Chapter 2

Mathematical Programming Models

Mathematical models are idealized representation that is expressed in term of mathematical symbols and expressions. Similarly, the mathematical model of an optimization problem is a system of equations and related mathematical expressions that describe the essence of the problem. Thus, if there are n related quantifiable decisions to be made, they are represented as *decision variables* (say, $x_1, x_2, ..., x_n$) which respective values are to be determined. The appropriate measure of performance (e.g., cost) is then expressed as a mathematical function of the decision variables (e.g., $F = 3x_1 + 2x_2 + ... + 5x_n$). The function called an *objective function*. Any restrictions on the values that can be assigned to these decision variables are also expressed mathematically, typically means of equations or inequations (e.g., $x_1 + 3x_1x_2 + 2x_2 \le 10$). Such mathematical expressions for the restrictions often are called *constraints* (Hillier and Lieberman, 1990). We then write the problem as

Optimize:	$f(x)$	objective function
Subject to:	$h(x) = 0$	equality constraints
	$g(x) \ge 0$	inequality constraints

For identifying optimal piping design problem, there are three mathematical programming models that have been widely proposed: Linear Programming (LP), Nonlinear Programming (NLP) and Dynamic Programming (DP).

2.1 Linear Programming Model.

Linear programming is one of the most widely used optimization techniques and one of the most effective. The term linear programming was coined by George Dantzig in 1947 refering to the procedure of optimization in problem in which both the objective function and the constrains are linear (Edgar and Himmelbau, 1988).

Kameli, Gadish and S. Meyers (1968) presented a method of piping design networks using the theory of linear programming. The linear programming model is based on an assumption of the pipes connecting the delivery points to the fluid source. The network and conditions under which it is assumed to operate are shown in Figure 2.1.

The network consists of a set of N demand points indexed by $n = 1, 2, 3, \ldots, N$. There are two types of demand points; delivery points and junction points. At a delivery point, fluid is to be removed from the system at a given rate, $q(n)$, and at a hydraulic head greater than a given minimum $\underline{H}(n)$. These demand points are connected to the source at O by a network of N pipe sections that also indexed by n.

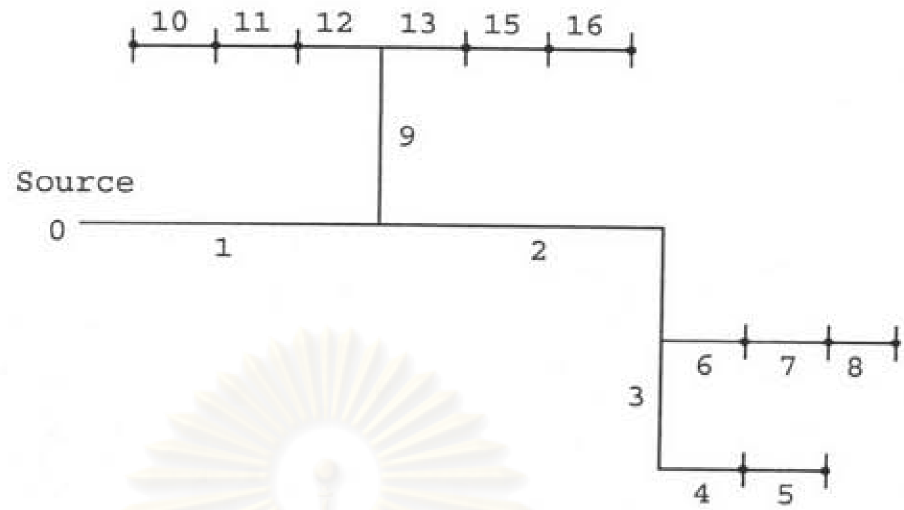

Figure 2.1: Network example.

a) Capital Cost.

In each section there are $G(n)$ different kinds of pipes and diameters that are to be considered for inclusion in the final design. These are chosen in relation to the discharge required, the maximum and minimum allowed flow velocities. The $G(n)$ different pipes are indexed by j with ranging from 1 to $G(n)$. The total cost of the section is

$$F(n) = \sum_{j=1}^{G(n)} l(n, j)c(n, j) \qquad (2.1)$$

in which

$c(n, j)$ = unit cost in jth alternative pipe in section n;

$l(n, j)$ = length of jth alternative pipe in section n;

b) Pumping Cost.

The head to be provided at the pump is given by $H(0)$, and its cost is assumed to be $c(0)$ per unit (This includes annual operating cost and capital costs of the pump expressed as annual amount by a suitable capital recovery factor).

c) Minimum Total Cost.

Thus, the optimization problem may be stated in standard LP form as follows:

Objective:

Minimize $$F = c(0)H(0) + \sum_{n=1}^{N}\sum_{j=1}^{G(n)} c(n, j)l(n, j) \qquad (2.2)$$

Subject to:

$$\sum_{j=1}^{G(n)} l(n, j) = L(n) \qquad (2.3)$$

$$H(0) - \sum_{n'}\sum_{j=1}^{G(n')} l(n', j)h_f(n', j) \geq \underline{H}(0) \qquad (2.4)$$

$$H(0) \geq \underline{H}(0) \qquad (2.5)$$

$$l(n, j) \geq 0 \qquad (2.6)$$

in which

F is the total annual cost of the system;

$H(n)$ = the total hydraulic head at point n;

$L(n)$ = total length of section n;

$h_f(n, j)$ = unit friction loss in jth alternative pipe in section n;

$G(n)$ = number of alternative pipes for consideration in section n;

and the summation over n' is taken over all sections in the flow path from 0 to n.

The aim is to choose the $c(n, j)$ so as to minimize the value of F in Eq. 2.2 while assuring the required pressures and discharge at all delivery.

For all sections the sum of the lengths of the pipes selected, $l(n, j)$ must equal the length of the section, $L(n)$. At each demand point the actual total hydraulic head, $H(n)$ must be equal to or greater than a given minimum head, $\underline{H}(n)$.

Each pair, (n, j), has associated with it a diameter, D, friction factor, C, as selected by the designer, and a unit friction loss denoted by $h_f(n, j)$.

The total friction loss in a section is given by

$$h(n) = \sum_{j=1}^{G(n)} l(n, j) h_f(n, j) \qquad (2.7)$$

Therefore, the values of $H(n)$ for $n > 0$ can be written with $H(0)$, and the $l(n, j)$ of the section lying in the flow path from the source to the point as Eq. 2.4.

The model is employed to minimize the total cost by determining the optimal commercially available pipe size for each link, and for the pump head at each node. The model is suitable for the both cases in which the fluid pressure is to be selected, and the case where pressure is given.

The LP model is found to be very useful for the analysis of branched networks. However, when the number of candidate pipe sizes is large and the configuration of the network becomes complicated, then computer resources such as memory and CPU time increase substantially that cause the LP model becoming too large to solve (Fujiwara and Dey, 1988).

For this reason, Alperovits and Shamir (1977, quoted in Perez, Matinez, and Vela, 1993) suggested the problem size could be reduced by considering a number of candidate diameters in a link and using only the strictly needed constraints. With the same purpose, Bhave (1979) used the concept of critical path for establishing a minimum number of candidate in each link.

2.2 Nonlinear Programming Model.

The LP approach has advantages over other methods in simplicity in application, especially in branched networks, but the application is a large system. For this reason, some authors have chosen nonlinear formulations where diameters are treated as continuous decision variables.

NLP problems come in many different shapes and forms. Unlike the simplex method for linear programming, no single algorithms have been developed for various individual classes of NLP problem (Hillier and Lieberman, 1990). One of the NLP approach for economic pipe size selection giving a quick analysis is developed as unconstrained optimization problems (Peter and Timmerhaus, 1968; Nolte, 1978; Coulson and Richardson, 1983; Edgar and Himmelblau, 1988). However, as the process model is made more accurate and complicated, it can lose the possibility of obtaining an analytical solution (Edgar and Himmelblau, 1988). For constrained problems, many authors present an optimization algorithm using Lagrangain Multiplier Technique.

Lagragians constitute the easiest solution subject constraints. Historically they are also the first. They may be considered an intermediate step to the general optimum conditions published by Kuhn and Tucker in 1951 (de Neufville, 1990). The Lagrangain Multiplier Technique appends the equality constraints to the objective function (cost function) through Lagrangian Multipliers to obtain the Lagrangian function.

Chiplunkar and Khanna (1983) presented a simple optimization algorithm for fluid supply systems using the nonlinear programming (NLP):

a) Capital Cost.

The capital cost of pipe per length (including laying and jointing cost) can be expressed as

$$C_p = KD^m \qquad (2.8)$$

in which

C_p = cost of pipe per length; D = internal diameter of pipe; and K and m are constants depending on the material of pipe.

Therefore, the total cost of pipe for N sections can be expressed as follows:

$$F = \sum_{i=1}^{N} K l_i D_i^{\,m} \tag{2.9}$$

where

l_i = the length of the ith section;

D_i = the diameter of the ith section;

N = the total number of sections;

b) Pumping Cost.

The relationship for estimating the friction head loss in ith section can be written in general as,

$$h_{fi} = \frac{l_i Q_i^p}{K_R C_R^p D_i^r} = C_i D_i^{-r} \tag{2.10}$$

in which

h_{fi} = friction head loss in the ith section; K_R = constant; Q_i = flow in ith section; C_R = pipe roughness coefficient; p, r = exponents; and $C_i = l_i Q_i{}^p / (K_R C_R{}^p)$.

The total head loss should be equal to available head so as to achieve the required terminal head H_n at point n. The constraint can therefore be expressed as

$$\sum_{i=1}^{N} C_i D_i^{\,-r} = H_0 - H_n \tag{2.11}$$

in which

H_0 = an initial head of source at the point 0; H_n = the terminal head required at point n.

The cost of pumping plant can be expressed as

$$C_{pump} = K'(HP)^{m_1} \tag{2.12}$$

in which

C_{pump} = cost of pumping plant; K' = a constant determined by analysis of the data correlating capital cost of pump with horse power; m_1 = exponent depending on type and range of pumping plant; and HP = horse power of pump which can be expressed as

$$HP = \frac{\rho g Q_1 H_0}{\eta 746} = A Q_1 H_0 \tag{2.13}$$

in which

Q_1 = total discharge, i.e., flow in the first section of branch; H_0 = initial pump head; and A = in which ρ = density of fluid; g = acceleration due to gravity; and η = combined efficiency of pump and motor.

$$C_{pump} = K_1(Q_1 H_0)^{m_1} \tag{2.14}$$

in which $K_1 = K'A^{m_1}$

The total cost of power can be expressed as

$$C_{power} = A_1 Q_1 H_0 \tag{2.15}$$

in which

C_{power} = capitalized cost of power; A_1 = a constant determined from the cost of energy, hours of pumping during a day, efficiency of pump and motor, yearly average consumption, and capitalizing factor over the design period.

c) Minimum Total Cost.

Therefore, the optimal problem can be expressed as

Objective:

$$\text{Minimize} \quad F = \sum_{i=1}^{N} Kl_i D_i^m + K_1 (Q_1 H_0)^{m_1} + A_1 Q_1 H_0 \qquad (2.16)$$

Subject to:

$$\sum_{i=1}^{N} C_i D_i^{-r} = H_0 - H_n \qquad (2.11)$$

$$D_i > 0 \text{ for } i = 1, \ldots, N \qquad (2.17)$$

The optimization problem can be stated as that of minimizing total cost F in (Eq. 16) subject to constraint (Eq. 11). The Lagrangian function of problem is

$$F(D_i, \lambda) = \sum_{i=1}^{N} Kl_i D_i^m + \sum_{i=1}^{N} \lambda_i \left[C_i D_i^{-r} - (H_0 - H_n) \right] \qquad (2.18)$$

where λ_i = the Lagrange multiplier.

The minimization of the Lagrangain function amounts to the minimization of the objective function as the constraint is satisfied. The optimal solution is achieved by taking the first partial derivative of Lagrangian function with respect to the decision variable (diameter and pumping head) and the Lagrangian multipliers and equating the expressions thus obtained to zero (Rao, 1979 quoted in Chiplunkar and Khanna, 1983). The optimal solution results in continuous diameters which neccessaries must be round off to discrete commercial pipe.

2.3 Dynamic Programming Model.

The linearization not only introduces results that are often far from the true optimum solution, but also hardly applies to complex piping networks (Liang, 1971).

The concept of DP was developed to solve a general separable, serial optimization problem. The approach is to decompose a mathematical programming problem into a series of individual subproblem connected in a serial manner. These individual subproblems are solved recursively to yield the optimal solution. The separable components of the problem are represented by the stages in DP formulation. These stages are tied together by transition functions that transmit information into and out of each subproblem based on decisions made at each stage. The current behavior or condition of the system at each stage is retained and recorded in the state variables (Martin, 1980).

Yang, Liang and Wu (1975) developed an optimization procedure, which selects an optimal system design. The configurations of fluid transport systems range from a simple one size pipe, delivering fluid from one point to another (Figure 2.2), to complex distribution system consisting of many branches (Figure 2.3). Each pipe segment of a system is treated as a dynamic programming stage.

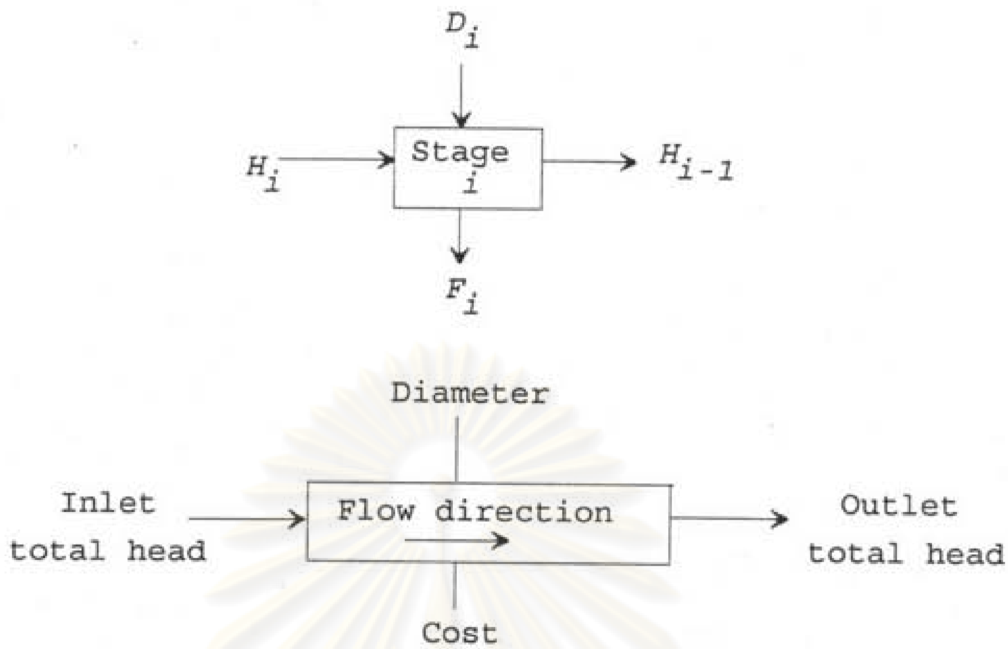

Figure 2.2: Corresponding relations between dynamic programming stage and segment.

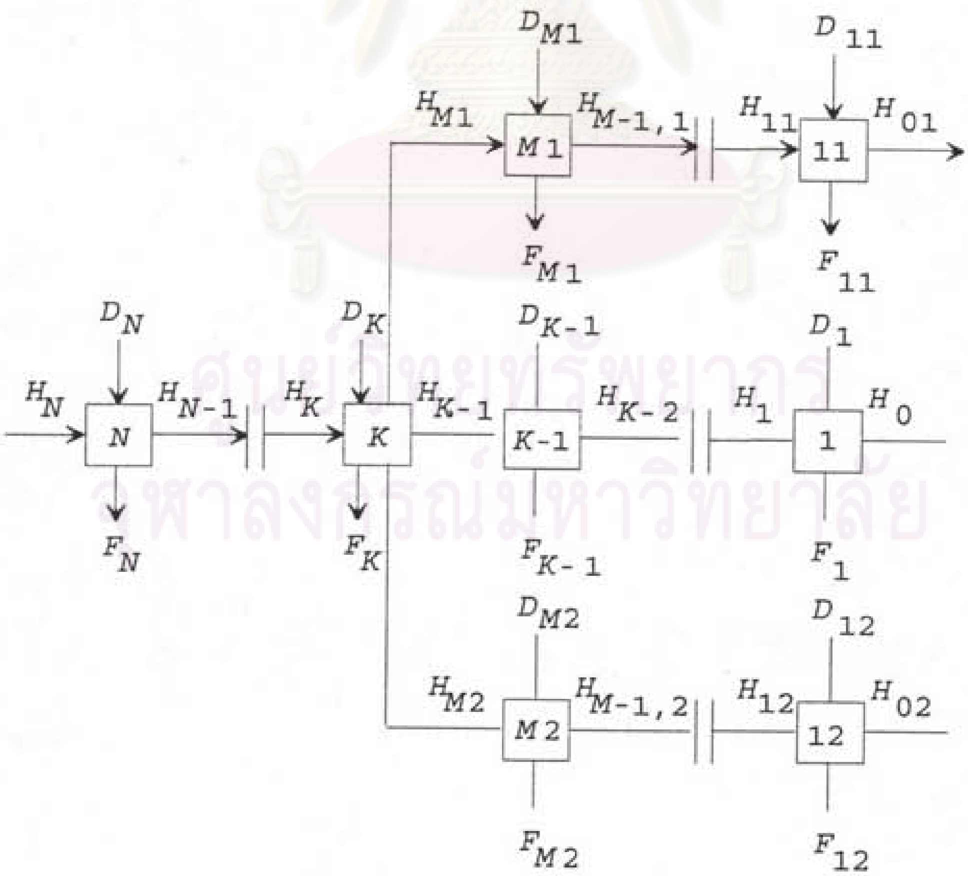

Figure 2.3: Diverging branch system.

Figure 2.2 shows the corresponding relation between dynamics programming stage and a pipe segment in which H_i = inlet total head of the ith stage, H_{i-1} = outlet total head of the ith stage, D_i = diameter of the ith stage, F_i = total cost due to the ith stage. The total heads at the inlet and outlet of each segment are the input and output state variable of each stage. The pipe diameter is the decision variable of each stage. The sum of all cost phases, which include pipe material cost, power cost, and wasted fluid cost of all stages, is used as a criterion for measuring the overall system effectiveness.

A design problem of branch type system is expressed as

Objective:

Minimize

$$F(H_i, D_i, H_0, H_{01}, \ldots, H_{0L}) = \sum_{i=1}^{N} \left(C_{pi} + C_{ei}\right) \tag{2.19}$$

Subject to:

$$H_{i-1} = t_i(H_i, D_i) \tag{2.20}$$

$$Q_i = F_i(Q_{i-1}, H_{i-1}, H_i) \tag{2.21}$$

$$Q_i = \sum_j Q_{(i-1)j} \quad \text{for elbow, tee, or cross} \tag{2.22}$$

where as

C_{pi} = the cost of pipes, elbows, tees, and crosses per year;

C_{ei} = the cost of energy put into fluid per year;

D_i = diameter of pipe, elbow, tee, or cross of the *i*th stage;

Q_i = flow rate in the *i*th stage;

H_i = inlet total head of the *i*th stage;

H_{i-1} = outlet total head of the *i*th stage and inlet total head of (*i* - 1) stage (a pipe and connector attached to its outlet end together are treated as one stage);

$H_{01}, H_{02}, \ldots, H_{0L}$ = final states of each branch.

The solution using the dynamic programming technique consists of two parts:

1. Moving backward to partial optimization at each component, the best choice of each diameter will be determined.

2. Until component is the initial component, the minimum cost of overall components and optimum diameter of this component are known. The best choice of this diameter is the next optimum diameter. According to this, the optimum diameters of next components will be get by moving forward.

The objective function in Eq. 2.19 is to minimum the sum of overall total cost that may be expressed as

$$\text{cost} = \min \sum C_{tj,i} = \min\{C_{tj,i}\} \tag{2.23}$$

where

$C_{tj,i}$ = total annual cost of the *j*th diameter of the i^{th} component.

The recurrence formula for additive function can be expressed as

$$F^{*}_{j,i} = \min\left\{F_{j,i}\right\} \qquad (2.24)$$

where

$F^{*}_{j,i}$ = minimum cumulative cost of the jth diameter of the i^{th} component;

$F_{j,i}$ = cumulative cost of the jth diameter of the i^{th} component.

The cumulative cost, $F_{j,i}$ is summation of total annual capital cost and minimum cumulative cost of its choice. In equation form,

$$F_{j,i} = C_{tj,i} + F^{*}_{i-1} \qquad (2.25)$$

where

F^{*}_{i-1} = minimum cumulative future cost (component n-1 onward) of the choice.

The $F_{j,i}$ in Eq. 2.24 may be replaced itself by Eq. 2.25. Therefore, the recurrence formula can be expressed as

$$F^{*}_{j,i} = \min\left\{C_{tj,i} + F^{*}_{i-1}\right\} \qquad (2.26)$$

When cost of connector is considerable, Yang, Liang and Wu(1975) added term of connector cost in Eq. 2.26, so the development equation may be expressed as

$$F^{*}_{j,i} = \min\left\{C_{tj,i} + F^{*}_{i-1} + C_{cj,i}\right\} \qquad (2.27)$$

where

$C_{cj,i}$ = total annual cost of jth connector for connection between i^{th} component and i-1^{th} component.

The connector has two different diameters, first is the inlet diameter that will be equal to diameter of i^{th} component, second is the outlet diameter will be equal to diameter of i-1^{th} component. The total annual cost of connector is also determined by the same method as total pipe cost.

Perez et al. (1993) presented a method based on a dynamic programming formulation. The method identified optimal strategies for use of pressure reducing valves (PRVs) in order to reduce pressure in downstream pipes, thereby reducing wall thickness and investment cost in those pipes. The strategy represented an opportunity to achieve considerable saving investment cost.