CHAPTER III

REGULAR MATRIX SEMIGROUPS OVER A SEMIFIELD

This chapter aims at generalizing the following well-known result : For any positive integer n and for any field F, the matrix semigroup $M_n(F)$ is regular. However, a theorem that gives a generalization of this result is given in [1] as follows : For any positive integer n and for any ring R, the matrix semigroup $M_n(R)$ is regular if and only if R is a regular ring. Semifields are a generalization of fields. A semifield need not be a ring, and conversely. In this chapter, we generalize the above well-known result by establishing necessary and sufficient conditions of a positive integer n and a semifield S such that the matrix semigroup $M_n(S)$ is regular.

Some examples of semifields which are not fields are as follows : $(\mathbb{R}^+\cup\{0\},+,\cdot)$, $(\mathbb{Q}^+\cup\{0\},+,\cdot)$, $(\mathbb{R}^+\cup\{0\},\max.,\cdot)$, $(\mathbb{Q}^+\cup\{0\},\max.,\cdot)$ where + and $\cdot$ are the usual addition and the usual multiplication of real numbers, respectively.

We first give some general properties of any commutative additively idempotent semiring with 0,1 which will be used later.

<u>Proposition 3.1</u>. Let S be a commutative additively idempotent semiring with 0,1. Then the following statements hold:

(1) For $x,y \in S$, $x+y = 1$ implies $1+x = 1+y = 1$.

(2) For $x,y \in S$, $x+y = 0$ implies $x = y = 0$.

Proof : (1) Let $x,y \in S$ be such that $x+y = 1$. Since $x+x = x$ and $y+y = y$, we have that $1+x = (x+y)+x = (x+x)+y = x+y = 1$ and $1+y = 1$ can be shown similary.

(2) If $x,y \in S$ are such that $x+y = 0$, then $x = x+0 = x+(x+y) = (x+x)+y = x+y = 0$ and $y = 0+y = (x+y)+y = x+(y+y) = x+y = 0$ since $x+x = x$ and $y+y = y$. #

To obtain the main theorem, the following lemma is required.

Lemma 3.2. Let S be a semifield which is not a field. Then the matrix semigroup $M_2(S)$ is regular if and only if S is an additively idempotent semifield.

Proof : Assume that $M_2(S)$ is a regular matrix semigroup. To prove that $x+x = x$ for all $x \in S$, it suffices to show that $1+1 = 1$. It follows from the regularity of the matrix semigroup $M_2(S)$ and Theorem 1.2 that S is a regular semiring. In particular, $(S,+)$ is a regular semigroup. Since S is a semifield which is not a field, $(S,+)$ is not a group. Therefore $(S,+)$ is a regular semigroup which is not a group and 0 is an idempotent of the semigroup $(S,+)$. Hence the semigroup $(S,+)$ contains more than one idempotent (see Chapter I, page 4). Thus there exists an element $a \in S$ such that $a \neq 0$ and $a+a = a$. Since $(S\setminus\{0\},\cdot)$ is a group, the inverse, a^{-1}, of a in the group $(S\setminus\{0\},\cdot)$ exists. Then $1 = aa^{-1} = (a+a)a^{-1} = aa^{-1}+aa^{-1} = 1+1$.

Conversely, assume that $x+x = x$ for all $x \in S$. To prove that the matrix semigroup $M_2(S)$ is regular, let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an element of

$M_2(S)$.

Case 1 : $ad+bc = 0$. By Proposition 3.1(2), $ad = bc = 0$. Let x,y,z,w be the elements of S defined by

$$x = \begin{cases} a^{-1} & \text{if } a \neq 0, \\ 0 & \text{if } a = 0, \end{cases} \qquad y = \begin{cases} b^{-1} & \text{if } b \neq 0, \\ 0 & \text{if } b = 0, \end{cases}$$

$$z = \begin{cases} c^{-1} & \text{if } c \neq 0, \\ 0 & \text{if } c = 0, \end{cases} \qquad w = \begin{cases} d^{-1} & \text{if } d \neq 0, \\ 0 & \text{if } d = 0. \end{cases}$$

Then $\begin{bmatrix} x & z \\ y & w \end{bmatrix} \in M_2(S)$ and

$$ax = \begin{cases} 1 & \text{if } a \neq 0, \\ 0 & \text{if } a = 0, \end{cases} \qquad by = \begin{cases} 1 & \text{if } b \neq 0, \\ 0 & \text{if } b = 0, \end{cases}$$

$$cz = \begin{cases} 1 & \text{if } c \neq 0, \\ 0 & \text{if } c = 0, \end{cases} \qquad dw = \begin{cases} 1 & \text{if } d \neq 0, \\ 0 & \text{if } d = 0. \end{cases} \qquad \ldots\ldots (*)$$

Claim that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & z \\ y & w \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We first consider

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & z \\ y & w \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ax+by & az+bw \\ cx+dy & cz+dw \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} a(ax+by)+c(az+bw) & b(ax+by)+d(az+bw) \\ a(cx+dy)+c(cz+dw) & b(cx+dy)+d(cz+dw) \end{bmatrix}$$

$$= \begin{bmatrix} a(ax+by+cz)+bcw & b(ax+by+dw)+adz \\ c(ax+cz+dw)+ady & d(by+cz+dw)+bcx \end{bmatrix}.$$

Since $ad = bc = 0$, we have that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & z \\ y & w \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a(ax+by+cz) & b(ax+by+dw) \\ c(ax+cz+dw) & d(by+cz+dw) \end{bmatrix}.$$

Since $1+1 = 1$, it follows easily from (*) that $ax+by+cz = 1$ if $a \neq 0$, $ax+by+dw = 1$ if $b \neq 0$, $ax+cz+dw = 1$ if $c \neq 0$ and $by+cz+dw = 1$ if $d \neq 0$. Then $a(ax+by+cz) = a$, $b(ax+by+dw) = b$, $c(ax+cz+dw) = c$ and $d(by+cz+dw) = d$. Hence $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & z \\ y & w \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

<u>Case 2 : ad+bc ≠ 0.</u> Since $(S\setminus\{0\},\cdot)$ is a group, there exists a $k \in S$ such that $(ad+bc)k = 1$. Then $\begin{bmatrix} dk & bk \\ ck & ak \end{bmatrix} \in M_2(S)$. Claim that

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} dk & bk \\ ck & ak \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Since $adk+bck = 1$ and $x+x = x$ for every $x \in S$, we have that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} dk & bk \\ ck & ak \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} adk+bck & abk+abk \\ cdk+cdk & adk+bck \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} 1 & abk \\ cdk & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} a+abck & b+abdk \\ c+acdk & d+bcdk \end{bmatrix}$$

$$= \begin{bmatrix} a(1+bck) & b(1+adk) \\ c(1+adk) & d(1+bck) \end{bmatrix}.$$

Because $adk+bck = 1$, it follows from Proposition 3.1(1) that $1+bck = 1+adk = 1$. Hence $\begin{bmatrix} a(1+bck) & b(1+adk) \\ c(1+adk) & d(1+bck) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, so we have the claim.

Hence the converse is proved. #

<u>Theorem 3.3.</u> Let S be a semifield and n a positive integer. Then the matrix semigroup $M_n(S)$ is regular if and only if one of the following conditions holds :

(i) $n = 1$.

(ii) S is a field.

(iii) $n = 2$ and S is an additively idempotent semifield.

<u>Proof</u> : If $n = 1$, then the matrix semigroup $M_1(S)$ can be considered as the multiplicative structure of S which is a regular

semigroup, so $M_1(S)$ is a regular matrix semigroup.

If S is a field, then by Theorem 1.1, the matrix semigroup $M_n(S)$ is regular.

If $n = 2$ and S is an additively idempotent semifield, we have by Lemma 3.2 that the matrix semigroup $M_2(S)$ is regular.

For the converse, assume that $M_n(S)$ is a regular matrix semigroup. To prove that one of (i), (ii) and (iii) holds, suppose that $n \geqslant 2$ and S is not a field. Since S is a semifield, S is not a ring. Then the matrix semigroup $M_n(S)$ is regular and S is not a ring. By Theorem 1.3, it follows that $n = 2$. Now we have that S is a semifield which is not a field and $M_2(S)$ is a regular matrix semigroup. Hence by Lemma 3.2, S is an additively idempotent semifield. Therefore (iii) holds. #

The four examples of semifields given above are semifields of nonnegative real numbers under the usual multiplication which are not fields. However, we can prove by Theorem 1.5 that there exists an operation $\oplus$ on $Q^+ \cup \{0\}$ such that $(Q^+ \cup \{0\}, \oplus, \cdot)$ is a field where $\cdot$ is the usual multiplication. By Theorem 1.5, there exists an operation $*$ on $N \cup \{0\}$ such that $(N \cup \{0\}, *, \cdot)$ is a ring where $\cdot$ is the usual multiplication. Define $\oplus$ on $Q^+ \cup \{0\}$ by

$$\frac{m}{n} \oplus \frac{p}{q} = \frac{mq*pn}{nq}$$

for all $m,n,p,q \in N \cup \{0\}$, $n \neq 0$, $q \neq 0$. To show that $\oplus$ is well-defined, let $\frac{m}{n} = \frac{m'}{n'}$ and $\frac{p}{q} = \frac{p'}{q'}$ where $m,m',p,p' \in N \cup \{0\}$ and $n,n',q,q' \in N$. Then $mn' = m'n$ and $pq' = p'q$, so $mn'qq' = m'nqq'$ and $nn'pq' = nn'p'q$. Thus $n'q'(mq*pn) = n'q'mq*n'q'pn = m'nqq'*nn'p'q = nq(m'q'*p'n')$ and hence

$$\frac{m}{n} \oplus \frac{p}{q} = \frac{mq*pn}{nq} = \frac{m'q'*p'n'}{n'q'} = \frac{m'}{n'} \oplus \frac{p'}{q'}.$$

It is straightforward to vertify that the operation $\oplus$ is associative on $Q^+\cup\{0\}$, the usual multiplication is distributive over the operation $\oplus$ on $Q^+\cup\{0\}$ and $0\oplus x = x\oplus 0 = x$ for all $x \in Q^+\cup\{0\}$. Because the operation $*$ is commutative on $N\cup\{0\}$, the operation $\oplus$ is commutative on $Q^+\cup\{0\}$. It remains to show that for each $x \in Q^+\cup\{0\}$, $x\oplus y = 0$ for some $y \in Q^+\cup\{0\}$. Let $\frac{m}{n} \in Q^+\cup\{0\}$ where $m \in N\cup\{0\}$ and $n \in N$. Since $(N\cup\{0\},*)$ is a group $m*p = p*m = 0$ for some $p \in N\cup\{0\}$. Then $\frac{p}{n} \in Q^+\cup\{0\}$ and

$$\frac{m}{n} \oplus \frac{p}{n} = \frac{mn*pn}{nn} = \frac{m*p}{n} = \frac{0}{n} = 0.$$

Therefore $(Q^+\cup\{0\},\oplus,\cdot)$ is a field.

The next corollary gives simple necessary and sufficient conditions of a positive integer n and a semifield S of nonnegative real numbers under the usual multiplication such that $M_n(S)$ is a regular matrix semigroup. The following lemma related to any semifield of nonnegative real numbers under the usual multiplication is required.

Lemma 3.4. Let S be a semifield of nonnegative real numbers under the usual multiplication. Assume that $\oplus$ is the addition on the semifield S. Then S is a field if and only if $1\oplus 1 = 0$.

Proof : Assume that S is a field. Then $1\oplus x = 0$ for some $x \in S$, so $x\oplus x^2 = 0$. Hence $1 = x^2$. But since $x \geqslant 0$, $x = 1$. Therefore $1\oplus 1 = 0$.

Conversely, if $1\oplus 1 = 0$, then $x\oplus x = 0$ for all $x \in S$, so S is a ring, and hence S is a field (since S is a semifield).

Corollary 3.5. Let S be a semifield of nonnegative real numbers under the usual multiplication and n a positive integer. Assume that $\oplus$ is the addition of the semifield S. Then the matrix semigroup $M_n(S)$ is regular if and only if one of the following conditions holds :

(i) $n = 1$.

(ii) $1 \oplus 1 = 0$.

(iii) $n = 2$ and $1 \oplus 1 = 1$.

Proof : It follows easily from Theorem 3.3 and Lemma 3.4. #

It then follows from Corollary 3.5 that if S is the semifield $(\mathbb{R}^+ \cup \{0\}, \max., \cdot)$ or the semifield $(\mathbb{Q}^+ \cup \{0\}, \max., \cdot)$ where $\cdot$ is the usual multiplication, then the matrix semigroups $M_1(S)$ and $M_2(S)$ are regular and for $n \geq 3$, the matrix semigroup $M_n(S)$ is not regular.

Let S be a semiring $(\mathbb{R}^+ \cup \{0\}, \max., \circledast)$ or a semiring $(\mathbb{Q}^+ \cup \{0\}, \max, \circledast)$ (the operation $\circledast$ need not be the usual multiplication). Then the matrix semigroup $M_1(S)$ is regular if and only if $(S, \circledast)$ is a regular semigroup, and for $n \geq 3$, by Theorem 1.3, the matrix semigroup $M_n(S)$ is not regular since S is not a ring. It follows from Theorem 3.3 that if S is a semifield, then $M_2(S)$ is a regular matrix semigroup. It is natural to ask whether this converse is true if S is commutative and has an identity. To answer this question, we give a general proposition as follows :

Proposition 3.6. Let S be a commutative semiring with 0,1 and $0 \neq 1$. Assume that for $x, y \in S$, either $x+y = x$ or $x+y = y$. Then for a positive integer n, the matrix semigroup $M_n(S)$ is regular if and only if

(i) $n = 1$ and $(S,\cdot)$ is a regular semigroup where $\cdot$ is the multiplication of the semiring S or

(ii) $n = 2$ and S is a semifield.

Proof : If (i) holds, then $M_1(S)$ is a regular matrix semigroup since the matrix semigroup $M_1(S)$ can be considered as the semigroup $(S,\cdot)$ which is regular.

If (ii) holds, by Theorem 3.3, $M_2(S)$ is a regular matrix semigroup.

For the converse, assume that the matrix semigroup $M_n(S)$ is regular and suppose that (i) does not hold. Suppose that $n = 1$. Since (i) does not hold, $(S,\cdot)$ is not a regular semigroup. But by assumption, $M_1(S)$ is a regular matrix semigroup. This implies that $(S,\cdot)$ is a regular semigroup, a contradiction. Hence $n > 1$. Since S is not a ring and $M_n(S)$ is a regular matrix semigroup, we have by Theorem 1.3 that $n \not\geq 3$. Hence $n = 2$.

Next, to show that $(S\setminus\{0\},\cdot)$ is a group. Let $a \in S\setminus\{0\}$. Then $\begin{bmatrix}1 & a\\ a & 0\end{bmatrix} \in M_2(S)$. Since the matrix semigroup $M_2(S)$ is regular, there exist $x,y,z,w \in S$ such that

$$\begin{bmatrix}1 & a\\ a & 0\end{bmatrix} = \begin{bmatrix}1 & a\\ a & 0\end{bmatrix}\begin{bmatrix}x & y\\ z & w\end{bmatrix}\begin{bmatrix}1 & a\\ a & 0\end{bmatrix}.$$

Thus

$$\begin{bmatrix}1 & a\\ a & 0\end{bmatrix} = \begin{bmatrix}x+az & y+aw\\ ax & ay\end{bmatrix}\begin{bmatrix}1 & a\\ a & 0\end{bmatrix}$$
$$= \begin{bmatrix}x+az+ay+a^2w & ax+a^2z\\ ax+a^2y & a^2x\end{bmatrix}$$

which implies that

$$1 = x+az+ay+a^2w \qquad \ldots\ldots\ (I)$$

$$a = ax+a^2z \qquad \ldots\ldots\ (II)$$

$$a = ax+a^2y \qquad \ldots\ldots\ (III)$$

$$0 = a^2x \qquad \ldots\ldots\ (IV)$$

Since the matrix semigroup $M_2(S)$ is regular, by Theorem 1.2, S is a regular semiring, so $a = ara$ for some $r \in S$. Since $a^2x = 0$ (from (IV)), we have that $0 = r0 = r(a^2x) = (ara)x = ax$. From (I), $x+(az+ay+a^2w) = 1$. By the assumption of the semiring S, $x = 1$ or $az+ay+a^2w = 1$. If $x = 1$, then $a = a1 = ax = 0$, a contradiction. Hence $az+ay+a^2w = 1$. Therefore $a(z+y+aw) = 1$.

This proves that for every $a \in S\setminus\{0\}$, $aa' = 1$ for some $a' \in S\setminus\{0\}$. It remains to show that $S\setminus\{0\}$ is closed under the multiplication of S. Let $a,b \in S\setminus\{0\}$. Then $aa' = 1$ for some $a' \in S\setminus\{0\}$ which implies that $a'(ab) = (a'a)b = b \neq 0$. Thus $ab \neq 0$.

Hence $(S\setminus\{0\},\cdot)$ is a group. Therefore S is a semifield. #