CHAPTER III

REDUCTION THEOREMS

Let S be a commutative inverse semigroup and F a field. We are interested in finding functions f, g from S to F such that

(*) $$g(xy^{-1}) = g(x)g(y) + f(x)f(y)$$

for all x in S. The purpose of this chapter is to provide theorems which reduce this problem into simpler problems.

Definition 3.1. Let A be any subset of a commutative inverse semigroup S, F a field and f, g functions from A into F. We say that (f,g) is a solution of (*) on A if f,g satisfy (*) for all x,y in A such that xy^{-1} is also in A. If f,g are functions on S into F such that (f,g) is a solution of (*) on S we simply say that (f,g) is a solution of (*).

Remark 3.2. Note that in Definition 3.1, we allow A to be empty. In such a case $(\emptyset,\emptyset)$ is the solution of (*) on A. In case $A \neq \emptyset$, the pair (f,g), where

$$f(x) = 0 \qquad \text{and} \qquad g(x) = 0$$

for all x in A, is always a solution of (*) on A. Such a solution is said to be trivial. Any solution of (*) on A which is not trivial is said to be non-trivial.

Lemma 3.3. Let A be an inverse subsemigroup of S and (f,g) a solution of (*) on A. Then for any x in A we have

(3.3.1) $$f(x)^2 + g(x)^2 = g(xx^{-1}),$$

(3.3.2) $$g(x) = g(x^{-1}),$$

(3.3.3) $$f(x)f(xx^{-1}) = g(x) - g(x)g(xx^{-1}),$$

(3.3.4) $$f(x) = f(x^{-1}) \quad \text{or} \quad f(x) = -f(x^{-1}).$$

Proof. By hypothesis, we have

(*) $$g(xy^{-1}) = g(x)g(y) + f(x)f(y)$$

for all x, y in A such that xy^{-1} is also in A.

Let x be any element in A. Hence it follows from (*) that

$$f(x)^2 + g(x)^2 = g(xx^{-1}).$$

Thus we have (3.3.1).

By using elementary properties of commutative inverse semigroups togther with (*), we see that the following hold:

(3.3.5) $$g(x) = g(xx^{-1}x) = g(x(xx^{-1})^{-1}) = g(x)g(xx^{-1}) + f(x)f(xx^{-1}),$$

(3.3.6) $$g(x^{-1}) = g(x^{-1}xx^{-1}) = g(xx^{-1})g(x) + f(xx^{-1})f(x).$$

It follows that

$$g(x) = g(x^{-1}).$$

Thus (3.3.2) holds for all x in A.

From (3.3.5) we have that for each x in A,

$$f(x)f(xx^{-1}) = g(x) - g(x)g(xx^{-1}).$$

Hence, we have (3.3.3).

By using (3.3.1) we have

$$f(x)^2 = g(xx^{-1}) - g(x)^2,$$

$$f(x^{-1})^2 = g(x^{-1}x) - g(x^{-1})^2.$$

It follows from (3.3.2) that

$$f(x)^2 \quad = \quad f(x^{-1})^2.$$

Therefore,

$$f(x) = f(x^{-1}) \quad \text{or} \quad f(x) = f(x^{-1}).$$

Thus (3.3.4) holds for all x in A. #

<u>Lemma 3.4</u>. Let A be an inverse subsemigroup of S and (f,g) a solution of (*) on A. Then the following hold:

(3.4.1) For any x in A, $f(xx^{-1}) = 0$ iff $g(xx^{-1}) = 0$ or $g(xx^{-1}) = 1$,

(3.4.2) For any e in E(A), $f(e) = 0$ iff $g(e) = 0$ or $g(e) = 1$,

(3.4.3) For any x in A, $f(x) = 0$ iff $f(x^{-1}) = 0$,

(3.4.4) For any x in A, if $g(xx^{-1}) = 0$ then $g(x) = 0 = f(x)$.

<u>Proof</u>. Let x be any element of A. Replacing x in (3.3.1) by xx^{-1}, we have

$$f(xx^{-1})^2 + g(xx^{-1})^2 = g(xx^{-1}(xx^{-1})^{-1}) = g(xx^{-1}xx^{-1}) = g(xx^{-1}).$$

Therefore,

$$f(xx^{-1})^2 = g(xx^{-1})\ [\,1-g(xx^{-1})\,].$$

Hence

$$f(xx^{-1}) = 0 \ \text{iff}\ g(xx^{-1}) = 0 \ \text{or}\ g(xx^{-1}) = 1.$$

Thus we have (3.4.1). By specializing x = e in (3.4.1) we have (3.4.2). From (3.3.4) we have that $f(x) = f(x^{-1})$ or $f(x) = -f(x^{-1})$. Hence for all x in A,

$$f(x) = o \text{ iff } f(x^{-1}) = 0.$$

Hence (3.4.3) holds for all x in A. To show (3.4.4), we assume that $g(xx^{-1}) = 0$. From (3.4.1) we have $f(xx^{-1}) = 0$. Therefore, by (3.3.3), we have

$$g(x) = g(x)g(xx^{-1}) + f(x)f(xx^{-1}) = 0.$$

But, by using (3.3.1), we have

$$f(x)^2 = g(xx^{-1}) - g(x)^2 = 0.$$

Therefore,

$$f(x) = 0.$$

Thus (3.4.4) holds for all x in A. #

Notation For any solution of (*) on S we associate a pair of disjoint subsets of S as follows. Let

$$S_1(f,g) = \{x \varepsilon S \,/\, g(xx^{-1}) = 1\},$$

and

$$S_2(f,g) = \{x \varepsilon S \,/\, g(xx^{-1}) \neq 1\}.$$

It is clear that $S_1(f,g)$, $S_2(f,g)$ are disjoint and $S_1(f,g) \cup S_2(f,g) = S$.

Lemma 3.5. Let (f,g) be any solution of (*). Then

(3.5.1) If $S_1(f,g) \neq \phi$, then $S_1(f,g)$ is a filter of S, and hence it is an inverse subsemigroup of S.

(3.5.2) If $S_2(f,g) \neq \phi$, then $S_2(f,g)$ is a completely prime ideal of S, and hence it is an inverse subsemigroup of S.

<u>Proof</u>. To show (3.5.1), we assume that $S_1(f,g) \neq \emptyset$. Let $x,y \in S_1(f,g)$. Then $g(xx^{-1}) = 1$ and $g(yy^{-1}) = 1$. Hence it follows from (3.4.1) that $f(xx^{-1}) = 0$ and $f(yy^{-1}) = 0$. Therefore

$$\begin{aligned} g(xy(xy)^{-1}) &= g(xx^{-1}(yy^{-1})^{-1}) \\ &= g(xx^{-1})g(yy^{-1}) + f(xx^{-1})f(yy^{-1}) \\ &= 1. \end{aligned}$$

Thus $xy \in S_1(f,g)$, so $S_1(f,g)$ is a subsemigroup of S. Now suppose that x,y are elements in S such that $xy \in S_1(f,g)$ we shall show that $x,y \in S_1(f,g)$. Since $xy \in S_1(f,g)$, so $g(xy(xy)^{-1}) = 1$. Therefore it follows from (3.4.1) that $f(xy(xy)^{-1}) = 0$. Observe that

$$\begin{aligned} xy(xy)^{-1} &= xx^{-1}xy\ (xy)^{-1} \\ &= (xx^{-1})^{-1}(xy(xy)^{-1}) \\ &= (xy\ (xy)^{-1})(xx^{-1})^{-1}. \end{aligned}$$

Hence we have

$$\begin{aligned} 1 &= g(xy(xy)^{-1}) \\ &= g((xy(xy^{-1}))(xx^{-1})^{-1}) \\ &= g(xy(xy)^{-1})g(xx^{-1}) + f(xy(xy)^{-1})f(xx^{-1}) \\ &= 1\ g(xx^{-1}) + o\ f(xx^{-1}) \\ &= g(xx^{-1}) \end{aligned}$$

Therefore $x \in S_1(f,g)$. By a similar argument, we can show that $y \in S_1(f,g)$. Thus $x,y \in S_1(f,g)$. So $S_1(f,g)$ is a filter of S. Therefore (3.5.1) holds. Hence, from Theorem 2.5, it follows that $S_2(f,g) = S \setminus S_1(f,g)$ is either a completely prime ideal of S or an empty set. Thus (3.5.2) holds. #

<u>Theorem 3.6</u>. Let (f,g) be any solution of (*). Then f,g must be of

the forms

$$f = f_1 \cup f_2 \quad \text{and} \quad g = g_1 \cup g_2$$

where (f_1,g_1), (f_2, g_2) are solutions of (*) on $S_1(f,g)$, $S_2(f,g)$ respectively.

Proof. For i = 1,2, let f_i, g_i be the restictions of f,g on $S_i(f,g)$, respectively. It is clear that, for i = 1,2, (f_i, g_i) is a solution of (*) on $S_i(f,g)$ and

$$f = f_1 \cup f_2 \quad \text{and} \quad g = g_1 \cup g_2. \qquad \#$$

Definition 3.7. Let (f,g) be any solution of (*) on S.

If $g(e) = 1$ for all $e \in E(S)$ we say that (f,g) is of class 1.

If $g(e) \neq 1$ for any $e \in E(S)$ we say that (f,g) is of class 2.

Remark 3.8. From Theorem 3.6, we see that (f_1,g_1) is of class 1 and (f_2,g_2) is of class 2. So from this theorem we see that any solution of (*) on S is either of class 1 or class 2 or can be written as a union of a class 1 solution of (*) on a filter and a class 2 solution of (*) on a completely prime ideal of S. Since both filter and completely prime ideal in a commutative inverse semigroup are themselves a commutative inverse semigroups, so it suffices to consider the following two problems:

Problem 1: Find all solutions of (*) of class 1.

Problem 2: Find all solutions of (*) of class 2.

From now on, we shall assume that F is a field of characteristic different from 2. With this restriction on F, we shall be able to classifly the solutions of (*) into two types. The classification will be based on the following theorem.

<u>Theorem 3.9</u>. Let (f,g) be a solution of (*). Then f satisfies either

(3.9.1) $$f(x) = f(x^{-1})$$

for all x in S, or

(3.9.2) $$f(x) = -f(x^{-1})$$

for all x in S.

<u>Proof</u>. Suppose that there exist x,y in S such that

$$f(x) \neq f(x^{-1}),$$

and

$$f(y) \neq -f(y^{-1}).$$

Hence, by (3.4.3), it follows that $f(x) \neq 0$ and $f(y) \neq 0$. By (3.3.4), it follows from the above supposition that

$$f(x) = -f(x^{-1}) \quad \text{and} \quad f(y) = f(y^{-1}).$$

Therefore

$$g(xy^{-1}) = g(x)g(y) + f(x)f(y),$$

and

$$g(y^{-1}x) = g(y^{-1})g(x^{-1}) + f(y^{-1})f(x^{-1})$$

$$= g(x)g(y) - f(x)f(y).$$

The last equality follows from (3.3.2). Hence

$$g(x)g(y) + f(x)f(y) = g(x)g(y) - f(x)f(y),$$

$$2f(x)f(y) = 0.$$

Therefore $f(x) = 0$ or $f(y) = 0$, which is a contradiction. #

<u>Definition 3.10</u>. Let (f,g) be a solution of (*) on S. If f satisfies (3.9.1) for all x in S we say that (f,g) is of <u>positive-type</u>. In the case f satisfies (3.9.2) for all x in S we say that (f,g) is of <u>negative-type</u>.