	a'	e'

and

+	a'	e'
a'	a'	a'
e'	e'	e'

or

(e)

$\cdot$	a'	e'
a'	a'	a'
e'	a'	e'

and

+	a'	e'
a'	a'	e'
e'	a'	e'

or

(f)

$\cdot$	a'	e'
a'	a'	a'
e'	a'	e'

and

+	a'	e'
a'	a'	a'
e'	a'	e'

or

(g)

$\cdot$	a'	e'
a'	a'	a'
e'	a'	e'

and

+	a'	e'
a'	a'	e'
e'	e'	a'

or

(h)

$\cdot$	a'	e'
a'	a'	a'
e'	a'	e'

and

+	a'	e'
a'	a'	a'
e'	a'	a'

.

Define $F : S \to K$ by $F(a) = e'$ and $F(b) = a'$. Then we can show that (13) $\cong$ (c), (14) $\cong$ (d), ..., (18) $\cong$ (h).

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Proposition 2.10. Let S be a Classification B seminear-ring w.r.t. a. If a is M.C. in S, then $ax \neq a$ and $xa \neq a$ for all $x \in S \setminus \{a\}$.

Proof. Suppose that there exists a $b \in S \setminus \{a\}$ such that $ab = a$ or $ba = a$. Then $ab = aa$ or $ba = aa$. Therefore a is not M.C. in S. Hence we have the proposition.

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Let S be a Classification B seminear-ring w.r.t. a. Then there is a $b \in S \setminus \{a\}$ such that $ab \neq a$ or $ba \neq a$. We shall now show that b may not be unique.

Example 2.11. Z^+ with the usual addition and multiplication is a Classification B seminear-ring w.r.t. 1 but $2,3 \in Z^+ \setminus \{1\}$ are such that $2.1 = 2 \neq 1$ and $3.1 = 3 \neq 1$.

Proposition 2.12. Let S be a Classification B seminear-ring w.r.t. a. If there exists a $b \in S \setminus \{a\}$ such that $ba = a$ and $xa = x$ for all $x \in S \setminus \{b\}$, then the following statements hold:

(i) $S\setminus\{b\}$ is a seminear-ring.

(ii) For all $x \in S\setminus\{b\}$, $b+x = b$ or $b+x = a+x$.

(iii) If S is A.C., then $|S| = 2$.

Proof. (i) Let $x,y \in S\setminus\{b\}$. If $x+y = b$, then $b = x+y = xa+ya = (x+y)a = ba = a$. This is a contradiction, so $x+y \neq b$. Similarly, we can show that $xy \neq b$. Hence $S\setminus\{b\}$ is a seminear-ring.

(ii) Let $x \in S\setminus\{b\}$. If $b+x = b$, then done. Assume that $b+x \neq b$. Then $b+x = (b+x)a = ba+xa = a+x$.

(iii) Assume that S is A.C.. To prove that $|S| = 2$, suppose not. Let $x \in S\setminus\{a,b\}$. By (ii), $b+x = a+x$ or $b+x = b$.

Case 1. $b+x = a+x$. Then x is not A.C., a contradiction.

Case 2. $b+x = b$. By (ii), $b+a = b$ or $b+a = a+a$.

Subcase 2.1. $b+a = a+a$. Then a is not A.C., a contradiction.

Subcase 2.2. $b+a = b$. Then $b+a = b = b+x$. Hence b is not A.C., a contradiction.

#

The converse of (iii) in Proposition 2.12 is not always true as the next example shows.

Example 2.13. Let $S = \{a,b\}$. Define $\cdot$ and $+$ on S as follow :

$\cdot$	a	b
a	a	b
b	a	b

and

$+$	a	b
a	a	a
b	b	b

By Proposition 2.8, S is a Classification B seminear-ring w.r.t. a. S is not A.C. because $a+a = a+b$ but $a \neq b$.

Proposition 2.14. Let S be a Classification C seminear-ring w.r.t. a. Let $b \in S \setminus \{a\}$ be such that $ab = a$. If there exists an element $c \in S$ such that $ac = a$, then $b = c$.

Proof. Clearly, $c \neq a$. Since $ab = a = ac$, $a^2b = a^2c$. Since $a^2 \neq a$ and $b \neq a$ and $c \neq a$, $b = c$. #

Corollary 2.15. Let S be a Classification C seminear-ring w.r.t. a. Let $b \in S \setminus \{a\}$ be such that $ab = a$. Then $ax \neq a$ for all $x \in S \setminus \{b\}$.

Proof. Follows directly from Proposition 2.14. #

Proposition 2.16. Let S be a Classification C seminear-ring w.r.t. a. Let $b \in S \setminus \{a\}$ be such that $ab = a$. Then the following statements hold :

(i) $b^2 = b$.

(ii) $bx = x$ for all $x \in S \setminus \{a\}$.

(iii) $xb = x$ for all $x \in S$.

Proof. (i) Since $ab = a$, $a^2b^2 = a^2b$. Since $a^2 \neq a$ and $b \neq a$, $b^2 = b$.

(ii) Let $x \in S \setminus \{a\}$. Then $b^2x = bx$. Since $b \neq a$ and $bx \neq a$, $bx = x$.

(iii) Using a proof similar to the proof of (ii), we have that $xb = x$ for all $x \in S \setminus \{a\}$. Since $ab = a$, $xb = x$ for all $x \in S$. #

Proposition 2.17. Let S be a finite Classification C seminear-ring w.r.t. a. Let $b \in S \setminus \{a\}$ be such that $ab = a$. Then $(S \setminus \{a\}, \cdot)$ is a group and b is the identity of $(S \setminus \{a\}, \cdot)$.

Proof. It follows from Proposition 1.6 and Proposition 2.16(i). #

Proposition 2.18. Let S be a finite Classification C seminear-ring w.r.t. a. Let $b \in S \setminus \{a\}$ be such that $ab = a$. If b is either an additive zero or an additive identity of $S \setminus \{a\}$, then $|S| = 2$.

Proof. It follows from Proposition 1.16 and Proposition 1.17. #

Proposition 2.19. A Classification C seminear-ring of order 2 must be isomorphic to a seminear-field with a category V special element.

Proof. Let S = {a,b} be a Classification C seminear-ring w.r.t. a. Then S must have one of the structures given below :

(1)

·	a	b
a	b	a
b	a	b

and

+	a	b
a	a	a
b	b	b

or

(2)

·	a	b
a	b	a
b	a	b

and

+	a	b
a	a	b
b	a	b

.

Let K = {a',e'} be a seminear-field with a' as a category V special element. Then K must have one of the structures given below :

(i)

·	a'	e'
a'	e'	a'
e'	a'	e'

and

+	a'	e'
a'	a'	a'
e'	e'	e'

or

(ii)

·	a'	e'
a'	e'	a'
e'	a'	e'

and

+	a'	e'
a'	a'	e'
e'	a'	e'

.

Define $f : S \to K$ by $f(a) = a'$ and $f(b) = e'$. Then we can show that $(1) \cong (i)$ and $(2) \cong (ii)$.

#

<u>Proposition 2.20</u>. Let S be a Classification C seminear-ring w.r.t. a. Let $b \in S \setminus \{a\}$ be such that $ab = a$. If $b+b \neq a$, then for all $x,y \in S\setminus\{a\}$, $x+x = y+y$ if and only if $x = y$.

<u>Proof</u>. Let $x,y \in S\setminus\{a\}$ be such that $x+x = y+y$. Then $(b+b)x = bx+bx = x+x = y+y = by+by = (b+b)y$. Since $b+b \neq a$ and $x \neq a$ and $y \neq a$, $x = y$.

The converse is obvious.

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<u>Remark</u>. If S is a Classification C seminear-ring w.r.t. a, then a may not be L.M.C. in S. First, we shall give an example where a is L.M.C. in S.

<u>Example 2.21</u>. $\mathbb{Z}^+$ with the usual addition and multiplication is a Classification C seminear-ring w.r.t. 2 and 2 is L.M.C. in $\mathbb{Z}^+$.

We shall now give an example of a Classification C seminear-ring w.r.t. a such that a is not L.M.C..

<u>Example 2.22</u>. $\mathbb{Z}^+$ with the usual addition and multiplication is a seminear-ring. Let a be a symbol not representing any element of $\mathbb{Z}^+$. Extend + and $\cdot$ from $\mathbb{Z}^+$ to $\mathbb{Z}^+ \cup \{a\}$ by defining

$$a+x = 2+x \text{ and } x+a = x+2 \quad \text{for all } x \in \mathbb{Z}^+ \cup \{a\},$$

$$1a = a1 = a,\ ax = 2x \text{ and } xa = x2 \quad \text{for all } x \in (\mathbb{Z}^+ \cup \{a\})\setminus\{1\}.$$

We shall show that $\mathbb{Z}^+ \cup \{a\}$ is a Classification C seminear-ring w.r.t. a. Let $x,y,z \in \mathbb{Z}^+ \cup \{a\}$.

To show that $(x+y)+z = x+(y+z)$, we shall consider the following cases :

<u>Case 1</u>. $x = y = z = a$.

$(x+y)+z = (a+a)+a = 4+a = 4+2 = 2+4 = a+4 = a+(a+a) = x+(y+z)$.

<u>Case 2</u>. $x = y = a$, $z \neq a$.

$(x+y)+z = (a+a)+z = (2+2)+z = 2+(2+z) = a+(2+z) = a+(a+z) = x+(y+z)$.

<u>Case 3</u>. $x = z = a$, $y \neq a$.

$(x+y)+z = (a+y)+a = (2+y)+a = (2+y)+2 = 2+(y+2) = a+(y+2) = a+(y+a) = x+(y+z)$.

<u>Case 4</u>. $x = a$, $y \neq a$, $z \neq a$.

$(x+y)+z = (a+y)+z = (2+y)+z = 2+(y+z) = a+(y+z) = x+(y+z)$.

<u>Case 5</u>. $x \neq a$, $y = z = a$.

$(x+y)+z = (x+a)+a = (x+2)+2 = x+(2+2) = x+(a+a) = x+(y+z)$.

<u>Case 6</u>. $x \neq a$, $y = a$, $z \neq a$.

$(x+y)+z = (x+a)+z = (x+2)+z = x+(2+z) = x+(a+z) = x+(y+z)$.

<u>Case 7</u>. $x \neq a$, $y \neq a$, $z = a$.

$(x+y)+z = (x+y)+a = (x+y)+2 = x+(y+2) = x+(y+a) = x+(y+z)$.

<u>Case 8</u>. $x \neq a$, $y \neq a$, $z \neq a$. This case is clear.

To show that $(xy)z = x(yz)$, we shall consider the following cases :

<u>Case 1</u>. $x = y = z = a$.

$(xy)z = (aa)a = 4a = 8 = a(4) = a(aa) = x(yz)$.

<u>Case 2</u>. $x = y = a,\ z \neq a$.

<u>Subcase 2.1</u>. $z \neq 1$. $(xy)z = (aa)z = 4z = 2(2z) = a(2z) = a(az) = x(yz)$.

<u>Subcase 2.2</u>. $z = 1$. $(xy)z = (xy)1 = (aa)1 = aa = a(a1) = x(yz)$.

<u>Case 3</u>. $x = z = a,\ y \neq a$.

<u>Subcase 3.1</u>. $y \neq 1$. $(xy)z = (ay)a = (2y)a = (2y)2 = 2(y2) = a(y2) = a(ya) = x(yz)$.

<u>Subcase 3.2</u>. $y = 1$. $(xy)z = (a1)a = aa = a(1a) = x(yz)$.

<u>Case 4</u>. $x = a,\ y \neq a,\ z \neq a$.

<u>Subcase 4.1</u>. $y \neq 1,\ z \neq 1$. $(xy)z = (ay)z = (2y)z = 2(yz) = a(yz) = x(yz)$.

<u>Subcase 4.2</u>. $y \neq 1,\ z = 1$. $(xy)z = (ay)1 = (2y)1 = 2(y1) = a(y1) = x(yz)$.

<u>Subcase 4.3</u>. $y = 1,\ z \neq 1$. $(xy)z = (a1)z = az = a(1z) = x(yz)$.

<u>Subcase 4.4</u>. $y = z = 1$. $(xy)z = (a1)1 = a1 = a(yz) = x(yz)$.

<u>Case 5</u>. $x \neq a,\ y = z = a$.

<u>Subcase 5.1</u>. $x \neq 1$. $(xy)z = (xa)a = (x2)a = (x2)2 = x4 = x(aa) = x(yz)$.

Subcase 5.2. $x = 1$. $(xy)z = (1a)a = aa = 4 = 1(4) = 1(aa) = x(yz)$.

Case 6. $x \neq a$, $y = a$, $z \neq a$.

Subcase 6.1. $x \neq 1$, $z \neq 1$. $(xy)z = (xa)z = (x2)z = x(2z) = x(az) = x(yz)$.

Subcase 6.2. $x \neq 1$, $z = 1$. $(xy)z = (xa)1 = (x2)1 = x2 = xa = x(a1) = x(yz)$.

Subcase 6.3. $x = 1$, $z \neq 1$. $(xy)z = (1a)z = az = 2z = 1(2z) = 1(az) = x(yz)$.

Subcase 6.4. $x = z = 1$. $(xy)z = (1a)1 = a1 = a = 1a = 1(a1) = x(yz)$.

Case 7. $x \neq a$, $y \neq a$, $z = a$.

Subcase 7.1. $x \neq 1$, $y \neq 1$. $(xy)z = (xy)a = (xy)2 = x(ya) = x(yz)$.

Subcase 7.2. $x \neq 1$, $y = 1$. $(xy)z = (x1)a = x2 = xa = x(1a) = x(yz)$.

Subcase 7.3. $x = 1$, $y \neq 1$. $(xy)z = (1y)a = ya = y2 = 1(y2) = 1(ya) = x(yz)$.

Subcase 7.4. $x = y = 1$. $(xy)z = 1a = 1(1a) = x(yz)$.

Case 8. $x \neq a$, $y \neq a$, $z \neq a$. This case is clear.

Lastly, we shall show that $(x+y)z = xz+yz$. Consider the following cases :

Case 1. $x = y = z = a$.

$(x+y)z = (a+a)a = 4a = 8 = 4+4 = 2a+2a = aa+aa = xz+yz$.

Case 2. $x = y = a$, $z \neq a$.

Subcase 2.1. $z \neq 1$. $(x+y)z = (a+a)z = 4z = 2z+2z = az+az = xz+yz$.

Subcase 2.2. $z = 1$. $(x+y)z = (a+a)1 = (4)1 = 4 = a+a = a1+a1 = xz+yz$.

Case 3. $x = z = a$, $y \neq a$.

Subcase 3.1. $y \neq 1$. $(x+y)z = (a+y)a = (2+y)a = (2+y)2 = 4+y2 = aa+ya = xz+yz$.

Subcase 3.2. $y = 1$. $(x+y)z = (a+1)a = (2+1)a = (2+1)2 = 4+2 = 4+a = aa+1a = xz+yz$.

Case 4. $x = a$, $y \neq a$, $z \neq a$.

Subcase 4.1. $y \neq 1$, $z \neq 1$. $(x+y)z = (a+y)z = (2+y)z = 2z+yz = az+yz = xz+yz$.

Subcase 4.2. $y \neq 1$, $z = 1$. $(x+y)z = (a+y)1 = (2+y)1 = 2+y = a+y = a1+y1 = xz+yz$.

Subcase 4.3. $y = 1$, $z \neq 1$. $(x+y)z = (a+1)z = (2+1)z = 2z+z = az+1z = xz+yz$.

Subcase 4.4. $y = z = 1$. $(x+y)z = (a+1)1 = (2+1)1 = 3 = 2+1 = a+1 = a1+yz = xz+yz$.

Case 5. $x \neq a$, $y = z = a$.

Subcase 5.1. $x \neq 1$. $(x+y)z = (x+a)a = (x+2)a = (x+2)2 = x2+4 = xa+aa = xz+yz$.

Subcase 5.2. $x = 1$. $(x+y)z = (1+a)a = (1+2)a = (1+2)2 = 2+4 = a+4 = 1a+aa = xz+yz$.

Case 6. $x \neq a$, $y = a$, $z \neq a$.

Subcase 6.1. $z \neq 1$. $(x+y)z = (x+a)z = (x+2)z = xz+2z = xz+az = xz+yz$.

Subcase 6.2. $z = 1$. $(x+y)z = (x+a)1 = (x+2)1 = x1+2 = x1+a1 = xz+yz$.

Case 7. $x \neq a$, $y \neq a$, $z = a$.

$(x+y)z = (x+y)a = (x+y)2 = x2+y2 = xa+ya = xz+yz$.

Case 8. $x \neq a$, $y \neq a$, $z \neq a$. This case is clear.

Thus $\mathbf{Z}^+ \cup \{a\}$ is a seminear-ring. Since $\mathbf{Z}^+$ is a cancellative semigroup and $1 \in \mathbf{Z}^+$ is such that $a1 = a$ and $a^2 = 4 \neq a$, $\mathbf{Z}^+ \cup \{a\}$ is a Classification C seminear-ring w.r.t. a. Furthermore, a is not L.M.C. in S because $aa = a2$ but $a \neq 2$.

Proposition 2.23. Let S be a Classification C seminear-ring w.r.t. a such that a is L.M.C. in S. Let $b \in S\smallsetminus\{a\}$ be such that $ab = a$. Then $ba = a$ and b is L.M.C. in S. Furthermore, if a is R.M.C. in S, then b is R.M.C. in S.

Proof. Assume that a is L.M.C. in S. To show that $ba = a$, suppose not. Since $a(ba) = (ab)a = aa$, a is not L.M.C. in S. This is a contradiction, so $ba = a$. Let $x,y \in S$ be such that $bx = by$. Then $(ab)x = (ab)y$ which implies that $ax = ay$. Hence $x = y$.

Furthermore, assume that a is R.M.C. in S. Let $x,y \in S$ be such that $xb = yb$. Then $xa = x(ba) = (xb)a = (yb)a = y(ba) = ya$. Hence $x = y$. Thus b is R.M.C. in S.

#

Theorem 2.24. Let S be a finite Classification C seminear-ring w.r.t. a. Let $b \in S\setminus\{a\}$ be such that $ab = a$. Then for any $c \in D = S\setminus\{a\}$, the following statements hold :

(i) $LI_D(c) = LI_D(b).\ c$.

(ii) $RI_D(c) = RI_D(b).\ c$.

(iii) $LI_D(b) \cdot LI_D(c) \subseteq LI_D(c)$.

(iv) $RI_D(b) \cdot RI_D(c) \subseteq RI_D(c)$.

Proof. By Proposition 2.17, b is the identity of $(D,\cdot)$.

(i) Let $x \in LI_D(b).\ c$. Then $x = yc$ for some $y \in LI_D(b)$, so $y+b = b$. Therefore $x+c = yc+bc = (y+b)c = bc = c$. Thus $x \in LI_D(c)$ and hence $LI_D(b) \cdot c \subseteq LI_D(c)$. On the other hand, let $x \in LI_D(c)$. Then $x+c = c$, so $b = cc^{-1} = xc^{-1}+b$. Therefore $xc^{-1} \in LI_D(b)$ and hence $x = xc^{-1} \cdot c \in LI_D(b).\ c$. Thus $LI_D(c) \subseteq LI_D(b).\ c$.

The proof of (ii) is similar to the proof of (i).

(iii) Let $x \in LI_D(b) \cdot LI_D(c)$. Then $x = yz$ for some $y \in LI_D(b)$ and some $z \in LI_D(c)$, so $y+b = b$ and $z+c = c$. Therefore $x+c = yz+c = yz+(z+c) = (yz+z)+c = (yz+bz)+c = (y+b)z+c = z+c = c$. Thus $x \in LI_D(c)$ and hence $LI_D(b) \cdot LI_D(c) \subseteq LI_D(c)$.

The proof of (iv) is similar to the proof of (iii).

#

<u>Theorem 2.25</u>. Let S be a Classification D seminear-ring w.r.t. a. Then a is not L.M.C. in S if and only if there exists a unique $d \in S \setminus \{a\}$ such that $ax = dx$ for all $x \in S \setminus \{a\}$.

<u>Proof</u>. Assume that a is not L.M.C. in S. Then there exist $d, y \in S$ such that $d \neq y$ and $ay = ad$, so $a^2y = a^2d$. If $y \neq a$ and $d \neq a$, then $y = d$, a contradiction. Therefore $y = a$ or $d = a$. Without loss of generality, assume that $y = a$. Then $d \neq a$ and $a^2 = ad$. Let $x \in S \setminus \{a\}$. Then $a^2(ax) = a^2(dx)$ which implies that $ax = dx$. To prove uniqueness, assume that there exists a $d' \in S \setminus \{a\}$ such that $ax = d'x$ for all $x \in S \setminus \{a\}$. Then $dd = ad = d'd$, so $d = d'$.

Conversely, assume that there exists a unique $d \in S \setminus \{a\}$ such that $ax = dx$ for all $x \in S \setminus \{a\}$. Then $(a^2)d = a(ad) = a(dd) = (ad)d$. Since $a^2 \neq a$ and $ad \neq a$ and $d \neq a$, $a^2 = ad$. Hence a is not L.M.C..

#

<u>Proposition 2.26</u>. Let S be a Classification D seminear-ring w.r.t. a such that a is not L.M.C. in S. Let $d \in S \setminus \{a\}$ be such that $ax = dx$ for all $x \in S \setminus \{a\}$. If $xa \neq a$ for all $x \in S \setminus \{a\}$, then $xa = xd$ for all $x \in S \setminus \{a\}$ and $ad = da$.

<u>Proof</u>. Let $x \in S \setminus \{a\}$. Then $ax = dx$, so $(xa)x = (xd)x$. Since $xa \neq a$ and $xd \neq a$ and $x \neq a$, $xa = xd$. Moreover, $ad = dd = da$.

#

<u>Theorem 2.27</u>. Let S be an A.M.C. seminear-ring w.r.t. a such that $a^2 \neq a$. Then S is a Classification E seminear-ring w.r.t. a if and only if $ax = x$ for all $x \in S \setminus \{a\}$.

Proof. Assume that S is a Classification E seminear-ring w.r.t. a. Let $x \in S \setminus \{a\}$. Then $a^2x = a^2(ax)$. Since $ax \neq a$ and $x \neq a$ and $a^2 \neq a$, $ax = x$.

Conversely, assume that $ax = x$ for all $x \in S \setminus \{a\}$. Let $x_o \in S \setminus \{a\}$. Then $ax_o = x_o \neq a$. Since $a^2 \neq a$, $ax \neq a$ for all $x \in S$. Since $a^2x_o = a^3x_o$, $a^2 = a^3$. By induction, $a^2 = a^n$ for all $n \in Z^+ \setminus \{1\}$. Thus S is a Classification E seminear-ring w.r.t. a.

#

Proposition 2.28. Let S be a Classification E seminear-ring w.r.t. a. Then $xa \neq a$ for all $x \in S$.

Proof. Suppose that there exists an $x \in S$ such that $xa = a$. Clearly, $x \neq a$. Since $xa^2 = a^2 = a^2a^2$, $x = a^2$. Thus $a = xa = a^2a = a^3$, a contradiction.

#

Corollary 2.29. Let S be a Classification E seminear-ring w.r.t. a. Then $xa = x$ for all $x \in S \setminus \{a\}$.

Proof. Let $x \in S \setminus \{a\}$. By Proposition 2.28, $xa \neq a$. Since $(xa)a^2 = xa^3 = xa^2$ and $xa \neq a$ and $a^2 \neq a$, $xa = x$.

#

Proposition 2.30. Let S be a Classification E seminear-ring w.r.t. a. If S is also a Classification E seminear-ring w.r.t. b, then a = b.

Proof. Suppose that $a \neq b$. Then $a \in S \setminus \{b\}$ and $b \in S \setminus \{a\}$. By Theorem 2.27, $ab = b$ and $ba = a$. Therefore $a = ba = (ab)a = a(ba) = aa$, a contradiction.

#

We shall now show that if S is a Classification B, C or D seminear-ring w.r.t. a, then a may not be unique. First, we shall give some examples where a is unique.

Example 2.31. $\mathbb{Z}^+$ with the usual addition and multiplication is a Classification B seminear-ring w.r.t. 1 and 1 is the unique essential element of $\mathbb{Z}^+$.

Example 2.32. Let S = {a,b}. Define · and + on S as follows :

·	a	b
a	b	a
b	a	b

and

+	a	b
a	a	a
b	b	b

Then (S,+,·) is a Classification C seminear-ring w.r.t. a and a is the unique essential element of S.

Example 2.33. Let S = {a,b,c}. Define · and + as follows :

·	a	b	c
a	c	c	b
b	c	c	b
c	b	b	c

and

+	a	b	c
a	a	a	a
b	b	b	b
c	c	c	c

Then S is a Classification D seminear-ring w.r.t. a and a is the unique essential element of S.

We shall now give some examples of Classification B, C and D seminear-rings w.r.t. a such that a is not unique.

Example 2.34. Let S = {a,b}. Define + and · on S by x+y = x = x·y for all x,y ε S. Then S is a Classification B seminear-ring w.r.t. a and a Classification B seminear-ring w.r.t. b.

Example 2.35. $\mathbb{Z}^+$ with the usual addition and multiplication is a Classification C seminear-ring w.r.t. 2 and a Classification C seminear-ring w.r.t. 3.

Example 2.36. $\mathbb{Z}^+\setminus\{1\}$ with the usual addition and multiplication is a Classification D seminear-ring w.r.t. 2 and a Classification D seminear-ring w.r.t. 3.

Proposition 2.37. Let S be a Classification D or E seminear-ring w.r.t. a. If S is finite, then $S\setminus\{a\}$ is a seminear-ring.

Proof. It suffices to show that $x+y \neq a$ for all $x,y \in S\setminus\{a\}$. Let e be the identity of the group $(S\setminus\{a\},\cdot)$. Let $x,y \in S\setminus\{a\}$. If $x+y = a$, then $a = x+y = xe+ye = (x+y)e = ae$, a contradiction. Thus $x+y \neq a$.

#

The converse of this proposition is not always true as the next examples show.

Example 2.38. Let $S = \{x \in \mathbb{Q}^+ \mid x > 1\}$. Define + on S by x+y = minimum of x,y for all $x,y \in S$. Then $(S,+,\cdot)$ is a seminear-ring where $\cdot$ is the usual multiplication. Let a be a symbol not representing any element of S. Extend + and $\cdot$ from S to $S \cup \{a\}$ by defining

$$ax = 2x \quad , \quad xa = x2,$$
$$a+x = 2+x \quad \text{and} \quad x+a = x+2$$

for all $x \in S \cup \{a\}$. Claim that $S \cup \{a\}$ is a Classification D seminear-ring w.r.t. a. The proof that $S \cup \{a\}$ is a seminear-ring is the same as the proof used in the proof of Example 2.22. Clearly, $(S,\cdot)$ is a cancellative semigroup, $ax \neq a$, $ax \neq x$ and $xa \neq x$ for all $x \in S \cup \{a\}$. Therefore $S \cup \{a\}$ is an infinite Classification D seminear-ring w.r.t. a.

Example 2.39. $(\mathbb{Z}^+, \oplus, \cdot)$ is a seminear-ring where $x \oplus y = x$ for all $x, y \in \mathbb{Z}^+$ and $\cdot$ is the usual multiplication. Let a be a symbol not representing any element of $\mathbb{Z}^+$. Extend $\oplus$ and $\cdot$ on $\mathbb{Z}^+$ to $\mathbb{Z}^+ \cup \{a\}$ by defining

$$a^2 = 1 \quad , \quad ax = x = xa \quad \text{for all } x \in \mathbb{Z}^+,$$

$$a \oplus x = a \quad \text{and} \quad x \oplus a = x \quad \text{for all } x \in \mathbb{Z}^+ \cup \{a\}.$$

We shall show that $\mathbb{Z}^+ \cup \{a\}$ is a Classification E seminear-ring w.r.t. a.

Let $x, y, z \in \mathbb{Z}^+ \cup \{a\}$. Then $(x \oplus y) \oplus z = x \oplus z = x = x \oplus y = x \oplus (y \oplus z)$ and $(x \oplus y)z = xz = xz \oplus yz$.

To show that $(xy)z = x(yz)$, we shall consider several cases :

Case 1. $x = y = z = a$.

$(xy)z = (aa)a = 1a = 1 = a1 = a(aa) = x(yz)$.

Case 2. $x = y = a$, $z \neq a$.

$(xy)z = (aa)z = 1z = z = az = a(az) = x(yz)$.

Case 3. $x = a = z$, $y \neq a$.

$(xy)z = (ay)a = ya = y = ay = a(ya) = x(yz)$.

Case 4. $x = a$, $y \neq a$, $z \neq a$.

$(xy)z = (ay)z = yz = a(yz) = x(yz)$.

Case 5. $x \neq a$, $y = z = a$.

$(xy)z = (xa)a = xa = x = x1 = x(aa) = x(yz)$.

Case 6. $x \neq a$, $y = a$, $z \neq a$.

$(xy)z = (xa)z = xz = x(az) = x(yz)$.

Case 7. $x \neq a, y \neq a, z = a$.

$$(xy)z = (xy)a = xy = x(ya) = x(yz).$$

Case 8. $x \neq a, y \neq a, z \neq a$. This case is clear.

Thus $Z^+ \cup \{a\}$ is a seminear-ring. Clearly, $(Z^+,\cdot)$ is a cancellative semigroup, $a^2 = 1 \neq a$, $ax = x \neq a$ for all $x \in S\setminus\{a\}$ and $a^2 = a^n = 1$ for all $n \in Z^+\setminus\{1\}$. Thus $Z^+ \cup \{a\}$ is an infinite Classification E seminear-ring w.r.t. a.

Proposition 2.40. Let S be a Classification D seminear-ring w.r.t. a. If a is L.M.C. in S, then S is infinite.

Proof. Assume that a is L.M.C. in S. To prove that S is infinite, suppose not, say $S = \{a,x_1,\ldots,x_n\}$. Then for any $i \in \{1,2,\ldots,n\}$ there exists a $j \in \{1,2,\ldots,n\}\setminus\{i\}$ such that $ax_i = x_j$.

Case 1. There exist distinct $i,j \in \{1,2,\ldots,n\}$ such that $ax_i = ax_j$. Then a is not L.M.C., a contradiction.

Case 2. For all distinct $i,j \in \{1,2,\ldots,n\}$, $ax_i \neq ax_j$. Then $\{ax_1,ax_2,\ldots,ax_n\} = \{x_1,x_2,\ldots,x_n\}$. Since $a^2 \neq a$, there exists a $k \in \{1,2,\ldots,n\}$ such that $a^2 = ax_k$. Hence a is not L.M.C., a contradiction.

Thus S is infinite.

#

The converse of this proposition is not always true as Example 2.38 showed.

Proposition 2.41. Let S be a Classification D seminear-ring w.r.t. a such that a is not L.M.C. in S. Let $d \in S\setminus\{a\}$ be such that $ax = dx$

for all $x \in S \setminus \{a\}$. Then the following statements hold:

(1) For all $x \in S$, $a+x = a$ or $a+x = d$ or $a+x = d+x$.

(2) For all $x \in S$, $x+a = a$ or $x+a = d$ or $x+a = x+d$.

(3) $a+a = a$ or $a+a = d$ or $a+a = d+d$.

Proof. (1) Let $x \in S$. Assume that $a+x \neq a$. Then $(a+x)d = ad+xd = dd+xd = (d+x)d$.

Case 1. $d+x \neq a$. Then $a+x = d+x$, so done.

Case 2. $d+x = a$. Then $(a+x)d = (d+x)d = ad = dd$, so $a+x = d$.

The proof of (2) is similar to the proof of (1).

(3) Suppose that $a+a \neq a$. Then $(a+a)d = ad+ad = dd+dd = (d+d)d$.

Case 1. $d+d \neq a$. Then $a+a = d+d$, so done.

Case 2. $d+d = a$. Then $(a+a)d = (d+d)d = ad = dd$, so $a+a = d$.

#

Proposition 2.42. Let S be a Classification E seminear-ring w.r.t. a. Then the following statements hold:

(1) For all $x \in S$, $a+x = a$ or $a+x = a^2$ or $a+x = a^2+x$.

(2) For all $x \in S$, $x+a = a$ or $x+a = a^2$ or $x+a = x+a^2$.

(3) $a^2+a^2 = a^2$ or $a^2+a^2 = a+a$.

Proof. The proofs of (1) and (2) are similar to the one used in Proposition 2.41 (substitute a^2 for d).

To show (3), suppose that $a^2+a^2 \neq a^2$. Then $a+a \neq a$. Since $(a+a)a^2 = (a^2+a^2)a^2$, $a+a = a^2+a^2$.

#

Proposition 2.43. Let S be a Classification D seminear-ring w.r.t. a such that a is not L.M.C. in S. Let $d \in S\setminus\{a\}$ be such that $ax = dx$ for all $x \in S\setminus\{a\}$. If $x+y \neq a$ for all $x,y \in S$, then we get that:

(1) $a+x = d+x$ and $x+a = x+d$ for all $x \in S$.

(2) $a+a = d+d$.

Proof. (1) Let $x \in S$. Then $(a+x)d = ad+xd = dd+xd = (d+x)d$. Since $a+x \neq a$ and $d+x \neq a$, $a+x = d+x$. Similarly, we can show that $x+a = x+d$.

(2) Since $(a+a)d = ad+ad = (d+d)d$, $a+a = d+d$.
#

Proposition 2.44. Let S be a Classification E seminear-ring w.r.t. a. If $x+y \neq a$ for all $x,y \in S$, then the following statements hold:

(1) $a+x = a^2+x$ and $x+a = x+a^2$ for all $x \in S$.

(2) $a+a = a^2+a^2$.

Proof. The proof of this proposition is similar to the proof of Proposition 2.43 (substitute a^2 for d).
#

Proposition 2.45. Let S be a Classification D or E seminear-ring w.r.t. a. Then $xy \neq a$ for all $x \in S$ and all $y \in S\setminus\{a\}$.

Proof. The proof is obvious.
#

Proposition 2.46. Let S be a Classification E seminear-ring w.r.t. a. Then $xy \neq a$ for all $x,y \in S$.

Proof. It follows from Proposition 2.28 and Proposition 2.45.
#

Theorem 2.47. There does not exist a Classification D seminear-ring which is also a seminear-field with a category I,II,III,IV or V special element.

Proof. Let S be a Classification D seminear-ring w.r.t. a. Then $ax \neq a$, $ax \neq x$ and $xa \neq x$ for all $x \in S$ and $|S| > 2$. Suppose that S is a seminear-field with a category I,II,III,IV or V special element.

Case 1. S is a seminear-field with b as a category I special element. Then $xb = b$ for all $x \in S$. Hence $ab = b$, a contradiction.

Case 2. S is a seminear-field with b as a category II special element. Then $xb = x$ for all $x \in S$. Hence $ab = a$, a contradiction.

Case 3. S is a seminear-field with a category III, IV or V special element. Then $|S| = 2$, a contradiction.

#

Now we shall give an example of a Classification D seminear-ring which is also a seminear-field with a category VI special element.

Example 2.48. Let $S = \{a,b,c\}$. Define $\cdot$ and + on S as follows :

·	a	b	c
a	b	c	b
b	c	b	c
c	b	c	b

and

+	a	b	c
a	a	b	c
b	a	b	c
c	a	b	c

Then S is a Classification D seminear-ring w.r.t. a which is a seminear-field with a as a category VI special element.

Proposition 2.49. Let S be a Classification D seminear-ring w.r.t. a such that a is not L.M.C. in S. Let $d \in S \setminus \{a\}$ be such that $ax = dx$ for all $x \in S \setminus \{a\}$. If $xa \neq a$ for all $x \in S \setminus \{a\}$, then $S \setminus \{d\}$ is M.C..

Proof. By Proposition 2.26, $xa = xd$ for all $x \in S \setminus \{a\}$. Let $x,y,z \in S \setminus \{d\}$ be such that $xy = xz$.

Case 1. $x = a$. Then $ay = az$.

Subcase 1.1. $y = a$. If $z \neq a$, then $dd = ad = aa = az = dz$. Thus $d = z$, a contradiction. Hence $z = a = y$.

Subcase 1.2. $z = a$. Using a proof similar to the proof of Subcase 1.1, we get that $y = a$.

Subcase 1.3. $y \neq a$ and $z \neq a$. Then $dy = ay = az = dz$, so $y = z$.

Case 2. $x \neq a$.

Subcase 2.1. $y = a$. If $z \neq a$, then $xz = xa = xd$. Thus $z = d$, a contradiction. Hence $z = a$.

Subcase 2.2. $z = a$. Using a proof similar to the proof of Subcase 2.1, we get that $y = a$.

Subcase 2.3. $y,z \in S \setminus \{a,d\}$. Since $S \setminus \{a\}$ is M.C., we are done.

Hence $S \setminus \{d\}$ is L.M.C.. Similarly, we can show that $S \setminus \{d\}$ is R.M.C..

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Proposition 2.50. Let S be a Classification E seminear-ring w.r.t. a. Then $S \setminus \{a^2\}$ is M.C..

Proof. The proof of this proposition is similar to the proof of Proposition 2.49 (substitute a^2 for d). #

Proposition 2.51. Let S be a Classification D seminear-ring w.r.t. a such that a is not L.M.C. in S. Let $d \in S \setminus \{a\}$ be such that $ax = dx$ for all $x \in S \setminus \{a\}$. If $xa \neq a$ for all $x \in S \setminus \{a\}$ and $u+v \neq d$ for all $u,v \in S$, then the following statements hold:

(1) $a+x = d+x$ and $x+a = x+d$ for all $x \in S$.

(2) $a+a = d+d$.

Proof. By Proposition 2.26, $xa = xd$ for all $x \in S \setminus \{a\}$ and $ad = da$. By Proposition 2.49, $S \setminus \{d\}$ is M.C..

(1) Let $x \in S$. Then $(a+x)a = aa+xa = da+xa = (d+x)a$. Since $a+x \neq d$ and $d+x \neq d$ and $a \neq d$, $a+x = d+x$. Similarly, we can show that $x+a = x+d$.

(2) Since $(a+a)a = aa+aa = da+da = (d+d)a$ and $a+a \neq d$ and $d+d \neq d$ and $a \neq d$, $a+a = d+d$. #

Proposition 2.52. Let S be an A.M.C. seminear-ring w.r.t. a such that $|S| > 2$. Then for any $x \in S \setminus \{a\}$, x is not a multiplicative zero of S.

Proof. Suppose that there exists an $x_o \in S \setminus \{a\}$ such that x_o is a multiplicative zero of S. Then $x_o x = x_o = x x_o$ for all $x \in S$. Let $y \in S \setminus \{a, x_o\}$. Then $x_o x_o = x_o = x_o y_o$. Thus $x_o = y_o$, a contradiction. #

Corollary 2.53. Let S be a Classification D seminear-ring. Then S contains no multiplicative zero.

Remark. Let $S = \{a,b\}$ be a Classification E seminear-ring w.r.t. a. Then $xy = b$ for all $x,y \in S$, so b is a multiplicative zero of S.

Proposition 2.54. Let S be a Classification D seminear-ring w.r.t. a such that a is not L.M.C. in S. Let $d \in S\setminus\{a\}$ be such that $ax = dx$ for all $x \in S\setminus\{a\}$. If $x+d \neq a$ for all $x \in LI_S(a)$, then we get that:

(1) Either $LI_S(d) = \phi$ or $LI_S(d)$ is an additive semigroup and either $LI_S(a) = \phi$ or $LI_S(a)$ is an additive subsemigroup of $LI_S(d)$.

(2) If $d \in LI_S(a)$, then $LI_S(a) = LI_S(d)$.

Proof. (1) Assume that $LI_S(d) \neq \phi$. Let $x,y \in LI_S(d)$. Then $x+d = d = y+d$. Therefore $(x+y)+d = x+(y+d) = x+d = d$. Thus $x+y \in LI_S(d)$. Hence $LI_S(d)$ is an additive semigroup.

Assume that $LI_S(a) \neq \phi$. Let $x \in LI_S(a)$. Then $x+a = a$. Therefore $dd = ad = (x+a)d = (x+d)d$. Thus $d = x+d$, so $LI_S(a) \subseteq LI_S(d)$. Let $x,y \in LI_S(a)$. Then $x+a = a = y+a$ and $(x+y)+a = x+(y+a) = x+a = a$, so $x+y \in LI_S(a)$. Thus $LI_S(a)$ is an additive subsemigroup of $LI_S(d)$.

(2) Assume that $d \in LI_S(a)$. Then $d+a = a$. By (1), $LI_S(a) \subseteq LI_S(d)$. Let $x \in LI_S(d)$. Then $x+d = d$. Therefore $a = d+a = (x+d)+a = x+(d+a) = x+a$. Hence $x \in LI_S(a)$. Thus $LI_S(d) \subseteq LI_S(a)$, so $LI_S(a) = LI_S(d)$. #

Proposition 2.55. Let S be a Classification D seminear-ring w.r.t. a such that a is not L.M.C. in S. Let $d \in S\setminus\{a\}$ be such that $ax = dx$ for all $x \in S\setminus\{a\}$. If $d+x \neq a$ for all $x \in RI_S(a)$, then we get that:

(1) Either $RI_S(d) = \phi$ or $RI_S(d)$ is an additive semigroup and either $RI_S(a) = \phi$ or $RI_S(a)$ is an additive subsemigroup of $RI_S(d)$.

(2) If $d \in RI_S(a)$, then $RI_S(a) = RI_S(d)$.

Proof. The proof of this proposition is similar to the proof of Theorem 2.54.

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The proofs of the following two propositions are similar to the proofs of Proposition 2.54 and Proposition 2.55, respectively (substitute a^2 for d).

Proposition 2.56. Let S be a Classification E seminear-ring w.r.t. a. If $x+a^2 \neq a$ for all $x \in LI_S(a)$, then the following statements hold:

(1) Either $LI_S(a^2) = \phi$ or $LI_S(a^2)$ is an additive semigroup and either $LI_S(a) = \phi$ or $LI_S(a)$ is an additive subsemigroup of $LI_S(a^2)$.

(2) If $a^2 \in LI_S(a)$, then $LI_S(a) = LI_S(a^2)$.

Proposition 2.57. Let S be a Classification E seminear-ring w.r.t. a. If $a^2+x \neq a$ for all $x \in RI_S(a)$, then the following statements hold:

(1) Either $RI_S(a^2) = \phi$ or $RI_S(a^2)$ is an additive semigroup and either $RI_S(a) = \phi$ or $RI_S(a)$ is an additive subsemigroup of $RI_S(a^2)$.

(2) If $a^2 \in RI_S(a)$, then $RI_S(a) = RI_S(a^2)$.