CHAPTER II

REGULAR MATRIX SEMIGROUPS OVER A BOOLEAN SEMIRING

It was proved in [2] that for a commutative idempotent semiring S with 0,1 and a positive integer n, the matrix semigroup $M_n(S)$ is regular if and only if either n = 1 or n = 2 and S is a Boolean algebra. The aim of this chapter is to generalize this result. We introduce necessary and sufficient conditions of a Boolean semiring S and a positive integer n that give the matrix semigroup $M_n(S)$ the regularity.

Some examples of Boolean semirings are as follows :

(1) Boolean algebras,

(2) Boolean rings with identity,

(3) the direct product B×R where B is a Boolean algebra and R is a Boolean ring,

(4) ([0,1],max.,min.)(where ([0,1],max.,min.) is the semiring $([0,1],\oplus,\otimes)$ with $x\oplus y$ = maximum $\{x,y\}$ and $x\otimes y$ = minimum $\{x,y\}$.)

The semirings in (1) and (4) are commutative idempotent semirings with 0,1. Under the assumption that Boolean rings with identity are nontrivial, the semirings in (2) and (3) are Boolean semirings which are not idempotent semirings. An other example of Boolean semirings which is not an idempotent semiring is the semiring $(\{0\}\cup[\frac{1}{2},1],\oplus,\min.)$ where $x\oplus y = \frac{1}{2}$ for all $x,y \in [\frac{1}{2},1]$ and $x\oplus 0 = 0\oplus x = x$ for all $x \in \{0\}\cup[\frac{1}{2},1]$. More generally, if S is a set of positive real numbers with minimum element m and maximum element M and

$m \neq M$, then $(S\cup\{0\},\oplus,\min.)$ where $x\oplus y = m$ for all $x,y \in S$ and $x\oplus 0 = 0\oplus x = x$ for all $x \in S\cup\{0\}$, is a Boolean semiring which is not an idempotent semiring.

In the next two propositions, we give some general properties of semirings which will be used later.

Proposition 2.1. Let S be a semiring and E^+ the set of all additive idempotents of S. And, let I^+ be the set of all additively invertible elements of S if the semiring S has a zero. Then the following statements hold :

(1) $SE^+ \subseteq E^+$ and $E^+S \subseteq E^+$.

(2) $SI^+ \subseteq I^+$ and $I^+S \subseteq I^+$ if S has a zero.

(3) $E^+ \cap I^+ = \{0\}$ if S has a zero 0.

(4) $E^+I^+ = \{0\} = I^+E^+$ if S has a zero 0.

Proof : (1) If $x \in S$ and $y \in E^+$, then $xy+xy = x(y+y) = xy$ and $yx+yx = (y+y)x = yx$. Hence $SE^+ \subseteq E^+$ and $E^+S \subseteq E^+$.

(2) Let $x \in S$ and $y \in I^+$. Then $y+z = z+y = 0$ for some $z \in S$ where 0 is the zero of the semiring S. Since $xy+xz = x(y+z) = x0 = 0$, $xz+xy = x(z+y) = x0 = 0$, $yx+zx = (y+z)x = 0x = 0$ and $zx+yx = (z+y)x = 0x = 0$, we have that $xy, yx \in I^+$. This proves that $SI^+ \subseteq I^+$ and $I^+S \subseteq I^+$.

(3) Clearly, $0 \in E^+ \cap I^+$. Let $x \in E^+ \cap I^+$. Then $x+x = x$ and $x+y = y+x = 0$ for some $y \in S$. Hence $x = x+0 = x+(x+y) = (x+x)+y = x+y = 0$. Therefore $E^+ \cap I^+ = \{0\}$.

(4) Since $E^+I^+ \subseteq E^+S$ and $E^+S \subseteq E^+$ (by (1)), it follows that $E^+I^+ \subseteq E^+$. Also, $E^+I^+ \subseteq I^+$ since $E^+I^+ \subseteq SI^+$ and $SI^+ \subseteq I^+$ (by (2)).

Then $E^{+}I^{+} \subseteq E^{+} \cap I^{+}$. By (3), $E^{+} \cap I^{+} = \{0\}$, so $E^{+}I^{+} = \{0\}$ since $0 \in E^{+}I^{+}$. It can be shown similarly that $I^{+}E^{+} = \{0\}$. #

Proposition 2.2. Let $\{S_\alpha\}_{\alpha\in\Lambda}$ be a nonempty family of additively commutative semirings and $S = \prod_{\alpha\in\Lambda} S_\alpha$ the direct product of the semirings $S_\alpha, \alpha\in\Lambda$. Then for each positive integer n, the matrix semigroup $M_n(S)$ is regular if and only if the matrix semigroup $M_n(S_\alpha)$ is regular for all $\alpha \in \Lambda$.

Proof : First, note that the semiring S is additively commutative since each S_α is additively commutative. Let n be a positive integer. Assume that the matrix semigroup $M_n(S)$ is regular. Let $\alpha \in \Lambda$. For each $\beta \in \Lambda$ and $\beta \neq \alpha$, let s_β be a fixed element in S_β. To show that the matrix semigroup $M_n(S_\alpha)$ is regular, let $A \in M_n(S_\alpha)$. Let the matrix $B \in M_n(S)$ be defined as follows : For $i,j \in \{1,2,\ldots,n\}$, $\beta \in \Lambda$, let

$$\text{the } \beta\underline{\text{th}} \text{ component of } B_{ij} = \begin{cases} A_{ij} & \text{if } \beta = \alpha, \\ s_\beta & \text{if } \beta \neq \alpha. \end{cases} \qquad \ldots\ldots. (*)$$

Since $M_n(S)$ is regular, there exists a matrix $C \in M_n(S)$ such that $B = BCB$. Define the matrix $D \in M_n(S_\alpha)$ by

$$D_{ij} = \text{the } \alpha\underline{\text{th}} \text{ component of } C_{ij} \qquad \ldots\ldots. (**)$$

for all $i,j \in \{1,2,\ldots,n\}$. Claim that A = ADA. To prove the claim, let $i,j \in \{1,2,\ldots,n\}$. Then

$$\begin{aligned} A_{ij} &= \text{the } \alpha\underline{\text{th}} \text{ component of } B_{ij} \\ &= \text{the } \alpha\underline{\text{th}} \text{ component of } (BCB)_{ij}. \end{aligned}$$

Since $(BCB)_{ij} = \sum_{k=1}^{n} B_{ik}(CB)_{kj} = \sum_{k=1}^{n} B_{ik}(\sum_{t=1}^{n} C_{kt}B_{tj})$, it follows from (*) and (**) that

$$\text{the } \underline{\alpha th} \text{ component of } (BCB)_{ij} = \sum_{k=1}^{n} A_{ik}(\sum_{t=1}^{n} D_{kt}A_{tj})$$
$$= \sum_{k=1}^{n} A_{ik}(DA)_{kj}$$
$$= (ADA)_{ij}.$$

Hence $A_{ij} = (ADA)_{ij}$.

For the converse, assume that the matrix semigroup $M_n(S_\alpha)$ is regular for all $\alpha \in \Lambda$. To prove $M_n(S)$ is regular, let $A \in M_n(S)$. For each $\alpha \in \Lambda$, let A_α be the matrix in $M_n(S_\alpha)$ defined by

$$(A_\alpha)_{ij} = \text{the } \underline{\alpha th} \text{ component of } A_{ij}$$

for all $i,j \in \{1,2,\ldots,n\}$. Since each $M_n(S_\alpha)$ is a regular semigroup, for each $\alpha \in \Lambda$, $A_\alpha = A_\alpha B_\alpha A_\alpha$ for some $B_\alpha \in M_n(S_\alpha)$. Define the matrix $B \in M_n(S)$ by

$$\text{the } \underline{\alpha th} \text{ component of } B_{ij} = (B_\alpha)_{ij}$$

for all $\alpha \in \Lambda$, $i,j \in \{1,2,\ldots,n\}$. Claim that $A = ABA$. To prove the claim, let $i,j \in \{1,2,\ldots,n\}$, $\alpha \in \Lambda$. Then

$$\text{the } \underline{\alpha th} \text{ component of } A_{ij} = (A_\alpha)_{ij}$$
$$= (A_\alpha B_\alpha A_\alpha)_{ij}$$
$$= \sum_{k=1}^{n} (A_\alpha)_{ik}(\sum_{t=1}^{n} (B_\alpha)_{kt}(A_\alpha)_{tj})$$
$$= \text{the } \underline{\alpha th} \text{ component of } (ABA)_{ij}.$$

Hence $A = ABA$ as required. #

In order to characterize regular matrix semigroups over any Boolean semiring, we need some general facts about Boolean semirings which are given in the next two propositions.

<u>Proposition 2.3.</u> If S is a Boolean semiring, then the following statements hold :

(1) For every $x \in S$, $2x$ is an additive idempotent of S.

(2) For $x,y \in S$, if $x+y = 0$, then $2x = 2y = 0$.

Proof : (1) If x is an element of S, then $2x = x+x = (x+x)^2 = x^2 + x^2 + x^2 + x^2 = x+x+x+x = 4x$ which implies that 2x is an additive idempotent of S.

(2) Let $x,y \in S$ be such that $x+y = 0$. Then $2x+2y = 0$ and by (1), $2x = 2x+2x$ and $2y = 2y+2y$. Hence $2x = 2x+0 = 2x+2x+2y = 2x+2y = 0$, and $2y = 0$ can be shown similarly. #

Proposition 2.4. Let S be a Boolean semiring and let E^+ and I^+ be the set of all additive idempotents of S and the set of all additively invertible elements of S, respectively. Then the following statements hold :

(1) $E^+ = \{2x \mid x \in S\}$ and under the addition and multiplication of S, E^+ is a commutative idempotent semiring having 0 and 1+1 as its zero and identity, respectively.

(2) $I^+ = \{x \in S \mid 2x = 0\}$ and under the addition and multiplication of S, I^+ is a Boolean ring having 0 as its zero.

Proof : (1) It follows from Proposition 2.3(1) that $\{2x \mid x \in S\} \subseteq E^+$. If $y \in E^+$, then $y = 2y \in \{2x \mid x \in S\}$. Hence $E^+ = \{2x \mid x \in S\}$.

Next, we shall show that under the addition and multiplication of S, E^+ is a commutative idempotent semiring which has 0 and 1+1 as its zero and identity, respectively. Clearly $0 \in E^+$. By the definition of E^+, $x+x = x$ for all $x \in E^+$. And $x^2 = x$ for all $x \in E^+$ since S is a Boolean semiring. Since $E^+ = \{2x \mid x \in S\}$, $1+1 \in E^+$. Also, for $x \in E^+$, $(1+1)x = x+x = x$. It remains to show that

$x+y$, $xy \in E^+$ for all $x,y \in E^+$. If $x,y \in E^+$, then $(x+y)+(x+y) = (x+x)+(y+y) = x+y$ and $xy+xy = x(y+y) = xy$, so $x+y$, $xy \in E^+$.

(2) It follows from the definition of I^+ that $\{x \in S | 2x = 0\} \subseteq I^+$. If $y \in I^+$, then $y+z = 0$ for some $z \in S$, so $2y = 0$ by Proposition 2.3(2), and therefore $y \in \{x \in S | 2x = 0\}$. Hence $I^+ = \{x \in S | 2x = 0\}$.

Now, we have $0 \in I^+$ and for any $x \in I^+$, $x+x = 0$ and $x^2 = x$. Then to show that under the addition and multiplication of S, I^+ is a Boolean ring having 0 as its zero, it suffices to show that I^+ is closed under the addition and multiplication of S. Let $x,y \in I^+$. Then $2(x+y) = (x+y)+(x+y) = (x+x)+(y+y) = 2x+2y = 0+0 = 0$ and $2(xy) = xy+xy = x(y+y) = x(2y) = x0 = 0$. Therefore $x+y$, $xy \in I^+$. #

The following lemma is required to obtain the two main theorems of this chapter.

Lemma 2.5. Let S be a Boolean semiring which is not a ring. Then the following statements are equivalent :

(1) The matrix semigroup $M_2(S)$ is regular.

(2) For every $x \in S$, $1+2x = 1$ and there exists a $y \in S$ such that $x+y = 1$ and $xy = 0$.

(3) $S \cong B \times R$ for some Boolean algebra B and Boolean ring R.

Proof : (1) implies (2). Assume that the matrix semigroup $M_2(S)$ is regular. To prove that (2) holds, let $x \in S$. Then $\begin{bmatrix} 1 & 2x \\ 1 & 0 \end{bmatrix} \in M_2(S)$. Since the matrix semigroup $M_2(S)$ is regular, there exist $a,b,c,d \in S$ such that

$$\begin{bmatrix}1 & 2x\\ 1 & 0\end{bmatrix} = \begin{bmatrix}1 & 2x\\ 1 & 0\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}1 & 2x\\ 1 & 0\end{bmatrix}.$$

Thus

$$\begin{bmatrix}1 & 2x\\ 1 & 0\end{bmatrix} = \begin{bmatrix}a+2cx+b+2dx & 2ax+2cx\\ a+b & 2ax\end{bmatrix}$$

since $x^2 = x$ and $4x = 2x$ (Proposition 2.3(1)). It then follows that

$$1 = a+2cx+b+2dx \qquad \ldots\ldots \text{(I)}$$

$$2x = 2ax+2cx \qquad \ldots\ldots \text{(II)}$$

$$1 = a+b \qquad \ldots\ldots \text{(III)}$$

$$0 = 2ax \qquad \ldots\ldots \text{(IV)}$$

Replacing a+b by 1 (from (III)) in (I), we have $1 = 1+2cx+2dx$. Since $2ax = 0$ (from (IV)), it follows from (II) that $2x = 2cx$. Then $1 = 1+2x+2dx$. Since $4x = 2x$, $1+2x = 1+2x+2dx+2x = 1+(2x+2x)+2dx = 1+2x+2dx = 1$.

Also, since the matrix semigroup $M_2(S)$ is regular and $\begin{bmatrix}x & 1\\ 0 & x\end{bmatrix} \in M_2(S)$, there are elements $e,f,g,h \in S$ such that

$$\begin{bmatrix}x & 1\\ 0 & x\end{bmatrix} = \begin{bmatrix}x & 1\\ 0 & x\end{bmatrix}\begin{bmatrix}e & f\\ g & h\end{bmatrix}\begin{bmatrix}x & 1\\ 0 & x\end{bmatrix}.$$

Thus

$$\begin{bmatrix}x & 1\\ 0 & x\end{bmatrix} = \begin{bmatrix}ex+gx & ex+g+fx+hx\\ gx & gx+hx\end{bmatrix}$$

(since $x^2 = x$) which implies that

$$x = ex+gx \qquad \ldots\ldots \text{(I}'\text{)}$$

$$1 = ex+g+fx+hx \qquad \ldots\ldots \text{(II}'\text{)}$$

$$0 = gx \qquad \ldots\ldots \text{(III}'\text{)}$$

$$x = gx+hx \qquad \ldots\ldots \text{(IV}'\text{)}$$

Since $gx = 0$ (from (III')), $x = ex$ by (I') and $x = hx$ by (IV'). Replacing ex and hx by x in (II'), we get $1 = x+g+fx+x$. Multiply this equality by x, so we have $x = x+gx+fx+x$ since $x^2 = x$. Then $x = x+fx+x$ because $gx = 0$. From $1 = x+g+fx+x$ and $x = x+fx+x$, we obtain $1 = x+g$.

(2) implies (3). Assume that (2) holds. Claim that under the addition and multiplication of S, E^+ (the set of all additive idempotents of S) is a Boolean algebra having 0 and 1+1 as its zero and identity, respectively. By Proposition 2.4(1), under the addition and multiplication of S, E^+ is a commutative idempotent semiring which has 0 and 1+1 as its zero and identity, respectively. To prove the claim, it remains to show that (i) for all $x,y,z \in E^+$, $x+yz = (x+y)(x+z)$ and (ii) for each $x \in E^+$, there exists an $x' \in E^+$ such that $x+x' = 1+1$ and $xx' = 0$.

To prove (i), let $x,y,z \in E^+$. By (2), $1+2(y+z) = 1$. Since $y+z \in E^+$, $2(y+z) = y+z$. Then $1+y+z = 1$. Hence

$$\begin{aligned} x+yz &= x1+yz \\ &= x(1+y+z)+yz \\ &= x+xy+xz+yz \\ &= x^2+xy+xz+yz \\ &= (x+y)(x+z). \end{aligned}$$

To prove (ii), let $x \in E^+$. By (2), there is a $y \in S$ such that $x+y = 1$ and $xy = 0$. Then $2y \in E^+$ (Proposition 2.3(1)), $x+2y = 2x+2y = x+x+y+y = (x+y)+(x+y) = 1+1$ and $x(2y) = x(y+y) = xy+xy = 0+0 = 0$. Hence we have the claim.

Let I^+ be the set of all additively invertible elements of S. Then, by Proposition 2.4(2), under the addition and multiplication of S, I^+ is a Boolean ring having 0 as its zero. By the assumption (2), there exists a $k \in S$ such that $1+1+k = 1$ and $(1+1)k = 0$. Then $k+k = 0$ which implies that $k \in I^+$. Then by Proposition 2.1(2), $xk \in I^+$

for all $x \in S$. By Proposition 2.3(1), $2x \in E^+$ for all $x \in S$. Since $1 = 1+1+k$, it follows that for every $x \in S$, $x = 2x+xk$, $2x \in E^+$ and $xk \in I^+$. .... (*)

To obtain (3), we shall show that $S \cong E^+ \times I^+$ where the addition and multiplication of E^+ and I^+ are those of S. Define $\Phi : S \to E^+ \times I^+$ by

$$\Phi(x) = (2x,xk)$$

for all $x \in S$ (see (*)).

To show that Φ is a homomorphism, let $x,y \in S$. Then

$$\begin{aligned}\Phi(x+y) &= (2(x+y),(x+y)k)\\ &= (2x+2y,xk+yk)\\ &= (2x,xk)+(2y,yk)\\ &= \Phi(x)+\Phi(y).\end{aligned}$$

Since $4xy = 2xy$ by Proposition 2.3(1) and $k^2 = k$, we have that

$$\begin{aligned}\Phi(xy) &= (2xy,xyk)\\ &= (4xy,xyk^2)\\ &= ((2x)(2y),(xk)(yk))\\ &= (2x,xk)(2y,yk)\\ &= \Phi(x)\Phi(y).\end{aligned}$$

To show that Φ is a one-to-one map, let $x,y \in S$ be such that $\Phi(x) = \Phi(y)$. Then $(2x,xk) = (2y,yk)$. Thus $2x = 2y$ and $xk = yk$. Since by (*), $x = 2x+xk$ and $y = 2y+yk$, it follows that $x = y$.

To show that Φ is an onto map, let $x \in E^+$ and $y \in I^+$. Then

$$\begin{aligned}\Phi(x+y) &= (2(x+y),(x+y)k)\\ &= (2x+2y,xk+yk).\end{aligned}$$

Since $x \in E^+$, $2x = x$. By Proposition 2.4(2), $2y = 0$ because $y \in I^+$. Since $E^+I^+ = \{0\}$ (by Proposition 2.1(4)), $x \in E^+$ and $k \in I^+$, we have that $xk = 0$. By (*), $y = 2y+yk$, hence $y = yk$ since $2y = 0$. Therefore

$\Phi(x+y) = (x,y)$.

(3) implies (1). Assume that $S \cong B \times R$ for some Boolean algebra B and Boolean ring R. Since B is a Boolean algebra, by Theorem 1.4, the matrix semigroup $M_2(B)$ is regular. The matrix semigroup $M_2(R)$ is regular since R is a regular ring (Theorem 1.1). It follows from Proposition 2.2 that the matrix semigroup $M_2(B \times R)$ is regular. This implies that the matrix semigroup $M_2(S)$ is regular. #

Note that the statements (1) and (2) are always true if S is a Boolean ring. These follow from the facts that every Boolean ring is a regular ring and in any Boolean ring R, for every $x \in R$, $2x = 0$, $x+(x+1) = 2x+1 = 0+1 = 1$ and $x(x+1) = x^2+x = x+x = 0$.

<u>Theorem 2.6</u>. Let S be a Boolean semiring and n a positive integer. Then the matrix semigroup $M_n(S)$ is regular if and only if one of the following conditions holds :

(i) $n = 1$.

(ii) S is a Boolean ring.

(iii) $n = 2$ and for every $x \in S$, $1+2x = 1$ and there exists a $y \in S$ such that $x+y = 1$ and $xy = 0$.

<u>Proof</u> : If $n = 1$, then the matrix semigroup $M_1(S)$ can be considered as the multiplicative structure of the Boolean semiring S, so $M_1(S)$ is a regular semigroup.

If S is a Boolean ring, then S is a regular ring, so by Theorem 1.1, $M_n(S)$ is a regular semigroup.

Assume that (iii) holds. If S is a ring, then S is Boolean ring, so $M_2(S)$ is a regular matrix semigroup. If S is not a ring, it follows from Lemma 2.5 that the matrix semigroup $M_2(S)$ is regular.

For the converse, assume that the matrix semigroup $M_n(S)$ is regular. To prove that one of the conditions (i), (ii) and (iii) holds, suppose that $n \geqslant 2$ and S is not a Boolean ring. Since $M_n(S)$ is a regular matrix semigroup, it follows from Theorem 1.3 that if $n \geqslant 3$, then S is a ring. But S is not a ring, so $n = 2$. Hence by Lemma 2.5, (iii) holds. #

<u>Theorem 2.7</u>. Let S be a Boolean semiring and n a positive integer. Then the matrix semigroup $M_n(S)$ is regular if and only if one of the following statements holds :

(i) $n = 1$.

(ii) S is a Boolean ring.

(iii) $n = 2$ and $S \cong B\times R$ for some Boolean algebra B and Boolean ring R.

<u>Proof</u> : If $n = 1$ or S is a Boolean ring, then the matrix semigroup $M_n(S)$ is regular (see the proof of Theorem 2.6).

Assume that $n = 2$ and $S \cong B\times R$ for some Boolean algebra B and Boolean ring R. Since B has more than one element, $B\times R$ is not a ring. Then S is not a ring. By Lemma 2.5, we have that $M_2(S)$ is a regular matrix semigroup.

For the converse, assume that the matrix semigroup $M_n(S)$ is regular, $n \geqslant 2$ and S is not a Boolean ring. By Theorem 1.3, $n = 2$. Then $M_2(S)$ is regular and S is not a ring. Thus by Lemma 2.5, $S \cong B\times R$ for some Boolean algebra B and Boolean ring R. Hence (iii) holds. #