CHAPTER IV

AFS_1-AFS_2 SANDWICH

The Hamiltonian of our AFS_1-AFS_2 sandwich is given by

$$H = H_{AFS_1} + H_{AFS_2} + H_t \tag{4.1}$$

where H_{AFS_1} (H_{AFS_2}) is the Hamiltonian of the first (second) thin antiferromagnetic superconductor (AFS) film and H_t is the tunneling Hamiltonian

$$H_{AFS_1} = \sum_{k_1\sigma} \varepsilon_{k_1} C^{+}_{k_1\sigma} C_{k_1\sigma} - \Delta_1 \sum_{k_1} (C^{+}_{k_1\uparrow} C^{+}_{-k_1\downarrow} + \text{h.c.})$$

$$- H_{Q_1} \sum_{k_1\beta\gamma} (\sigma_Z)_{\beta\gamma} (C^{+}_{k_1\beta} C_{k_1+Q_1\gamma} + \text{h.c.}) \tag{4.2}$$

$$H_{AFS_2} = \sum_{k_2\sigma} \varepsilon_{k_2} C^{+}_{k_2\sigma} C_{k_2\sigma} - \Delta_2 \sum_{k_2} (C^{+}_{k_2\uparrow} C^{+}_{-k_2\downarrow} + \text{h.c.})$$

$$- H_{Q_2} \sum_{k_2\beta\gamma} (\sigma_Z)_{\beta\gamma} (C^{+}_{k_2\beta} C_{k_2+Q_2\gamma} + \text{h.c.}) \tag{4.3}$$

$$H_t = V\Big[\sum_{k_1 k_2\sigma} (C^{+}_{k_1\sigma} C_{k_2\sigma} + C^{+}_{k_1\sigma} C_{k_2+Q_2\sigma} + \text{h.c.})$$

$$+ \sum_{k_2 k_1\sigma} (C^{+}_{k_2\sigma} C_{k_1\sigma} + C^{+}_{k_2\sigma} C_{k_1+Q_1\sigma} + \text{h.c.}) \Big] \tag{4.4}$$

where $C^{+}_{k_1\sigma}$ ($C^{+}_{k_2\sigma}$) is the creation operator for electron with momentum k and spin σ in the first (second) AFS. ε_{k_1}(ε_{k_2}) is the band energy of the conduction electron in the 1st (2nd) AFS measured from the Fermi energy. H_{Q_1}(H_{Q_2}) is the staggered molecular of the 1st (2nd) AFS which is taken to be temperature independent and $\vec{Q}_1(\vec{Q}_2)$ is the wave vector of 1st (2nd) AF ordering. V is the tunneling matrix element. The physical model we consider is the same as that described by McMillan. We also take V to be independent of k_1 and k_2. σ ,β and γ are spin indices, σ_Z is a Pauli matrix. $\Delta_1(\Delta_2)$ is the BCS superconducting order parameter of the 1st (2nd) AFS which is defined by

$$\Delta_1 \;=\; g_1 \sum_{k_1} \langle C^{+}_{k_1}\uparrow C^{+}_{-k_1}\downarrow \rangle \;=\; g_1 \sum_{k_1} \langle C_{-k_1}\downarrow C_{k_1}\uparrow \rangle \qquad (4.5)$$

here g_1 is the BCS coupling constant and the bracket < > denotes the thermal average.

In order to calculate Δ, we introduce the finite-temperature Green's function

$$G(k,\omega_n) \;=\; -\langle T_\tau\, \psi_k(\tau)\psi^{+}_k(0) \rangle \qquad (4.6)$$

where $\psi^{+}_k = (\; C^{+}_{k}\uparrow \;\; C_{-k}\downarrow \;\; C^{+}_{k+Q}\uparrow \;\; C_{-k-Q}\downarrow \;)$ and T_τ is the time ordering operator for the imaginary time $\tau = it$. Thus

$$G(k,\omega_n) =$$

$$- < T_\tau \begin{bmatrix} C_{k\uparrow}(\tau)C^{+}_{k\uparrow}(0) & C_{k\uparrow}(\tau)C_{-k\downarrow}(0) & C_{k\uparrow}(\tau)C^{+}_{k+Q\uparrow}(0) & C_{k\uparrow}(\tau)C_{-k-Q\downarrow}(0) \\ C^{+}_{-k\downarrow}(\tau)C^{+}_{k\uparrow}(0) & C^{+}_{-k\downarrow}(\tau)C_{-k\downarrow}(0) & C^{+}_{-k\downarrow}(\tau)C^{+}_{k+Q\uparrow}(0) & C^{+}_{-k\downarrow}(\tau)C_{-k-Q\downarrow}(0) \\ C_{k+Q\uparrow}(\tau)C^{+}_{k\uparrow}(0) & C_{k+Q\uparrow}(\tau)C_{-k\downarrow}(0) & C_{k+Q\uparrow}(\tau)C^{+}_{k+Q\uparrow}(0) & C_{k+Q\uparrow}(\tau)C_{-k-Q\downarrow}(0) \\ C^{+}_{-k-Q\downarrow}(\tau)C^{+}_{k\uparrow}(0) & C^{+}_{-k-Q\downarrow}(\tau)C_{-k\downarrow}(0) & C^{+}_{-k-Q\downarrow}(\tau)C^{+}_{k+Q\uparrow}(0) & C^{+}_{-k-Q\downarrow}(\tau)C_{-k-Q\downarrow}(0) \end{bmatrix} >$$

$$= \begin{bmatrix} G_{11} & G_{12} & G_{13} & G_{14} \\ G_{21} & G_{22} & G_{23} & G_{24} \\ G_{31} & G_{32} & G_{33} & G_{34} \\ G_{41} & G_{42} & G_{43} & G_{44} \end{bmatrix} \qquad (4.7)$$

Using the Heisenberg ' s equation of motion for the creation and annihilation operators with the pure AFS Hamiltonian given by Eq.(4.2), and by letting $\hbar = 1$, we get

$$i \frac{d}{dt} C_{k_1\uparrow} = [\, C_{k_1\uparrow} \, , \, H_{AFS_1} \,]$$

$$= \varepsilon_{k_1} C_{k_1\uparrow} - \Delta_1 C^{+}_{-k_1\downarrow} - H_{Q_1} C_{k_1+Q_1\uparrow}$$

$$(i \frac{d}{dt} - \varepsilon_{k_1}) C_{k_1\uparrow} + \Delta_1 C^{+}_{-k_1\downarrow} + H_{Q_1} C_{k_1+Q_1\uparrow} = 0 \qquad (4.8.1)$$

Proceeding in the same manner for each operator we obtain the following equations :

$$\left(i\frac{d}{dt} + \varepsilon_{k1}\right) C^{+}_{-k1\downarrow} + \Delta_1 C_{k1\uparrow} + H_{Q1} C^{+}_{-k-Q\downarrow} = 0 \tag{4.8.2}$$

$$\left(i\frac{d}{dt} - \varepsilon_{k1+Q1\uparrow}\right) C_{k1+Q1\uparrow} + \Delta_1 C^{+}_{-k-Q\downarrow} + H_{Q1} C_{k1\uparrow} = 0 \tag{4.8.3}$$

$$\left(i\frac{d}{dt} + \varepsilon_{k1+Q1\uparrow}\right) C^{+}_{-k-Q\downarrow} + \Delta_1 C_{k1+Q1\uparrow} + H_{Q1} C_{-k1\downarrow} = 0 \tag{4.8.4}$$

By Fourier transforming Eq. (4.8.1), we obtain the matrix elements of the Green ' s function as

$$(i\omega_n - \varepsilon_{k1}) \langle -T_\tau C_{k1\downarrow} C^{+}_{k1\uparrow} \rangle + \Delta_1 \langle -T_\tau C^{+}_{-k1\downarrow} C^{+}_{k1\uparrow} \rangle + H_{Q1} \langle -T_\tau C_{k1+Q1\uparrow} C^{+}_{k1\uparrow} \rangle$$

$$= [C_{k1\uparrow}, C^{+}_{k1\uparrow}]$$

or

$$(i\omega_n - \varepsilon_{k1}) G^{\circ}_{11} + \Delta_1 G^{\circ}_{21} + H_{Q1} G^{\bullet}_{31} = 1 \tag{4.9.1}$$

Similarly from Eqs. (4.8.2) - (4.8.4) , we can show that

$$(i\omega_n + \varepsilon_{k1}) G^{\circ}_{21} + \Delta_1 G^{\bullet}_{11} + H_{Q1} G^{\bullet}_{41} = 0 \tag{4.9.2}$$

$$(i\omega_n - \varepsilon_{k1+Q1}) G^{\circ}_{31} + \Delta_1 G^{\circ}_{41} + H_{Q1} G^{\bullet}_{11} = 0 \tag{4.9.3}$$

$$(i\omega_n + \varepsilon_{k1+Q1}) G^{\circ}_{41} + \Delta_1 G^{\circ}_{31} + H_{Q1} G^{\bullet}_{21} = 0 \tag{4.9.4}$$

where $\omega_n = (2n+1)\pi T$ and n is the integer, with T as temperature. Eq. (4.9) can therefore be written in matrix form as

$$\begin{bmatrix} i\omega_n-\varepsilon_{k1} & \Delta_1 & H_{Q1} & 0 \\ \Delta_1 & i\omega_n+\varepsilon_{k1} & 0 & H_{Q1} \\ H_{Q1} & 0 & i\omega_n-\varepsilon_{k1+Q1} & \Delta_1 \\ 0 & H_{Q1} & \Delta_1 & i\omega_n+\varepsilon_{k1+Q1} \end{bmatrix} \begin{bmatrix} \mathring{G}_{11} & \mathring{G}_{12} & \mathring{G}_{13} & \mathring{G}_{14} \\ \mathring{G}_{21} & \mathring{G}_{22} & \mathring{G}_{23} & \mathring{G}_{24} \\ \mathring{G}_{31} & \mathring{G}_{32} & \mathring{G}_{33} & \mathring{G}_{34} \\ \mathring{G}_{41} & \mathring{G}_{42} & \mathring{G}_{43} & \mathring{G}_{44} \end{bmatrix} = 1 \quad (4.10)$$

Assuming a one-dimensional-like half-filled band which satisfies the condition $\varepsilon_{k1} = -\varepsilon_{k1+Q1}$,we write Eq.(4.10) in short hand notation as

$$[\, i\omega_n\rho_0\sigma_0 - \varepsilon_{k1}\rho_3\sigma_3 + \Delta_1\rho_0\sigma_1 + H_{Q1}\,\rho_1\sigma_0]\; \mathring{G}_1(k,\omega_n) = 1 \qquad (4.11)$$

or

$$\mathring{G}_1(k,\omega_n) = \frac{1}{[\, i\omega_n - \varepsilon_{k1}\rho_3\sigma_3 + \Delta_1\sigma_1 + H_{Q1}\,\rho_1]} \qquad (4.12)$$

where ρ_i and σ_i (i = 1,2,3) are Pauli matrices acting on spin states and electron-hole states respectively.

In the same manner, we have a single particle Green ' s function for the 2nd AFS as

$$G_2^{\bullet}(k,\omega_n) = \frac{1}{[\, i\omega_n - \varepsilon_{k2}\rho_3\sigma_3 + \Delta_2\sigma_1 + H_{Q2}\,\rho_1]} \qquad (4.13)$$

The tunneling effect can be taken into account by evaluating the electron self energy ($\Sigma(i\omega_n)$). By treating the tunneling Hamiltonian in first-order self-consistent perturbation theory, the equation for the self-energy of the electron in the 1st AFS film is obtained in analytical form as (Shiba,1968)

$$\Sigma_1(i\omega_n) = A\, d_2\, V^2\, \rho_3\sigma_3 \sum_{k_2} G_2(k_2,\omega_n)\ \rho_3\sigma_3 \qquad (4.14)$$

In Eq.(4.14), d_2 is the thickness of the second AFS, A is the contact area of the junction. The self-energy $\Sigma_2(i\omega_n)$ can be obtained from Eq. (4.14) by interchanging labels 1 and 2.

We make the ansatz

$$G(k,\omega_n) = \frac{1}{[\, i\omega_n Z(i\omega_n) - \varepsilon_k\rho_3\sigma_3 + \Phi(i\omega_n)\sigma_1 + \tilde{H}_Q(i\omega_n)\ \rho_1]} \qquad (4.15)$$

here $G(k,\omega_n)$ is the renormalized Green ' s function which includes the proximity effect. $Z(i\omega_n)$ is the renormalizing function, $\Phi(i\omega_n)$ and $\tilde{H}_Q(i\omega_n)$ are renormalized order parameter and staggered field, respectively.

We find the usual form for the matrix Green ' s function of Eq. (4.15)

$$G(\mathbf{k},\omega_n) = \frac{\begin{aligned}\{ [(\omega_n Z)^2 + \varepsilon_k^2 + \Phi^2 + \tilde{H}_Q^2][i\omega_n Z + \varepsilon_k\rho_3\sigma_3 - \Phi\sigma_1 - \tilde{H}_Q\, \rho_1] \\ + 2\Phi\, \tilde{H}_Q\, [-i\omega_n Z\rho_1\sigma_1 + \varepsilon_k\rho_2\sigma_2 + \Phi\rho_1 + \tilde{H}_Q\sigma_1] \}\end{aligned}}{[(\omega_n Z)^2 + \varepsilon_k^2 + (\tilde{H}_Q - \Phi)^2] [(\omega_n Z)^2 + \varepsilon_k^2 + (\tilde{H}_Q + \Phi)^2]} \tag{4.16}$$

By inserting Eq. (4.16) into Eq. (4.14), we find

$$\Sigma_1(i\omega_n) = -Ad_2 V^2 \sum_{k_2} \frac{\begin{aligned}\{[(\omega_n Z_2)^2+\varepsilon_{k2}^2+\Phi_2^2+\tilde{H}_{Q2}^2][i\omega_n Z_2+\varepsilon_{k2}\rho_3\sigma_3+\Phi_2\sigma_1+\tilde{H}_{Q2}\,\rho_1] \\ +2\Phi_2\tilde{H}_{Q2}\,[-i\omega_n Z_2\rho_1\sigma_1+\varepsilon_{k2}\rho_2\sigma_2 -\Phi_2\rho_1 - \tilde{H}_{Q2}\sigma_1]\}\end{aligned}}{[(\omega_n Z_2)^2 + \varepsilon_{k2}^2 + (\tilde{H}_{Q2}-\Phi_2)^2][(\omega_n Z_2)^2 +\varepsilon_{k2}^2 +(\tilde{H}_{Q2}+ \Phi_2)^2]}$$

$$= -Ad_2 V^2 \sum_{k_2} \Big\{ \frac{[i\omega_n Z_2(1- \rho_1\sigma_1)+\varepsilon_{k2}(\rho_3\sigma_3+\rho_2\sigma_2)+\Phi_2(\sigma_1-\rho_1)+\tilde{H}_{Q2}(\rho_1-\sigma_1)]}{[(\omega_n Z_2)^2 + \varepsilon_{k2}^2 + (\tilde{H}_{Q2}-\Phi_2)^2]}$$

$$- \frac{[i\omega_n Z_2(1+ \rho_1\sigma_1)+\varepsilon_{k2}(\rho_3\sigma_3-\rho_2\sigma_2)-\Phi_2(\sigma_1+\rho_1)-\tilde{H}_{Q2}(\rho_1+\sigma_1)]}{[(\omega_n Z_2)^2 + \varepsilon_{k2}^2 + (\tilde{H}_{Q2}+\Phi_2)^2]} \Big\} \tag{4.17}$$

We replace $\sum$ in doing the summation over the k_2 states by $N_2(0) \int d\, \varepsilon_{k2}$, where $N_2(0)$ is the bulk density of states (per unit volume) of the 2nd AFS at the Fermi surface. Eq. (4.17) gives

$$\Sigma_1(i\omega_n) = (-Ad_2V^2N_2(0)\pi/2)\left\{ \frac{[i\omega_n Z_2(1-\rho_1\sigma_1)+\Phi_2(\sigma_1-\rho_1)+\tilde{H}_{Q_2}(\rho_1-\sigma_1)]}{\sqrt{[(\omega_n Z_2)^2+(\tilde{H}_{Q_2}-\Phi_2)^2]}} \right.$$

$$\left. - \frac{[-i\omega_n Z_2(1+\rho_1\sigma_1)-\Phi_2(\sigma_1+\rho_1)-\tilde{H}_{Q_2}(\rho_1+\sigma_1)]}{\sqrt{[(\omega_n Z_2)^2+(\tilde{H}_{Q_2}+\Phi_2)^2]}} \right\}$$

$$= (-Ad_2V^2N_2(0)\pi/2)\Big\{$$

$$i\omega_n Z_2\ [\,1/(\sqrt{(\omega_n Z_2)^2+(\tilde{H}_{Q_2}-\Phi_2)^2}) + 1/(\sqrt{(\omega_n Z_2)^2+(\tilde{H}_{Q_2}+\Phi_2)^2})]$$

$$- i\omega_n Z_2\rho_1\sigma_1[1/(\sqrt{(\omega_n Z_2)^2+(\tilde{H}_{Q_2}-\Phi_2)^2}) + 1/(\sqrt{(\omega_n Z_2)^2+(\tilde{H}_{Q_2}+\Phi_2)^2})]$$

$$+\sigma_1[(\Phi_2-\tilde{H}_{Q_2})/(\sqrt{(\omega_n Z_2)^2+(\tilde{H}_{Q_2}-\Phi_2)^2}) + (\Phi_2-\tilde{H}_{Q_2})/(\sqrt{(\omega_n Z_2)^2+(\tilde{H}_{Q_2}+\Phi_2)^2})]$$

$$+\rho_1[(\tilde{H}_{Q_2}-\Phi_2)/(\sqrt{(\omega_n Z_2)^2+(\tilde{H}_{Q_2}-\Phi_2)^2}) + (\tilde{H}_{Q_2}+\Phi_2)/(\sqrt{(\omega_n Z_2)^2+(\tilde{H}_{Q_2}+\Phi_2)^2})]\Big\} \tag{4.18}$$

From the Dyson ' s equation, the self-energy is related to the Green 's function as

$$\Sigma_1(i\omega_n) \quad = \quad [G_1^{\circ}(k_1,\omega_n)]^{-1} \ - \ [G_1(k_1,\omega_n)]^{-1} \tag{4.19}$$

By inserting Eqs. (4.12) and (4.15) into Eq. (4.19), we find

$$\Sigma_1(i\omega_n) \quad = \quad i\omega_n(1-Z_1)+\sigma_1(\Delta_1-\Phi_1)+\rho_1(H_{Q_1}-\tilde{H}_{Q_1}) \tag{4.20}$$

By comparing Eq. (4.20) to Eq. (4.18), we obtain the following equations

$$Z_1(i\omega_n) = 1 + [\,\Gamma_1 Z_2(i\omega_n)/2\,]\left\{ \frac{1}{\sqrt{[(\omega_n Z_2(i\omega_n))^2+(\tilde{H}_{Q_2}(i\omega_n)+\Phi_2(i\omega_n))^2]}} + \frac{1}{\sqrt{[(\omega_n Z_2(i\omega_n))^2+(\tilde{H}_{Q_2}(i\omega_n)-\Phi_2(i\omega_n))^2]}} \right\} \quad (4.21.1)$$

$$\Phi_1(i\omega_n) = \Delta_1 + [\Gamma_1/2]\{ (\tilde{H}_{Q_2}(i\omega_n)+\Phi_2(i\omega_n))/(\sqrt{[(\omega_n Z_2(i\omega_n))^2+(\tilde{H}_{Q_2}(i\omega_n)+\Phi_2(i\omega_n))^2]}) - (\tilde{H}_{Q_2}(i\omega_n)-\Phi_2(i\omega_n))/(\sqrt{[(\omega_n Z_2(i\omega_n))^2+(\tilde{H}_{Q_2}(i\omega_n)-\Phi_2(i\omega_n))^2]})\} \quad (4.21.2)$$

$$\tilde{H}_{Q_1}(i\omega_n) = H_{Q_1} + [\Gamma_1/2]\{ (\tilde{H}_{Q_2}(i\omega_n)+\Phi_2(i\omega_n))/(\sqrt{[(\omega_n Z_2(i\omega_n))^2+(\tilde{H}_{Q_2}(i\omega_n)+\Phi_2(i\omega_n))^2]}) + (\tilde{H}_{Q_2}(i\omega_n)-\Phi_2(i\omega_n))/(\sqrt{[(\omega_n Z_2(i\omega_n))^2+(\tilde{H}_{Q_2}(i\omega_n)-\Phi_2(i\omega_n))^2]})\} \quad (4.21.3)$$

where $\Gamma_1 = A\, d_2\, V^2 N_2(0)\,\pi$, $\Gamma_2 = A\, d_1\, V^2 N_1(0)\,\pi$.

For the second AFS slab there is a set of equations identical to Eqs.(4.17)-(4.21) with the subscripts 1 and 2 interchanged.

In general, the solution of Eq. (4.21) is exceedingly difficult to solve. To simplify the notations, we define the order parameter in the α^{th} ($\alpha = 1,2$) film to be given by

$$\Delta_\alpha(i\omega_n) = \Phi_\alpha(i\omega_n) / Z_\alpha(i\omega_n) \quad (4.22)$$

and the pseudo-staggered molecular field as

$$K_{\alpha}(i\omega_n) \;=\; \widetilde{H}_{Q\alpha}(i\omega_n) \;/\; Z_{\alpha}(i\omega_n) \tag{4.23}$$

Substitution of Eqs. (4.22) and (4.23) into Eqs. (4.21), we have

$$\Delta_{\alpha}(i\omega_n) = \frac{\left\{\Delta_{\alpha} + [\Gamma_{\alpha}/2]\{\,(K\beta(i\omega_n)+\Delta\beta(i\omega_n)) / (\sqrt{[\,\omega_n{}^2+(K\beta(i\omega_n)+\Delta\beta(i\omega_n))^2]}\,) - (K\beta(i\omega_n)-\Delta\beta(i\omega_n)) / (\sqrt{[\,\omega_n{}^2+(K\beta(i\omega_n)-\Delta\beta(i\omega_n))^2]}\,)\,\}\right\}}{\left\{1 + [\Gamma_{\alpha}/2]\{\,1 / (\sqrt{[\,\omega_n{}^2+(K\beta(i\omega_n)+\Delta\beta(i\omega_n))^2]}\,) + 1/ (\sqrt{[\,\omega_n{}^2+(K\beta(i\omega_n)-\Delta\beta(i\omega_n))^2]}\,)\,\}\right\}} \tag{4.24.1}$$

$$K_{\alpha}(i\omega_n) = \frac{\left\{H_{Q\alpha} + [\Gamma_{\alpha}/2]\{\,(K\beta(i\omega_n)+\Delta\beta(i\omega_n)) / (\sqrt{[\,\omega_n{}^2+(K\beta(i\omega_n)+\Delta\beta(i\omega_n))^2]}\,) + (K\beta(i\omega_n)-\Delta\beta(i\omega_n)) / (\sqrt{[\,\omega_n{}^2+(K\beta(i\omega_n)-\Delta\beta(i\omega_n))^2]}\,)\,\}\right\}}{\left\{1 + [\Gamma_{\alpha}/2]\{1/ (\sqrt{[\,\omega_n{}^2+(K\beta(i\omega_n)+\Delta\beta(i\omega_n))^2]}\,) + 1/ (\sqrt{[\,\omega_n{}^2+(K\beta(i\omega_n)-\Delta\beta(i\omega_n))^2]}\,)\,\}\right\}} \tag{4.24.2}$$

We note that Δ_{α} is the usual BCS order parameter which is given in finite temperature formalism as

$$\Delta_{\alpha} = A d_{\alpha} g_{\alpha} T \sum_{k_{\alpha},\omega_n}^{\omega_{D\alpha}/2\pi T} G^{21}_{\alpha}(k_{\alpha},\omega_n) \tag{4.25}$$

here $\omega_{D\alpha}$ is the Debye cuttoff frequency in the α^{th} film (α=1,2), and $G^{21}_{\alpha}(k_{\alpha},\omega_n)$ is the element 21 of the matrix G_{α} . G^{21}_{α} can been found from Eq. (4.16) as

$$G^{21}_{\alpha}(k_{\alpha},\omega_n) = \frac{-\,[(\omega_n Z_{\alpha})^2 + \varepsilon_{k\alpha}^2 + \tilde{H}_{Q\alpha}^2 + \Phi_{\alpha}^2][-\Phi_{\alpha} + 2\Phi_{\alpha}\tilde{H}_{Q\alpha}^2]}{[(\omega_n Z_{\alpha})^2 + \varepsilon_{k\alpha}^2 + (\tilde{H}_{Q\alpha} - \Phi_{\alpha})^2][(\omega_n Z_{\alpha})^2 + \varepsilon_{k\alpha}^2 + (\tilde{H}_{Q\alpha} + \Phi_{\alpha})^2]}$$

$$= (-1/2)\{ (\tilde{H}_{Q\alpha} - \Phi_{\alpha}) / [(\omega_n Z_{\alpha})^2 + \varepsilon_{k\alpha}^2 + (\tilde{H}_{Q\alpha} - \Phi_{\alpha})^2]$$

$$- (\tilde{H}_{Q\alpha} + \Phi_{\alpha}) / [(\omega_n Z_{\alpha})^2 + \varepsilon_{k\alpha}^2 + (\tilde{H}_{Q\alpha} + \Phi_{\alpha})^2] \} \tag{4.26}$$

Substitution of Eq. (4.26) into Eq. (4.25) yields

$$\Delta_{\alpha} = (-A d_{\alpha} g_{\alpha} T/2) \sum_{k_{\alpha},\omega_n}^{\omega_{D\alpha}/2\pi T} \{ (\tilde{H}_{Q\alpha} - \Phi_{\alpha}) / [(\omega_n Z_{\alpha})^2 + \varepsilon_{k\alpha}^2 + (\tilde{H}_{Q\alpha} - \Phi_{\alpha})^2]$$

$$- (\tilde{H}_{Q\alpha} + \Phi_{\alpha}) / [(\omega_n Z_{\alpha})^2 + \varepsilon_{k\alpha}^2 + (\tilde{H}_{Q\alpha} + \Phi_{\alpha})^2] \} \tag{4.27}$$

Summation over k states yields the following equation

$$\Delta_{\alpha} = (-Ad_{\alpha}g_{\alpha}TN_{\alpha}(0)\pi/2) \sum_{n}^{\omega_{D\alpha}/2\pi T} \Big\{ (\tilde{H}_{Q\alpha}-\Phi_{\alpha})/\sqrt{[(\omega_n Z_{\alpha})^2 + (\tilde{H}_{Q\alpha}-\Phi_{\alpha})^2]} - (\tilde{H}_{Q\alpha}+\Phi_{\alpha})/\sqrt{[(\omega_n Z_{\alpha})^2 + (\tilde{H}_{Q\alpha}+\Phi_{\alpha})^2]} \Big\} \tag{4.28}$$

By substituting Eq. (4.21) into Eq. (4.28), we write

$$\Delta_{\alpha} = (-\lambda_{\alpha}\pi T) \sum_{n}^{\omega_{D\alpha}/2\pi T} \Big\{ (K_{\alpha}(i\omega_n)-\Delta_{\alpha}(i\omega_n))/\sqrt{[\omega_n^2 + (K_{\alpha}(i\omega_n)-\Delta_{\alpha}(i\omega_n))^2]} - (K_{\alpha}(i\omega_n)+\Delta_{\alpha}(i\omega_n))/\sqrt{[\omega_n^2 + (K_{\alpha}(i\omega_n)+\Delta_{\alpha}(i\omega_n))^2]} \Big\} \tag{4.29}$$

where $\lambda_{\alpha} = Ad_{\alpha}g_{\alpha}N_{\alpha}(0)$ is the BCS coupling constant in the α^{th} film ($\alpha = 1,2$).

Using the BCS relation

$$f(T) = \ln[2\gamma\omega_{D1}/\pi T] = \pi T \sum_{n}^{\omega_{D1}/2\pi T} 1/|\omega_n| \tag{4.30.1}$$

and

$$g(T) = \ln[2\gamma\omega_{D2}/\pi T] = \pi T \sum_{n}^{\omega_{D2}/2\pi T} 1/|\omega_n| \tag{4.30.2}$$

where $\gamma = 1.781$. For $\omega_{D2} > \omega_{D1}$, we have the identity

$$g(T) \;=\; f(T) + \ln[\omega_{D2}/\omega_{D1}] \tag{4.31}$$

Near the transition temperature T_c, $\Delta_\alpha(i\omega_n)$ is small, we do the following approximations

$$[\,\omega_n{}^2+(\,K_\alpha(i\omega_n)-\Delta_\alpha(i\omega_n)\,)^2]$$

$$= [\,\omega_n{}^2+(\,K_\alpha(i\omega_n))^2-2\Delta_\alpha(i\omega_n)K_\alpha(i\omega_n)]^{-1/2}$$

$$= [\,\omega_n{}^2+(K_\alpha(i\omega_n))^2]^{-1/2}\,[1 - 2\Delta_\alpha(i\omega_n)K_\alpha(i\omega_n)/[\,\omega_n{}^2+(K_\alpha(i\omega_n))^2]]^{-1/2}$$

$$= [\,\omega_n{}^2+(K_\alpha(i\omega_n))^2]^{-1/2}\,[1 + \Delta_\alpha(i\omega_n)K_\alpha(i\omega_n)/[\,\omega_n{}^2+(K_\alpha(i\omega_n))^2] \tag{4.32.1}$$

and

$$[\,\omega_n{}^2+(\,K_\alpha(i\omega_n)+\Delta_\alpha(i\omega_n)\,)^2]$$

$$=[\,\omega_n{}^2+(K_\alpha(i\omega_n))^2]^{-1/2}\,[1 - \Delta_\alpha(i\omega_n)K_\alpha(i\omega_n)/[\,\omega_n{}^2+(K_\alpha(i\omega_n))^2] \tag{4.32.2}$$

Then Eq. (4.29) gives

$$\Delta_\alpha \;=\; (\lambda_\alpha \pi T)\sum_{n}^{\omega_{D\alpha}/2\pi T} \Delta_\alpha(i\omega_n)\,\omega_n{}^2 \,/\, [\,\omega_n{}^2+(K_\alpha(i\omega_n))^2]^{3/2} \tag{4.33}$$

By inserting Eq. (4.32) into Eq. (4.24), we find that

$$\Delta_1\,(i\omega_n) = \frac{[\,\Delta_1 + \Delta_2(i\omega_n)\Gamma_1(\omega_n)^2 \;/\; [\,\omega_n{}^2 + (K_2(i\omega_n))^2]^{3/2}\,]}{[\,1 + \Gamma_1 \,/\, [\,\omega_n{}^2 + (K_2(i\omega_n))^2]^{1/2}} \tag{4.34.1}$$

and

$$\Delta_2\,(i\omega_n) = \frac{[\,\Delta_2 + \Delta_1(i\omega_n)\Gamma_2(\omega_n)^2 \;/\; [\,\omega_n{}^2 + (K_1(i\omega_n))^2]^{3/2}\,]}{[\,1 + \Gamma_2 \,/\, [\,\omega_n{}^2 + (K_1(i\omega_n))^2]^{1/2}} \tag{4.34.2}$$

By eliminating $\Delta_1\,(i\omega_n)$ and $\Delta_2\,(i\omega_n)$ from Eqs. (4.34.1) and (4.34.2), we finally get

$$(\lambda_1)^{-1} \;-\; P_1 \;-\; f(T) \;=\; [\,\Delta_2/\Delta_1\,]\; R_1 \tag{4.36.1}$$

$$(\lambda_2)^{-1} \;-\; P_2 \;-\; g(T) \;=\; [\,\Delta_1/\Delta_2\,]\; R_2 \tag{4.36.2}$$

where

$$P_\alpha = \pi T \sum_n^{\omega_{D\alpha}/2\pi T} \Bigg[\frac{[\omega_n{}^2(\sqrt{\omega_n{}^2 + K\beta^2}\,) \,/\, [\omega_n{}^2 + K_\alpha{}^2]^{3/2}}{\Gamma_\alpha + [\omega_n{}^2 + K\beta^2]^{1/2} - \omega_n{}^4\Gamma_\alpha\Gamma_\beta \,/[\omega_n{}^2 + K_\alpha{}^2][\omega_n{}^2 + K\beta^2][\Gamma_\beta + \sqrt{[\omega_n{}^2 + K_\alpha{}^2]}]} - 1/\omega_n \Bigg] \tag{4.37.1}$$

$$R_\alpha = \pi T \sum_{n}^{\omega_{D\alpha}/2\pi T} \left[\frac{\omega_n^4 \Gamma_\alpha}{[\omega_n^2 + K_\alpha^2][\omega_n^2 + K_\beta^2][\Gamma_\beta + \sqrt{\omega_n^2 + K_\alpha^2}][\Gamma_\alpha + \sqrt{\omega_n^2 + K_\beta^2}] - \omega_n^4 \Gamma_\alpha \Gamma_\beta} \right] \quad (4.37.2)$$

here $\alpha = 1,2$: $\beta = 1,2$ and $\alpha \neq \beta$.

By eliminating Δ_1/Δ_2 in Eq. (4.36) and with the help of Eq. (4.31), we have a quadratic equation for f(T)

$$[(\lambda_1)^{-1} - P_1 - f(T)]\,[(\lambda_2)^{-1} - P_2 - f(T) - \ln(\omega_{D2}/\omega_{D1})] = R_1 R_2 \quad (4.38)$$

The T_c formula can be obtained from this quadratic equation. It is valid for the case when the effective coupling constant of both films are not equal to zero and it includes the possibility of different cutoff frequencies for the 1st and 2nd films. The T_c formula will be derived in the next chapter.