# CHAPTER II

## PARTIALLY ORDERED DISTRIBUTIVE RATIO SEMINEAR-RINGS

In this chapter, some fundamental theorems of partially ordered distributive ratio seminear-rings are given and we also classify all complete ordered distributive ratio seminear-rings.

*Definition 2.1*. A partial order $\leqslant$ on a distributive ratio seminear-ring D is said to be *compatible* if it satisfies the following property: For any $x, y, z \in D$, $x \leqslant y$ implies $x + z \leqslant y + z$, $z + x \leqslant z + y$, $xz \leqslant yz$ and $zx \leqslant zy$.

*Definition 2.2*. A system $(D,+,\cdot,\leqslant)$ is called a *partially ordered distributive ratio seminear-ring* if $(D,+,\cdot)$ is a distributive ratio seminear-ring and $\leqslant$ is a compatible partial order on D. If the compatible partial order on D is a total order then $(D,+,\cdot,\leqslant)$ is called an *ordered distributive ratio seminear-ring*.

*Example 2.3*. (1) Every distributive ratio seminear-ring is a partially ordered distributive ratio seminear-ring with respect to the trivial partial order, that is, $x \leqslant y$ if and only if $x = y$.

(2) Every ratio subseminear-ring of a partially ordered distributive ratio seminear-ring is a partially ordered distributive ratio seminear-ring.

(3) If $(D,+,\cdot,\leqslant)$ is a partially ordered distributive ratio seminear-ring then $(D,+,\cdot,\leqslant_{opp})$ is a partially ordered distributive

ratio seminear-ring.

(4) $(\mathbb{Q}^+,+,\cdot,\leqslant)$ and $(\mathbb{R}^+,+,\cdot,\leqslant)$ are ordered distributive ratio seminear-rings.

(5) Let $(G,\cdot,\leqslant)$ be a partially ordered group. Define the operation + on G by

5.1) $x + y = x$ for all $x, y \in G$ or

5.2) $x + y = y$ for all $x, y \in G$.

Then $(G,+,\cdot,\leqslant)$ is a partially ordered distributive ratio seminear-ring.

(6) Let $(G,\cdot,\leqslant)$ be a lattice ordered group. Define the operation + on G by

6.1) $x + y = \sup \{x,y\}$ for all $x, y \in G$ or

6.2) $x + y = \inf \{x,y\}$ for all $x, y \in G$.

Then $(G,+,\cdot,\leqslant)$ is a partially ordered distributive ratio seminear-ring.

Proposition 2.4. $\mathbb{Q}^+$ has only three compatible partial orders, the usual order, the dual of the usual order and the trivial partial order.

Proof: Let $\overset{*}{\leqslant}$ be a compatible partial order on $\mathbb{Q}^+$.

Case 1: $1 \overset{*}{<} 2$. Then $2 = 1 + 1 \overset{*}{<} 2 + 1 = 3$. It follows by induction that $n \overset{*}{<} n + 1$ for all $n \in \mathbb{Z}^+$. This implies that $n \overset{*}{<} n + l$ for all $n, l \in \mathbb{Z}^+$. Since for each $m, n \in \mathbb{Z}^+$, $m < n$ implies $n = m + l$ for some $l \in \mathbb{Z}^+$, we get that for each $m, n \in \mathbb{Z}^+$, $m < n$ if and only if $m \overset{*}{<} n$.

Let $x, y \in \mathbb{Q}^+$. Then $x = \frac{m}{n}$ and $y = \frac{r}{s}$ for some $m,n,r,s \in \mathbb{Z}^+$. Then we have that $ms \overset{*}{<} nr$ if and only if $ms < nr$ which implies that

$\frac{m}{n} \overset{*}{<} \frac{s}{r}$ if and only if $\frac{m}{n} < \frac{r}{s}$. Therefore for each $x, y \in Q^+$, $x \overset{*}{\leqslant} y$ if and only if $x < y$. Hence $\overset{*}{\leqslant}$ is the usual order on $Q^+$.

<u>Case 2</u>: $2 \overset{*}{<} 1$. The proof of this case is similar to the proof of Case 1 and shows that $\overset{*}{\leqslant}$ is the dual of the usual order.

<u>Case 3</u>: 1 is incomparable to 2. We shall show that $\overset{*}{\leqslant}$ is the trivial partial order.

First, we claim that for any $m, n \in Z^+$, $m \overset{*}{\leqslant} n$ if and only if $m = n$. Suppose not. Then there exist $m, n \in Z^+$ such that $m \overset{*}{<} n$.

<u>Subcase 3.1</u>: $m < n$. Then there exist $q, r \in Z_0^+$ such that $n = mq + r$ and $0 \leqslant r < m$.

<u>Subcase 3.1.1</u>: $r = 0$. Then $m \overset{*}{<} mq$, so $1 \overset{*}{<} q$. Since 1 is incomparable to 2 and $q \in Z^+$, so $2 < q$, we have that $0 < q - 2$. Therefore $1 + (q-2) \overset{*}{\leqslant} q + (q-2)$, so $q - 1 \overset{*}{\leqslant} 2(q-1)$. Hence $1 \overset{*}{\leqslant} 2$, a contradiction.

<u>Subcase 3.1.2</u>: $r > 0$. Then $m \overset{*}{<} mq + r$, so we get that $m+(m-r) \overset{*}{<} (mq+r)+(m-r)$. Thus $2m-r \overset{*}{<} m(q+1)$, hence $1 \overset{*}{<} \frac{q+1}{2} + \frac{r}{2m}$.

Let $x = \frac{q+1}{2} + \frac{r}{2m}$. Then $1 \overset{*}{<} x$. Also, $1 \overset{*}{<} x^n$ for all $n \in Z^+$. Since $q \geqslant 1$ and $\frac{r}{2m} > 0$, $x > 1$. Then there exists an $n \in Z^+$ such that $x^n > 2$. Therefore $1 + (x^n-2) \overset{*}{\leqslant} x^n + (x^n-2)$, so we have that $x^n-1 \overset{*}{\leqslant} 2(x^n-1)$. Hence $1 \overset{*}{\leqslant} 2$, a contradiction.

<u>Subcase 3.2</u>: $n < m$. The proof of this subcase is similar to the proof of Subcase 3.1 and shows that this subcase cannot occur.

Hence we have the claim.

Let x, y $\varepsilon$ $\mathbb{Q}^+$. Then $x = \frac{m}{n}$ and $y = \frac{r}{s}$ for some m,n,r,s $\varepsilon$ $\mathbb{Z}^+$. By the claim, we have that $ms \overset{*}{\leqslant} nr$ if and only if ms = nr which implies that $\frac{m}{n} \overset{*}{\leqslant} \frac{r}{s}$ if and only if $\frac{m}{n} = \frac{r}{s}$. Therefore for each x,y $\varepsilon$ $\mathbb{Q}^+$, $x \overset{*}{\leqslant} y$ if and only if x = y. Hence $\overset{*}{\leqslant}$ is the trivial partial order on $\mathbb{Q}^+$. #

<u>Remark 2.5</u>. Let D be a partially ordered distributive ratio seminear-ring. Then the following statements clearly hold:

(1) For any x, y, z $\varepsilon$ D, $x < y$ implies $xz < yz$ and $zx < zy$.

(2) For any u, v, x, y $\varepsilon$ D, $u < v$ and $x < y$ imply $ux < vy$ and $u + x \leqslant v + y$.

(3) For any x, y $\varepsilon$ D, $x < y$ implies $y^{-1} < x^{-1}$.

Let D be a partially ordered distributive ratio seminear-ring and A a subset of D. The <u>positive cone</u> of A, denoted by $P_A$, is $\{x \in A \mid x \geqslant 1\}$. The following statements hold:

(1) $(P_D, \cdot)$ is a semigroup with identity.

(2) $(P_D, +)$ is a semigroup if and only if $1 \leqslant 1+1$

(3) $P_D \cap P_D^{-1} = \{1\}$

(4) $xP_Dx^{-1} = P_D$ for all x $\varepsilon$ D.

(5) $P_H = P_D \cap H$ where H is a subset of D.

<u>Proposition 2.6</u>. Let D be a partially ordered distributive ratio seminear-ring. Then the ratio subseminear-ring H is convex in D if and only if $P_H$ is a convex subset of $P_D$.

Proof: It is clear that if H is convex in D then $P_H$ is convex in $P_D$.

Conversely, assume that $P_H$ is a convex subset of $P_D$. To show that H is convex in D, let x, y $\varepsilon$ H and z $\varepsilon$ D be such that $x \leqslant z \leqslant y$. Then $1 \leqslant zx^{-1} \leqslant yx^{-1}$, so $zx^{-1} \varepsilon P_D$ and $yx^{-1} \varepsilon P_H$. By assumption, $zx^{-1} \varepsilon P_H$. Thus $zx^{-1} \varepsilon H$, so $z = (zx^{-1})x \varepsilon H$. Hence H is convex in D. #

Definition 2.7. Let D be a partially ordered distributive ratio seminear-ring. D is called upper additive if for any x, y $\varepsilon$ D, $1 \leqslant x$ and $1 \leqslant y$ imply $1 \leqslant x + y$, lower additive if for any x, y $\varepsilon$ D, $x \leqslant 1$ and $y \leqslant 1$ imply $x + y \leqslant 1$, left [right] increasing if $x \leqslant x + y$ [$x \leqslant y + x$] for all x, y $\varepsilon$ D. Left and right decreasing are defined dually.

In Example 2.3, (5) and (6) are both upper and lower additive, $(\mathbb{R}^+,+,\cdot,\leqslant)$ is upper additive but not lower additive, $(\mathbb{R}^+,+,\cdot,\leqslant_{opp})$ is lower additive but not upper additive, 6.1) is both left and right increasing, 6.2) is both left and right decreasing, 5.1) is left increasing but not right increasing and 5.2) is right increasing but not left increasing.

Proposition 2.8. Let D be a partially ordered distributive ratio seminear-ring. Then the following statements hold:

(1) D is upper [lower] additive if and only if $1 \leqslant 1 + 1$ [$1+1 \leqslant 1$] (hence D is both upper and lower additive if and only if

$1 + 1 = 1$).

(2) D is left [right] increasing if and only if $1 + D \subseteq P_D$ [$D+1 \subseteq P_D$] (hence D is both left and right increasing if and only if $(1+D) \cup (D+1) \subseteq P_D$).

(3) D is left [right] decreasing if and only if $1 + D \subseteq P_D^{-1}$ [$D+1 \subseteq P_D^{-1}$] (hence D is both left and right decreasing if and only if $(1+D) \cup (D+1) \subseteq P_D^{-1}$).

(4) D is directed if and only if $P_D$ generates $(D,\cdot)$.

(5) D is a lattice if and only if it is directed and $P_D$ is a lattice.

(6) D is complete if and only if every subset of $P_D$ has an infimum.

(7) D is totally ordered if and only if $D = P_D \cup P_D^{-1}$.

<u>Proof</u>: (1) It is clear that if D is upper additive then $1 \leqslant 1 + 1$. Conversely, assume that $1 \leqslant 1 + 1$. Let $x, y \in D$ be such that $x \geqslant 1$ and $y \geqslant 1$. Then $x + y \geqslant 1 + 1 \geqslant 1$. Hence D is upper additive.

(2) It is clear that if D is left increasing then $1 + D \subseteq P_D$. Conversely, assume that $1 + D \subseteq P_D$. Let $x, y \in D$. Then $1 + yx^{-1} \in P_D$, so $1 + yx^{-1} \geqslant 1$. Hence $x + y \geqslant x$. Therefore D is left increasing.

(3) The proof is similar to the proof of (2).

(4) Since $(D,\cdot)$ is a partially ordered group, it follows from Proposition 1.17(1) that D is directed if and only if $P_D$ generates $(D,\cdot)$.

(5) It is clear that if D is a lattice then it is directed and $P_D$ is a lattice.

Conversely, assume that D is directed and $P_D$ is a lattice. Let $x \in D$. Since D is directed, $U(x,1)$ is nonempty. Let $y \in U(x,1)$. Then $x \leqslant y$ and $1 \leqslant y$, so $yx^{-1}, y \in P_D$. Since $P_D$ is a lattice, $\sup \{yx^{-1},y\}$ exists. By Proposition 1.17(3), $\sup \{yx^{-1},y\} = y\cdot\sup \{x^{-1},1\}$, so $y^{-1}\cdot\sup \{yx^{-1},y\} = \sup \{x^{-1},1\} = x^{-1}\cdot\sup \{1,x\}$. Hence we have that $xy^{-1}\cdot\sup \{yx^{-1},y\} = \sup \{1,x\}$. Therefore $\sup \{x,1\}$ exists for all $x \in D$. By Proposition 1.17(2), D is a lattice.

(6) It is clear that if D is complete then every subset of $P_D$ has an infimum. For the converse, assume that every subset of $P_D$ has an infimum. To show that D is complete, let A be a subset of D which has a lower bound, say x. Then $x \leqslant a$ for all $a \in A$, so $1 \leqslant ax^{-1}$ for all $a \in D$. Hence $Ax^{-1} \subseteq P_D$. By assumption, $\inf (Ax^{-1})$ exists, say y. Then we have $y \leqslant ax^{-1}$ for all $a \in A$, so $yx \leqslant a$ for all $a \in A$. Thus yx is a lower bound of A. Let z be a lower bound of A. Then $z \leqslant a$ for all $a \in A$, so $zx^{-1} \leqslant ax^{-1}$ for all $a \in A$. It follows that $zx^{-1} \leqslant y$, so $z \leqslant yx$. Hence $yx = \inf (A)$. Therefore D is complete.

(7) If D is totally ordered then for each $x \in D$, $1 \leqslant x$ or $x \leqslant 1$ which implies that $D = P_D \cup P_D^{-1}$. Conversely, assume that $D = P_D \cup P_D^{-1}$. Let $x, y \in D$. Then $yx^{-1} \in P_D \cup P_D^{-1}$, so we get that $1 \leqslant yx^{-1}$ or $yx^{-1} \leqslant 1$. Hence $x \leqslant y$ or $y \leqslant x$. Therefore D is totally ordered. #

<u>Definition 2.9</u>. A subset A of a distributive ratio seminear-ring D is called an <u>O-set</u> of D if it satisfies the following conditions:

(i) $A \cap A^{-1} = \{1\}$.

(ii) $A^2 \subseteq A$.

(iii) $xAx^{-1} \subseteq A$ for all $x \in D$.

(iv) $(x+1)^{-1}(x+a)$, $(1+x)^{-1}(a+x) \in A$ for all $x \in D$, $a \in A$.

Note that for any distributive ratio seminear-ring D, $\{1\}$ is an O-set of D and for any partially ordered distributive ratio seminear-ring $D'$, the positive cone of $D'$ is an O-set of $D'$.

Theorem 2.10. Every distributive ratio seminear-ring has a maximal O-set.

Proof: Let D be a distributive ratio seminear-ring and let $\mathcal{A} = \{A \subseteq D \mid A \text{ is an O-set of } D\}$. Note that $\mathcal{A}$ is nonempty since $\{1\}$ belongs to $\mathcal{A}$ and $\mathcal{A}$ is a partially ordered set with respect to set inclusion. Let $\{A_\alpha\}_{\alpha\in I}$ be a chain in $\mathcal{A}$ and $J = \bigcup_{\alpha\in I} A_\alpha$. Clearly, J is an upper bound of the chain $\{A_\alpha\}_{\alpha\in I}$.

We shall show that J is an O-set of D. Let $x \in J \cap J^{-1}$. Then $x \in A_\alpha$ for some $\alpha \in I$ and $x = y^{-1}$ for some $y \in J$. We have that $y \in A_\beta$ for some $\beta \in I$. Since $\{A_\alpha\}_{\alpha\in I}$ is a chain in $\mathcal{A}$, $A_\alpha \subseteq A_\beta$ or $A_\beta \subseteq A_\alpha$. Without loss of generality, assume that $A_\alpha \subseteq A_\beta$. Then $x \in A_\beta$, so $x \in A_\beta \cap A_\beta^{-1}$. But $A_\beta$ is an O-set of D, so $x = 1$. Hence $J \cap J^{-1} = \{1\}$.

To show that $J^2 \subseteq J$, let $x, y \in J$. Then $x \in A_\alpha$ and $y \in A_\beta$ for some $\alpha, \beta \in I$. Without loss of generality, assume that $A_\alpha \subseteq A_\beta$. Then $x \in A_\beta$. Since $A_\beta^2 \subseteq A_\beta$, $xy \in A_\beta$. Hence $xy \in J$. Therefore $J^2 \subseteq J$.

To show that $xJx^{-1} \subseteq J$ for all $x \in D$, let $x \in D$ and $y \in J$. Then $y \in A_\alpha$ for some $\alpha \in I$. Since $xA_\alpha x^{-1} \subseteq A_\alpha$, $xyx^{-1} \in A_\alpha$. Hence $xyx^{-1} \in J$. Therefore $xJx^{-1} \subseteq J$.

Let $x \in D$ and $y \in J$. Then $y \in A_\alpha$ for some $\alpha \in I$. Since $A_\alpha$ is an O-set of D, $(x+1)^{-1}(x+y)$, $(1+x)^{-1}(y+x) \in A_\alpha$. Hence $(x+1)^{-1}(x+y)$, $(1+x)^{-1}(y+x) \in J$.

Therefore J is an O-set of D, that is, $J \in \mathcal{A}$. By Zorn's Lemma, $\mathcal{A}$ contains a maximal element. #

Let D be a distributive ratio seminear-ring and A an O-set of D. Define a relation $\leqslant$ on D by $x \leqslant y$ if and only if $x^{-1}y \in A$ for all $x, y \in D$. Using the same proof as in Theorem 1.19, we get that $\leqslant$ is a partial order on D and for any $x, y, z \in D$, $x \leqslant y$ implies that $xz \leqslant yz$ and $zx \leqslant zy$.

To prove that for any $x, y, z \in D$, $x \leqslant y$ implies $x + z \leqslant y + z$ and $z + x \leqslant z + y$, it suffices to prove that for any $x, y \in D$, $x \leqslant y$ implies $x + 1 \leqslant y + 1$ and $1 + x \leqslant 1 + y$. Let $x, y \in D$ be such that $x \leqslant y$. Then $x^{-1}y \in A$. Since A is an O-set of D, $(x+1)^{-1}(y+1) = (x(1+x^{-1}))^{-1}(y+1) = (1+x^{-1})^{-1}(x^{-1}y+x^{-1}) \in A$. Hence $x + 1 \leqslant y + 1$. Similarly, $1 + x \leqslant 1 + y$.

Therefore $\leqslant$ is a compatible partial order on D. Note that the relation $\overset{*}{\leqslant}$ on D which is defined by $x \overset{*}{\leqslant} y$ if and only if $yx^{-1} \in A$ for all $x, y \in D$ is also a compatible partial order on D and $\leqslant \, = \, \overset{*}{\leqslant}$ since A has the property that $xAx^{-1} \subseteq A$ for all $x \in D$. The proof that $\leqslant$ is the unique compatible partial order on D having A as its positive cone is the same as the proof given in the note, page 9. Hence we have the following theorem.

Theorem 2.11. A subset A of a distributive ratio seminear-ring D is an O-set of D if and only if there exists a unique compatible partial order $\leqslant$ on D such that A is the positive cone induced by $\leqslant$.

Note that for a distributive ratio seminear-ring D, the set of all O-sets of D and the set of all compatible partial orders on D are partially ordered set with respect to set inclusion. Then we have two corollaries, the first one is obtained from Theorem 2.11 by using the same proof given in Corollary 1.20.

Corollary 2.12. Let D be a distributive ratio seminear-ring, $\mathcal{A}$ the set of all O-sets of D and $\mathcal{B}$ the set of all compatible partial orders on D. Then $\mathcal{A}$ and $\mathcal{B}$ are order isomorphic.

Corollary 2.13. Every distributive ratio seminear-ring has a maximal compatible partial order.

Definition 2.14. Let D and D′ be partially ordered distributive ratio seminear-rings. A map $f: D \to D'$ is called an order homomorphism of D into D′ if f is isotone and a homomorphism. An order homomorphism $f: D \to D'$ is called an order monomorphism if f is injective and $f(P_D) = P_{f(D)}$, an order epimorphism if f is onto and $f(P_D) = P_{D'}$, and an order isomorphism if f is a bijection and $f^{-1}$ is isotone. D and D′ are said to be an order isomprphic if there exists an order isomorphism of D onto D′ and we denote this by $D \simeq D'$.

Proposition 2.15. Let $(D,\leqslant)$ and $(D',\leqslant')$ be partially ordered distributive ratio seminear-rings. Then the following statements hold :

(1) If $f: D \to D'$ is a homomorphism then $f$ is isotone if and only if $f(P_D) \subseteq P_{D'}$.

(2) If $f: D \to D'$ is an order homomorphism then ker $f$ is a convex C-set of D.

Proof: (1) Assume that $f: D \to D'$ is a homomorphism. It is clear that if $f$ is isotone then $f(P_D) \subseteq P_{D'}$. Conversely, assume that $f(P_D) \subseteq P_{D'}$. Let $x, y \in D$ be such that $x \leqslant y$. Then $yx^{-1} \in P_D$, so that $f(y)f(x)^{-1} = f(yx^{-1}) \in P_{D'}$. Hence $f(x) \leqslant' f(y)$. Therefore $f$ is isotone.

(2) Assume that $f: D \to D'$ is an order homomorphism. By Proposition 1.33(1), ker $f$ is a C-set of D. Let $x, y \in$ ker $f$ and $z \in D$ be such that $x \leqslant z \leqslant y$. Then $1' = f(x) \leqslant' f(z) \leqslant' f(y) = 1'$, so $f(z) = 1'$. Hence $z \in$ ker $f$. Therefore ker $f$ is convex. #

Theorem 2.16. Let $(D,\leqslant)$ be a partially ordered distributive ratio seminear-ring and C a convex C-set of D. Then there exists a compatible partial order on $D/C$ such that the projection map $\pi$ is an order epimorphism.

Proof: Define a relation $\overset{*}{\leqslant}$ on $D/C$ as follows: For $\alpha$, $\beta \in D/C$, $\alpha \overset{*}{\leqslant} \beta$ if and only if there exist $a \in \alpha$ and $b \in \beta$ such that $a \leqslant b$. Clearly, $\overset{*}{\leqslant}$ is reflexive. Let $\alpha, \beta \in D/C$ be such that $\alpha \overset{*}{\leqslant} \beta$ and $\beta \overset{*}{\leqslant} \alpha$. Then there exist $a, d \in \alpha$ and $b, c \in \beta$ such that $a \leqslant b$ and $c \leqslant d$. Then $d^{-1}a \leqslant d^{-1}b \leqslant c^{-1}b$. By definition of $\alpha$ and $\beta$, we have $d^{-1}a, c^{-1}b \in C$. But C is convex, so $d^{-1}b \in C$. Then

$\alpha = [d] = [b] = \beta$. Hence $\overset{*}{\leqslant}$ is anti-symmetric. Let $\alpha, \beta, \gamma \in D/C$ be such that $\alpha \overset{*}{\leqslant} \beta$ and $\beta \overset{*}{\leqslant} \gamma$. Then there exist $a \in \alpha$, $b, c \in \beta$ and $d \in \gamma$ such that $a \leqslant b$ and $c \leqslant d$. Hence $a \leqslant b = c(c^{-1}b) \leqslant d(c^{-1}b)$. This implies that $\alpha = [a] \overset{*}{\leqslant} [dc^{-1}b] = [d][c^{-1}][b] = \gamma\beta^{-1}\beta = \gamma$. Thus $\overset{*}{\leqslant}$ is transitive. Therefore $\overset{*}{\leqslant}$ is a partial order on $D/C$.

Next, we shall show that $\overset{*}{\leqslant}$ is compatible. Let $\alpha, \beta, \gamma \in D/C$ be such that $\alpha \overset{*}{\leqslant} \beta$. Then there exist $a \in \alpha$ and $b \in \beta$ such that $a \leqslant b$. Choose $c \in \gamma$. Thus $a + c \leqslant b + c$ and $ac \leqslant bc$. Then we have that $[a]+[c] = [a+c] \overset{*}{\leqslant} [b+c] = [b]+[c]$ and $[a][c] = [ac] \overset{*}{\leqslant} [bc] = [b][c]$. Hence $\alpha + \gamma \overset{*}{\leqslant} \beta + \gamma$ and $\alpha\gamma \overset{*}{\leqslant} \beta\gamma$. Similarly, $\gamma + \alpha \overset{*}{\leqslant} \gamma + \beta$ and $\gamma\alpha \overset{*}{\leqslant} \gamma\beta$. Therefore $\overset{*}{\leqslant}$ is compatible.

We have that $\pi : D \rightarrow D/C$ is an epimorphism. By definition of $\overset{*}{\leqslant}$, $\pi$ is isotone. Then $\pi(P_D) \subseteq P_{D/C}$ by Proposition 2.15(1). To show that $P_{D/C} \subseteq \pi(P_D)$, let $\alpha \in P_{D/C}$. Then $[1] \overset{*}{\leqslant} \alpha$, so that there exist $a \in [1]$ and $b \in \alpha$ such that $a \leqslant b$. Thus $ba^{-1} \in P_D$. Now, $\pi(ba^{-1}) = [ba^{-1}] = [b][a]^{-1} = [b][1] = [b] = \alpha$ which implies that $\alpha \in \pi(P_D)$. Then $P_{D/C} \subseteq \pi(P_D)$. Therefore $\pi(P_D) = P_{D/C}$. Hence $\pi$ is an order epimorphism.#

<u>Definition 2.17</u>. Let D be a distributive ratio seminear-ring and C a C-set of D. A <u>compatible partial order on C</u> is a partial order $\leqslant$ on C such that

(i) for any $x, y, z \in C$, $x \leqslant y$ implies $xz \leqslant yz$ and $zx \leqslant zy$,

(ii) for any $x \in D$, $xP_C^*x^{-1} \subseteq P_C^*$ where $P_C^* = \{x \in C \;/\; x \geqslant 1\}$ and

(iii) $(x+1)^{-1}(x+y)$, $(1+x)^{-1}(y+x) \in P_C^*$ for all $x \in D$, $y \in P_C^*$.

Remark 2.18. (1) If D is a partially ordered distributive ratio seminear-ring and C is a C-set of D then the restriction of the partial order on D to C gives a compatible partial order on C

(2) Let D be a distributive ratio seminear-ring and C a ratio subseminear-ring of D which is also a C-set and let $\leqslant$ be a partial order on C. If $\leqslant$ is a compatible partial order on C as a C-set then $\leqslant$ is a partial order compatible with the ratio subseminear-ring structure of C.

Proof: (1) Obvious.

(2) Assume that $\leqslant$ is a compatible partial order on C as a C-set. Let $x,y,z \in C$ be such that $x \leqslant y$. By assumption, $xz \leqslant yz$ and $zx \leqslant zy$. Since $yx^{-1} \in P_C^*$, so $(zx^{-1}+1)^{-1}(zx^{-1}+yx^{-1}) \in P_C^*$ which implies that $zx^{-1}+1 \leqslant zx^{-1}+yx^{-1}$. Hence $z+x \leqslant z+y$. Similarly, $x+z \leqslant y+z$. Therefore $\leqslant$ is a partial order compatible with the ratio seminear-ring structure of C. #

Theorem 2.19. Let D be a distributive ratio seminear-ring and C a prime C-set of D. Assume that C has a compatible partial order $\overset{*}{\leqslant}$ and $D/C$ has a compatible partial order $\leqslant$. Then there exists a compatible partial order on D such that $\overset{*}{\leqslant}$ is the restriction of the partial order on D and the projection map $\pi$ is an order epimorphism.

Proof: Let $A = P_C^* \cup (\bigcup_{\alpha \in P_{D/C} \setminus \{C\}} \alpha)$. We shall show that A is an O-set of D. Let $a \in A \cap A^{-1}$. Then $a^{-1} \in A$. Claim that $a \in P_C^*$.

Suppose that $a \in \bigcup_{\alpha \in P_{D/C} \setminus \{C\}} \alpha$. Then $a \in \alpha$ for some $\alpha \in P_{D/C} \setminus \{C\}$. Thus $[a] = \alpha > [1]$. If $a^{-1} \in P_C^*$ then $a^{-1} \in C$, so $a = (a^{-1})^{-1} \in C$ which implies that $\alpha = [a] = [1]$, a contradiction. Hence $a^{-1} \in \bigcup_{\alpha \in P_{D/C} \setminus \{C\}} \alpha$. Then $a^{-1} \in \beta$ for some $\beta \in P_{D/C} \setminus \{C\}$. Thus $[a^{-1}] = \beta > [1]$. It follows that $[1] > [a^{-1}]^{-1} = [a]$, a contradiction. Therefore $a \in P_C^*$, so we have the claim. Then $a \in C$, so $a^{-1} \in C$. But $a^{-1} \in A$, so $a^{-1} \in P_C^*$. This implies that $a \in P_C^* \cap (P_C^*)^{-1}$, hence $a = 1$. Therefore $A \cap A^{-1} = \{1\}$.

To show that $A^2 \subseteq A$, let $a, b \in A$. If $a, b \in P_C^*$ then $ab \in P_C^*$, so we are done. Assume that $a \notin P_C^*$ or $b \notin P_C^*$.

Case 1: $a, b \in \bigcup_{\alpha \in P_{D/C} \setminus \{C\}} \alpha$. Then $a \in \alpha$ and $b \in \beta$ for some $\alpha$, $\beta \in P_{D/C} \setminus \{C\}$. Also, $[ab] = [a][b] = \alpha\beta > [1]$. Hence $[ab] \in P_{D/C} \setminus \{C\}$, so $ab \in \bigcup_{\alpha \in P_{D/C} \setminus \{C\}} \alpha$.

Case 2: $a \in P_C^*$ and $b \in \bigcup_{\alpha \in P_{D/C} \setminus \{C\}} \alpha$. Then $a \in C$ and $b \in \alpha$ for some $\alpha \in P_{D/C} \setminus \{C\}$. Thus $[ab] = [a][b] = [1][b] = [b] = \alpha \in P_{D/C} \setminus \{C\}$, and hence $ab \in \bigcup_{\alpha \in P_{D/C} \setminus \{C\}} \alpha$.

Therefore $A^2 \subseteq A$.

To show that $xAx^{-1} \subseteq A$ for all $x \in D$, let $x \in D$ and $a \in A$. If $a \in P_C^*$ then $xax^{-1} \in P_C^*$, so we are done. Assume that $a \in \bigcup_{\alpha \in P_{D/C} \setminus \{C\}} \alpha$.

Then $a \in \alpha$ for some $\alpha \in P_{D/C} \setminus \{C\}$. Thus $[xax^{-1}] = [x][a][x]^{-1} = [x]\alpha[x]^{-1} > [x][x]^{-1} = [1]$, so $[xax^{-1}] \in P_{D/C} \setminus \{C\}$. Hence $xax^{-1} \in \bigcup_{\alpha \in P_{D/C} \setminus \{C\}} \alpha$, so $xax^{-1} \in A$. Therefore $xAx^{-1} \subseteq A$.

Let $x \in D$ and $a \in A$. If $a \in P_C^*$ then $(x+1)^{-1}(x+a)$, $(1+x)^{-1}(a+x) \in P_C^*$, so we are done. Assume that $a \in \bigcup_{\alpha \in P_{D/C} \setminus \{C\}} \alpha$. Then $a \in \alpha$ for some $\alpha \in P_{D/C} \setminus \{C\}$. Thus $[a] = \alpha > [1]$, so we get that $[x]+[a] \geqslant [x]+[1]$. Hence $[(x+1)^{-1}(x+a)] = ([x]+[1])^{-1}([x]+[a]) \geqslant [1]$. Since $a \notin C$ and $C$ is a prime C-set of D, so $(x+1)^{-1}(x+a) \notin C$ which implies that $[(x+1)^{-1}(x+a)] > [1]$. Hence $(x+1)^{-1}(x+a) \in \bigcup_{\alpha \in P_{D/C} \setminus \{C\}} \alpha$, so we get that $(x+1)^{-1}(x+a) \in A$. Similarly, $(1+x)^{-1}(a+x) \in A$.

Therefore A is an O-set of D. By Theorem 2.11, there exists a compatible partial order $\leqslant'$ on D such that $A = P_D$. We shall show that $\leqslant'|_{C\times C} = \leqslant^*$. Let $x, y \in C$. Assume that $x \leqslant' y$. Then $yx^{-1} \in P_D$, so $yx^{-1} \in A$. Since $yx^{-1} \in C$, $yx^{-1} \in P_C^*$ which implies that $x \leqslant^* y$. Assume that $x \leqslant^* y$. Then $yx^{-1} \in P_C^*$, so $yx^{-1} \in A$. Hence $x \leqslant' y$. Therefore $\leqslant'|_{C\times C} = \leqslant^*$.

We shall show that $\pi(P_D) = P_{D/C}$. Let $x \in P_D$. If $x \in P_C^*$ then $\pi(x) = [x] = [1] \in P_{D/C}$. Assume that $x \in \bigcup_{\alpha \in P_{D/C} \setminus \{C\}} \alpha$. Then $x \in \alpha$ for some $\alpha \in P_{D/C} \setminus \{C\}$. Thus $\pi(x) = [x] = \alpha \in P_{D/C}$. Hence $\pi(P_D) \subseteq P_{D/C}$.

Let $\alpha \in P_{D/C}$. If $\alpha = [1]$ then $\pi(1) = [1] = \alpha \in \pi(P_D)$. Assume that

$\alpha > [1]$. Choose $a \in \alpha$. Then $a \in \bigcup_{\alpha \in P_{D/C} \setminus \{C\}} \alpha$, so we have that $a \in P_D$.

It follows that $\pi(a) = [a] = \alpha \in \pi(P_D)$. Thus $P_{D/C} \subseteq \pi(P_D)$. Hence

$\pi(P_D) = P_{D_C}$. Therefore $\pi$ is an order epimorphism. #

From now on, for a partially ordered distributive ratio seminear-ring $(D, \leqslant)$ and a convex C-set C of D, the partial order on $D/C$ will mean the partial order $\leqslant^*$ which is defined by $\alpha \leqslant^* \beta$ if and only if there exist $a \in \alpha$ and $b \in \beta$ such that $a \leqslant b$.

<u>Theorem 2.20</u>. A C-set C of a partially ordered distributive ratio seminear-ring D is the kernel of an order homomorphism if and only if it is convex.

<u>Proof</u>: By Proposition 2.15(2), the kernel of an order homomorphism is convex. Conversely, if C is convex then the projection map $\pi: D \to D/C$ is an order homomorphism by Theorem 2.16 and we have that C is the kernel of $\pi$. #

<u>Theorem 2.21</u> (First Isomorphism Theorem). Let $(D, \leqslant)$ and $(D', \leqslant')$ be partially ordered distributive ratio seminear-rings and $f: D \to D'$ an order epimorphism. Then $D/_{\ker f} \simeq D'$. Furthermore, there exists an order isomorphism between the set of all ratio subseminear-rings

of D containing ker f and the set of all ratio subseminear-rings of $D'$ and there exists an order isomorphism between the set of all C-sets of D containing ker f and the set of all C-sets of $D'$.

<u>Proof</u>: By Proposition 2.15(2), ker f is a convex C-set of D, so $D/_{\ker f}$ has a compatible partial order $\leqslant^*$. Define $\psi: D/_{\ker f} \to D'$ as follows: Let $\alpha \in D/_{\ker f}$. Choose $x \in \alpha$. Define $\psi(\alpha) = f(x)$. Then $\psi$ is well-defined, bijective and a homomorphism.

To show that $\psi$ is isotone, let $\alpha, \beta \in D/_{\ker f}$ be such that $\alpha \leqslant^* \beta$. Then there exist $a \in \alpha$ and $b \in \beta$ such that $a \leqslant b$. Since f is an order epimorphism, f is isotone. Thus $\psi(\alpha) = f(a) \leqslant' f(b) = \psi(\beta)$. Hence $\psi$ is isotone.

To show that $P_{D'} \subseteq \psi(P_{D/_{\ker f}})$, let $y \in P_{D'}$. Since $f(P_D) = P_{D'}$, $y = f(x)$ for some $x \in P_D$. Then $[x] \in P_{D/_{\ker f}}$, so we get that $\psi([x]) = f(x) = y \in \psi(P_{D/_{\ker f}})$. Hence $P_{D'} \subseteq \psi(P_{D/_{\ker f}})$. Thus $\psi^{-1}(P_{D'}) \subseteq P_{D/_{\ker f}}$. By Proposition 2.15(1), $\psi^{-1}$ is isotone. Therefore $\psi$ is an order isomorphism.

Let $\mathcal{D} = \{H \subseteq D \mid H$ is a ratio subseminear-ring of D containing ker f$\}$ and $\mathcal{D}' = \{L \subseteq D' \mid L$ is a ratio subseminear-ring of $D'\}$. Since f is a homomorphism, $f(H) \in \mathcal{D}'$ for all $H \in \mathcal{D}$. Define $\Phi_1 : \mathcal{D} \to \mathcal{D}'$ by $\Phi_1(H) = f(H)$ for all $H \in \mathcal{D}$. Since $1' \in L$ for all $L \in \mathcal{D}'$, $f^{-1}(L) \in \mathcal{D}$ for all $L \in \mathcal{D}'$. Define $\Phi_2 : \mathcal{D}' \to \mathcal{D}$ by $\Phi_2(L) = f^{-1}(L)$ for all $L \in \mathcal{D}'$. Since f is onto, $\Phi_1 \cdot \Phi_2(L) = \Phi_1(f^{-1}(L)) = f(f^{-1}(L)) = L = I_{\mathcal{D}'}(L)$ for all $L \in \mathcal{D}'$. Hence $\Phi_1 \cdot \Phi_2 = I_{\mathcal{D}'}$

We shall show that $f^{-1}(f(H)) = H$ for all $H \in \mathcal{D}$. Let $H \in \mathcal{D}$. It is clear that $H \subseteq f^{-1}(f(H))$. Let $x \in f^{-1}(f(H))$. Then $f(x) \in f(H)$, so $f(x) = f(h)$ for some $h \in H$. It follows that $xh^{-1} \in \ker f$. But $\ker f \subseteq H$, so $x \in H$. Hence $f^{-1}(f(H)) \subseteq H$. Therefore $f^{-1}(f(H)) = H$.

Then we have that $\Phi_2 \circ \Phi_1(H) = f^{-1}(f(H)) = H = I_{\mathcal{D}}(H)$ for all $H \in \mathcal{D}$. Hence $\Phi_2 \circ \Phi_1 = I_{\mathcal{D}}$. Therefore $\Phi_1$ is bijective and $\Phi_1^{-1} = \Phi_2$. For each $H_1, H_2 \in \mathcal{D}$, if $H_1 \subseteq H_2$ then $\Phi_1(H_1) = f(H_1) \subseteq f(H_2) = \Phi_1(H_2)$ and for each $L_1, L_2 \in \mathcal{D}'$, if $L_1 \subseteq L_2$ then $\Phi_1^{-1}(L_1) = \Phi_2(L_1) = f^{-1}(L_1) \subseteq f^{-1}(L_2) = \Phi_2(L_2) = \Phi_1^{-1}(L_2)$. This implies that $\Phi_1$ and $\Phi_1^{-1}$ are isotone. Hence $\Phi_1$ is an order isomorphism.

Let $\mathcal{C} = \{C \subseteq D \mid C$ is a C-set of D containing $\ker f\}$ and $\mathcal{C}' = \{C' \subseteq D' \mid C'$ is a C-set of $D'\}$. Since f is onto, by Proposition 1.33(3), for any C-set C of D, f(C) is a C-set of D′. Define $\eta_1 : \mathcal{C} \to \mathcal{C}'$ by $\eta_1(C) = f(C)$ for all $C \in \mathcal{C}$. Since $1' \in C$ for all $C' \in \mathcal{C}'$, by Proposition 1.33(2), $f^{-1}(C') \in \mathcal{C}$ for all $C' \in \mathcal{C}'$. Define $\eta_2 : \mathcal{C}' \to \mathcal{C}$ by $\eta_2(C') = f^{-1}(C')$ for all $C' \in \mathcal{C}'$. Using the same proof as above, we get that $\eta_1$ is an order isomorphism.

Hence the theorem is proved. #

Remark 2.22. Let D be a partially ordered distributive ratio seminear-ring, H a ratio subseminear-ring of D and C a convex C-set of D. Then $H \cap C$ is a convex C-set of H and HC is a ratio subseminear-ring of D.

Proof: It is clear that $H \cap C$ is convex in $H$. Since $H$ is a ratio subseminear-ring of $D$ and $C$ is a multiplicative normal subgroup of $D$, $H \cap C$ is a multiplicative normal subgroup of $H$. Let $x \in H$ and $y \in H \cap C$. Then $(x+1)^{-1}(x+y) \in H$. Since $C$ is a C-set of $D$, $(x+1)^{-1}(x+y) \in C$. Thus $(x+1)^{-1}(x+y) \in H \cap C$. Similarly, $(1+x)^{-1}(y+x) \in H \cap C$. Hence $H \cap C$ is a C-set of $H$.

To show that $HC$ is a ratio subseminear-ring of $D$, let $x$, $y \in HC$. Then $x = h_1c_1$ and $y = h_2c_2$ for some $h_1, h_2 \in H$, $c_1, c_2 \in C$. Thus $xy^{-1} = (h_1c_1)(h_2c_2)^{-1} = h_1c_1c_2^{-1}h_2^{-1} = (h_1h_2^{-1})(h_2(c_1c_2^{-1})h_2^{-1}) \in HC$. Since $C$ is a C-set, $(h_2^{-1}h_1+1)^{-1}(h_2^{-1}h_1+c_2c_1^{-1}) \in C$. Hence we have that $x + y = h_1c_1 + h_2c_2 = h_2(h_2^{-1}h_1 + c_2c_1^{-1})c_1 =$ $[h_2(h_2^{-1}h_1+1)][(h_2^{-1}h_1+1)^{-1}(h_2^{-1}h_1+c_2c_1^{-1})c_1] \in HC$. Therefore $HC$ is a ratio subseminear-ring of $D$. #

Theorem 2.23 (Second Isomorphism Theorem). Let $(D,\leqslant)$ be a partially ordered distributive ratio seminear-ring, $H$ a ratio subseminear-ring of $D$ and $C$ a convex C-set of $D$ such that $P_{HC} = P_H$. Then $H/_{H\cap C} \simeq HC/_C$.

Proof: Define $f : H \to HC/_C$ by $f(x) = [x]$ for all $x \in H$. Then $f$ is onto and a homomorphism. It follows from the definition of the partial order $\leqslant^*$ on $HC/_C$ that for each $x \in H$, $x \geqslant 1$ implies $f(x) = [x] \ {}^*\!\!\geqslant [1]$, hence $f(P_H) \subseteq P_{HC/_C}$.

To show that $P_{HC/_C} \subseteq f(P_H)$, let $\alpha \in P_{HC/_C}$. By Theorem 2.16, the projection map $\pi : HC \to HC/_C$ is an order epimorphism, so

$\pi(P_{HC}) = P_{HC/C}$. Then $\alpha = \pi(x) = [x]$ for some $x \in P_{HC}$. Since $P_{HC} = P_H$, $x \in P_H$ which implies that $f(x) = [x] = \alpha \in f(P_H)$. Thus $P_{HC/C} \subseteq f(P_H)$.

Therefore $f(P_H) = P_{HC/C}$. Then f is an order epimorphism. By Theorem 2.21, $H/\ker f \cong HC/C$. But for each $x \in H$, $f(x) = [x] = [1]$ if and only if $x \in C$, so we have that $\ker f = H \cap C$.

Hence the theorem is proved. #

Remark 2.24 Let $(D,\leqslant)$ be a partially ordered distributive ratio seminear-ring, K a ratio subseminear-ring of D and H a subset of D such that H and K are convex C-sets of D and $H \subseteq K$. Then $K/H$ is a convex C-set of $D/H$.

Proof: To show that $K/H$ is convex in $D/H$, let $\alpha, \beta \in K/H$ and $\gamma \in D/H$ be such that $\alpha \leqslant^* \gamma \leqslant^* \beta$ where $\leqslant^*$ is a partial order on $D/H$. Then there exist $a \in \alpha$, $b, c \in \gamma$ and $d \in \beta$ such that $a \leqslant b$ and $c \leqslant d$. Then $a(b^{-1}c) \leqslant b(b^{-1}c) = c \leqslant d$. By definition of $\gamma$, $b^{-1}c \in H$. Since $H \subseteq K$ and $a \in K$, $ab^{-1}c \in K$. But K is convex, so $c \in K$. Hence $\gamma = [c] \in K/H$. Therefore $K/H$ is convex in $D/H$.

Since K is a multiplicative normal subgroup of D, $K/H$ is a multiplicative normal subgroup of $D/H$. Let $x \in D$ and $y \in K$. Since K is a C-set of D, $(x+1)^{-1}(x+y) \in K$. Hence $([x]+[1])^{-1}([x]+[y]) = [(x+1)^{-1}(x+y)] \in K/H$. Similarly, $([1]+[x])^{-1}([y]+[x]) \in K/H$. Therefore $K/H$ is a C-set of $D/H$. #

Theorem 2.25 (Third Isomorphism Theorem). Let $(D,\leqslant)$ be a partially ordered distributive ratio seminear-ring, K a ratio subseminear-ring of D and H a subset of D such that H and K are convex C-sets of D and $H \subseteq K$. Then $(D/H)/(K/H) \simeq D/K$.

Proof: Define $f: D/H \to D/K$ by $f(xH) = xK$ for all $x \in D$. Then f is well-defined, onto and a homomorphism. To show that f is isotone, let $\alpha, \beta \in D/H$ be such that $\alpha \leqslant^* \beta$ where $\leqslant^*$ is a partial order on $D/H$. Then there exist $a \in \alpha$ and $b \in \beta$ such that $a \leqslant b$. It follows from the definition of the partial order $\leqslant^{**}$ on $D/K$ that $f(\alpha) = f(aH) = aK \leqslant^{**} bK = f(bH) = f(\beta)$. Hence f is isotone. By Proposition 2.15(1), $f(P_{D/H}) \subseteq P_{D/K}$.

Next, to show that $P_{D/K} \subseteq f(P_{D/H})$, let $\alpha \in P_{D/K}$. Then $K \leqslant^{**} \alpha$, so there exist $a \in K$ and $b \in \alpha$ such that $a \leqslant b$. This implies that $aH \leqslant^* bH$, so $H \leqslant^* a^{-1}bH$. Thus $a^{-1}bH \in P_{D/H}$. Since $a \in K$, we get that $f(a^{-1}bH) = a^{-1}bK = a^{-1}KbK = bK = \alpha \in f(P_{D/H})$. Hence $P_{D/K} \subseteq f(P_{D/H})$.

Thus $f(P_{D/H}) = P_{D/K}$. Therefore f is an order epimorphism. By Theorem 2.21, $(D/H)/\ker f \simeq D/K$. But for each $x \in D$, $f(xH) = xK = K$ if and only if $x \in K$, so we have that $\ker f = K/H$.

Hence the theorem is proved. #

Theorem 2.26. Let $(D,\leqslant)$ and $(D',\leqslant')$ be partially ordered distributive ratio seminear-rings and $f: D \to D'$ an order epimorphism.

If $C'$ is a convex C-set of $D'$ then $D/_{f^{-1}(C')} \simeq D'/_{C'}$.

<u>Proof</u>: Assume that $C'$ is a convex C-set of $D'$. By Proposition 1.9 and 1.33(2), $f^{-1}(C')$ is a convex C-set of D. Define $g: D \to D'/_{C'}$ by $g(x) = [f(x)]$ for all $x \varepsilon D$. Since f is an order homomorphism of D onto $D'$, g is an order homomorphism of D onto $D'/_{C'}$. Then $g(P_D) \subseteq P_{D'/_{C'}}$ by Proposition 2.15(1).

To show that $P_{D'/_{C'}} \subseteq g(P_D)$, let $\alpha \varepsilon P_{D'/_{C'}}$. By Theorem 2.16, the projection map $\pi : D' \to D'/_{C'}$ is an order epimorphism, so $\pi(P_{D'}) = P_{D'/_{C'}}$. Then $\alpha = \pi(y) = [y]$ for some $y \varepsilon P_{D'}$. Since f is an order epimorphism, so $f(P_D) = P_{D'}$. Then $y = f(x)$ for some $x \varepsilon P_D$. Thus $g(x) = [f(x)] = [y] = \alpha \varepsilon g(P_D)$. Hence $P_{D'/_{C'}} \subseteq g(P_D)$.

Thus $g(P_D) = P_{D'/_{C'}}$. Therefore g is an order epimorphism. By Theorem 2.21, $D/_{\ker g} \simeq D'/_{C'}$. But for each $x \varepsilon D$, $g(x) = [f(x)] = [1']$ if and only if $f(x) \varepsilon C'$, so we have that $\ker g = f^{-1}(C')$.

Hence the theorem is proved. #

<u>Definition 2.27</u>. Let $\{(D_\alpha, \leqslant_\alpha)\}_{\alpha \varepsilon I}$ be a family of partially ordered distributive ratio seminear-rings. The <u>direct product</u> of the family $\{(D_\alpha, \leqslant_\alpha)\}_{\alpha \varepsilon I}$, denoted by $\prod_{\alpha \varepsilon I} D_\alpha$, is the set of all elements $(x_\alpha)_{\alpha \varepsilon I}$ in the Cartesian product of the family $\{(D_\alpha, \leqslant_\alpha)\}_{\alpha \varepsilon I}$ together with operations + and · and the partial order $\leqslant$ on $\prod_{\alpha \varepsilon I} D_\alpha$ which are defined by

$$(x_\alpha)_{\alpha \varepsilon I} + (y_\alpha)_{\alpha \varepsilon I} = (x_\alpha + y_\alpha)_{\alpha \varepsilon I} ,$$

$(x_\alpha)_{\alpha\varepsilon I} \cdot (y_\alpha)_{\alpha\varepsilon I} = (x_\alpha y_\alpha)_{\alpha\varepsilon I}$ and

$(x_\alpha)_{\alpha\varepsilon I} \leqslant (y_\alpha)_{\alpha\varepsilon I}$ if and only if $x_\alpha \leqslant_\alpha y_\alpha$ for all $\alpha \varepsilon I$.

Note that $(\prod_{\alpha\varepsilon I} D_\alpha, +, \cdot, \leqslant)$ is a partially ordered distributive ratio seminear-ring and $P_{\prod_{\alpha\varepsilon I} D_\alpha} = \prod_{\alpha\varepsilon I} P_{D_\alpha}$. So we see that given some examples of partially ordered distributive ratio seminear-rings we can construct new examples of partially ordered distributive ratio seminear-rings using the direct product.

<u>Proposition 2.28</u>. Let $\{(D_\alpha, \leqslant_\alpha)\}_{\alpha\varepsilon I}$ be a family of partially ordered distributive ratio seminear-rings. Then the following statements hold:

(1) $\prod_{\alpha\varepsilon I} D_\alpha$ is upper [lower] additive if and only if $D_\alpha$ is upper [lower] additive for all $\alpha \varepsilon I$.

(2) $\prod_{\alpha\varepsilon I} D_\alpha$ is left [right] increasing if and only if $D_\alpha$ is left [right] increasing for all $\alpha \varepsilon I$.

(3) $\prod_{\alpha\varepsilon I} D_\alpha$ is left [right] decreasing if and only if $D_\alpha$ is left [right] decreasing for all $\alpha \varepsilon I$.

(4) $\prod_{\alpha\varepsilon I} D_\alpha$ is directed if and only if $D_\alpha$ is directed for all $\alpha \varepsilon I$.

(5) $\prod_{\alpha\varepsilon I} D$ is a lattice if and only if $D_\alpha$ is a lattice for all $\alpha \varepsilon I$.

(6) $\prod_{\alpha\varepsilon I} D_\alpha$ is complete if and only if $D_\alpha$ is complete for all $\alpha \varepsilon I$.

(7) $\prod_{\alpha\varepsilon I} D_\alpha$ is totally ordered if and only if either $I = \{\alpha\}$ and $D_\alpha$ is totally ordered or there exists an $\alpha_o \varepsilon I$ such that $D_{\alpha_o}$ is totally ordered and $|D_\alpha| = 1$ for all $\alpha \varepsilon I \setminus \{\alpha_o\}$.

<u>Proof</u>: (1) It follows from the fact that $(1_\alpha)_{\alpha\varepsilon I} \leqslant (1_\alpha)_{\alpha\varepsilon I} + (1_\alpha)_{\alpha\varepsilon I}$ if and only if $1_\alpha \leqslant_\alpha 1_\alpha + 1_\alpha$ for all $\alpha \varepsilon I$ and Proposition 2.8(1).

(2) Assume that $(1_\alpha)_{\alpha\varepsilon I} + \prod_{\alpha\varepsilon I} D_\alpha \subseteq \prod_{\alpha\varepsilon I} P_{D_\alpha}$. To show that that $1_\alpha + D_\alpha \subseteq P_{D_\alpha}$ for all $\alpha \varepsilon I$, let $\alpha_o \varepsilon I$ and $x_{\alpha_o} \varepsilon D_{\alpha_o}$. Let $x_\alpha = 1_\alpha$ for all $\alpha \varepsilon I \setminus \{\alpha_o\}$. Then $(x_\alpha)_{\alpha\varepsilon I} \varepsilon \prod_{\alpha\varepsilon I} D_\alpha$. By assumption, $(1_\alpha + x_\alpha)_{\alpha\varepsilon I} = (1_\alpha)_{\alpha\varepsilon I} + (x_\alpha)_{\alpha\varepsilon I} \varepsilon \prod_{\alpha\varepsilon I} P_{D_\alpha}$, so $1_{\alpha_o} + x_{\alpha_o} \varepsilon P_{D_{\alpha_o}}$. Hence $1_{\alpha_o} + D_{\alpha_o} \subseteq P_{D_{\alpha_o}}$.

Conversely, assume that $1_\alpha + D_\alpha \subseteq P_{D_\alpha}$ for all $\alpha \varepsilon I$. Let $(x_\alpha)_{\alpha\varepsilon I} \varepsilon \prod_{\alpha\varepsilon I} D_\alpha$. Then $1_\alpha + x_\alpha \varepsilon P_{D_\alpha}$ for all $\alpha \varepsilon I$, so that $(1_\alpha)_{\alpha\varepsilon I} + (x_\alpha)_{\alpha\varepsilon I} = (1_\alpha + x_\alpha)_{\alpha\varepsilon I} \varepsilon \prod_{\alpha\varepsilon I} P_{D_\alpha}$. Hence $(1_\alpha)_{\alpha\varepsilon I} + \prod_{\alpha\varepsilon I} D_\alpha \subseteq \prod_{\alpha\varepsilon I} P_{D_\alpha}$.

This proves that $(1_\alpha)_{\alpha\varepsilon I} + \prod_{\alpha\varepsilon I} D_\alpha \subseteq \prod_{\alpha\varepsilon I} P_{D_\alpha}$ if and only if $1_\alpha + D_\alpha \subseteq P_{D_\alpha}$ for all $\alpha \varepsilon I$. By Proposition 2.8(2), $\prod_{\alpha\varepsilon I} D_\alpha$ is left increasing if and only if $D_\alpha$ is left increasing for all $\alpha \varepsilon I$.

(3) The proof is similar to the proof of (2) by using Proposition 2.8(3).

(4) Assume that $\prod_{\alpha\varepsilon I} D_\alpha$ is directed. To show that $D_\alpha$ is directed for all $\alpha \varepsilon I$, let $\alpha_o \varepsilon I$ and $x_{\alpha_o} \varepsilon D_{\alpha_o}$. Let $x_\alpha = 1_\alpha$ for all $\alpha \varepsilon I \setminus \{\alpha_o\}$. By assumption, $U((x_\alpha)_{\alpha\varepsilon I}, (1_\alpha)_{\alpha\varepsilon I})$ is nonempty. Let $(y_\alpha)_{\alpha\varepsilon I} \varepsilon U((x_\alpha)_{\alpha\varepsilon I}, (1_\alpha)_{\alpha\varepsilon I})$. Then $(x_\alpha)_{\alpha\varepsilon I} \leqslant (y_\alpha)_{\alpha\varepsilon I}$ and $(1_\alpha)_{\alpha\varepsilon I} \leqslant (y_\alpha)_{\alpha\varepsilon I}$, so $x_{\alpha_o} \leqslant_{\alpha_o} y_{\alpha_o}$ and $1_{\alpha_o} \leqslant_{\alpha_o} y_{\alpha_o}$. Hence $y_{\alpha_o} \varepsilon U(x_{\alpha_o}, 1_{\alpha_o})$. Therefore $U(x_{\alpha_o}, 1_{\alpha_o})$ is nonempty for all $x_{\alpha_o} \varepsilon D_{\alpha_o}$. By Proposition 1.16, $D_{\alpha_o}$ is directed.

Conversely, assume that $D_\alpha$ is directed for all $\alpha \varepsilon I$. Let $(x_\alpha)_{\alpha\varepsilon I} \varepsilon \prod_{\alpha\varepsilon I} D_\alpha$. Then $U(x_\alpha, 1_\alpha)$ is a nonempty subset of $D_\alpha$ for all $\alpha \varepsilon I$. For each $\alpha \varepsilon I$, let $y_\alpha \varepsilon U(x_\alpha, 1_\alpha)$. Thus for any $\alpha \varepsilon I$, $x_\alpha \leqslant_\alpha y_\alpha$ and $1_\alpha \leqslant_\alpha y_\alpha$, it follows that $(x_\alpha)_{\alpha\varepsilon I} \leqslant (y_\alpha)_{\alpha\varepsilon I}$ and $(1_\alpha)_{\alpha\varepsilon I} \leqslant (y_\alpha)_{\alpha\varepsilon I}$. Then $(y_\alpha)_{\alpha\varepsilon I} \varepsilon U((x_\alpha)_{\alpha\varepsilon I}, (1_\alpha)_{\alpha\varepsilon I})$. Hence $U((x_\alpha)_{\alpha\varepsilon I}, (1_\alpha)_{\alpha\varepsilon I})$ is nonempty for all $(x_\alpha)_{\alpha\varepsilon I} \varepsilon \prod_{\alpha\varepsilon I} D_\alpha$. By Proposition 1.16, $\prod_{\alpha\varepsilon I} D_\alpha$ is directed.

(5) Assume that $\prod_{\alpha\varepsilon I} D_\alpha$ is a lattice. To show that $D_\alpha$ is a lattice for all $\alpha \varepsilon I$, let $\alpha_o \varepsilon I$ and $x_{\alpha_o} \varepsilon D_{\alpha_o}$. Let $x_\alpha = 1_\alpha$ for all $\alpha \varepsilon I \setminus \{\alpha_o\}$. By assumption and Proposition 1.17(2), $\sup \{(x_\alpha)_{\alpha\varepsilon I}, (1_\alpha)_{\alpha\varepsilon I}\}$ exists, say $(y_\alpha)_{\alpha\varepsilon I}$. Then $x_{\alpha_o} \leqslant_{\alpha_o} y_{\alpha_o}$ and $1_{\alpha_o} \leqslant_{\alpha_o} y_{\alpha_o}$, so $y_{\alpha_o}$ is an upper bound of $x_{\alpha_o}$ and $1_{\alpha_o}$. Let $z_{\alpha_o} \varepsilon D_{\alpha_o}$ be

an upper bound of $x_{\alpha_o}$ and $1_{\alpha_o}$. Let $z_\alpha = 1_\alpha$ for all $\alpha \in I\setminus\{\alpha_o\}$. Then $(x_\alpha)_{\alpha\in I} \leqslant (z_\alpha)_{\alpha\in I}$ and $(1_\alpha)_{\alpha\in I} \leqslant (z_\alpha)_{\alpha\in I}$ which implies that $(y_\alpha)_{\alpha\in I} \leqslant (z_\alpha)_{\alpha\in I}$. Thus $y_{\alpha_o} \leqslant z_{\alpha_o}$, hence $y_{\alpha_o} = \sup\{x_{\alpha_o}, 1_{\alpha_o}\}$. Therefore $\sup\{x_{\alpha_o}, 1_{\alpha_o}\}$ exists for all $x_{\alpha_o} \in D_{\alpha_o}$. By Proposition 1.17(2), $D_{\alpha_o}$ is a lattice.

Conversely, assume that $D_\alpha$ is a lattice for all $\alpha \in I$. Let $(x_\alpha)_{\alpha\in I} \in \prod_{\alpha\in I} D_\alpha$. For each $\alpha \in I$, let $y_\alpha = \sup\{x_\alpha, 1_\alpha\}$. Then for any $\alpha \in I$, $x_\alpha \leqslant_\alpha y_\alpha$ and $1_\alpha \leqslant_\alpha y_\alpha$, so we get that $(x_\alpha)_{\alpha\in I} \leqslant (y_\alpha)_{\alpha\in I}$ and $(1_\alpha)_{\alpha\in I} \leqslant (y_\alpha)_{\alpha\in I}$. Thus $(y_\alpha)_{\alpha\in I}$ is an upper bound of $(x_\alpha)_{\alpha\in I}$ and $(1_\alpha)_{\alpha\in I}$. Let $(z_\alpha)_{\alpha\in I}$ be an upper bound of $(x_\alpha)_{\alpha\in I}$ and $(1_\alpha)_{\alpha\in I}$. Then for each $\alpha \in I$, $x_\alpha \leqslant_\alpha z_\alpha$ and $1_\alpha \leqslant_\alpha z_\alpha$, so we get that $y_\alpha \leqslant_\alpha z_\alpha$ for all $\alpha \in I$. Thus $(y_\alpha)_{\alpha\in I} \leqslant (z_\alpha)_{\alpha\in I}$. Hence $(y_\alpha)_{\alpha\in I} = \sup\{(x_\alpha)_{\alpha\in I}, (1_\alpha)_{\alpha\in I}\}$. By Proposition 1.17(2), $\prod_{\alpha\in I} D_\alpha$ is a lattice.

(6) Assume that every subset of $\prod_{\alpha\in I} P_{D_\alpha}$ has an infimum. To show that for each $\alpha \in I$, every subset of $P_{D_\alpha}$ has an infimum, let $\alpha_o \in I$ and let $A_{\alpha_o}$ be a subset of $P_{D_{\alpha_o}}$. Let $A_\alpha = \{1_\alpha\}$ for all $\alpha \in I\setminus\{\alpha_o\}$. Then $\prod_{\alpha\in I} A_\alpha \subseteq \prod_{\alpha\in I} P_{D_\alpha}$. By assumption, $\inf(\prod_{\alpha\in I} A_\alpha)$ exists, say $(x_\alpha)_{\alpha\in I}$. Thus $(x_\alpha)_{\alpha\in I} \leqslant (a_\alpha)_{\alpha\in I}$ for all $(a_\alpha)_{\alpha\in I} \in \prod_{\alpha\in I} A_\alpha$ which implies that $x_{\alpha_o} \leqslant_{\alpha_o} a_{\alpha_o}$ for all $a_{\alpha_o} \in A_{\alpha_o}$. Let $y_{\alpha_o}$ be a lower

bound of $A_{\alpha_o}$. Let $y_\alpha = 1_\alpha$ for all $\alpha \in I \setminus \{\alpha_o\}$. Then $(y_\alpha)_{\alpha\in I}$ is a lower bound of $\prod_{\alpha\in I} A_\alpha$, so $(y_\alpha)_{\alpha\in I} \leqslant (x_\alpha)_{\alpha\in I}$. Hence $y_{\alpha_o} \leqslant_{\alpha_o} x_{\alpha_o}$, so we get that $x_{\alpha_o} = \inf (A_{\alpha_o})$.

Conversely, assume that for each $\alpha \in I$, every subset of $P_{D_\alpha}$ has an infimum. Let A be a subset of $\prod_{\alpha\in I} P_{D_\alpha}$. Then $A = \prod_{\alpha\in I} A_\alpha$ where $A_\alpha \subseteq P_{D_\alpha}$ for all $\alpha \in I$. For each $\alpha \in I$, let $x_\alpha = \inf(A_\alpha)$. Let $(a_\alpha)_{\alpha\in I} \in A$. Then for each $\alpha \in I$, $x_\alpha \leqslant_\alpha a_\alpha$, so $(x_\alpha)_{\alpha\in I} \leqslant (a_\alpha)_{\alpha\in I}$. Hence $(x_\alpha)_{\alpha\in I} \leqslant (a_\alpha)_{\alpha\in I}$ for all $(a_\alpha)_{\alpha\in I} \in A$, so that $(x_\alpha)_{\alpha\in I}$ is a lower bound of A. Let $(y_\alpha)_{\alpha\in I}$ be a lower bound of A. Then we have that $y_\alpha$ is a lower bound of $A_\alpha$ for all $\alpha \in I$. Hence $y_\alpha \leqslant_\alpha x_\alpha$ for all $\alpha \in I$, so $(y_\alpha)_{\alpha\in I} \leqslant (x_\alpha)_{\alpha\in I}$. Therefore $(x_\alpha)_{\alpha\in I} = \inf (A)$. Also, every subset of $\prod_{\alpha\in I} P_{D_\alpha}$ has an infimum.

This proves that every subset of $\prod_{\alpha\in I} P_{D_\alpha}$ has an infimum if and only if for each $\alpha \in I$, every subset of $P_{D_\alpha}$ has an infimum. By Proposition 2.8(6), $\prod_{\alpha\in I} D_\alpha$ is complete if and only if $D_\alpha$ is complete for all $\alpha \in I$.

(7) Assume that $\prod_{\alpha\in I} D_\alpha$ is totally ordered. Suppose that $|I| > 1$. Claim that for each $\alpha \in I$, if $|D_\alpha| > 1$ then the partial order on $D_\alpha$ is a total order. Let $\alpha_o \in I$ be such that $|D_{\alpha_o}| > 1$. Let $x_{\alpha_o} \in D_{\alpha_o}$ and $x_\alpha = 1_\alpha$ for all $\alpha \in I \setminus \{\alpha_o\}$. Then

$(1_\alpha)_{\alpha\varepsilon I} \leqslant (x_\alpha)_{\alpha\varepsilon I}$ or $(x_\alpha)_{\alpha\varepsilon I} \leqslant (1_\alpha)_{\alpha\varepsilon I}$, so $1_{\alpha_o} \leqslant_{\alpha_o} x_{\alpha_o}$ or $x_{\alpha_o} \leqslant_{\alpha_o} 1_{\alpha_o}$. This implies that $D_{\alpha_o} \subseteq P_{D_{\alpha_o}} \cup P^{-1}_{D_{\alpha_o}}$. Hence $D_{\alpha_o} = P_{D_{\alpha_o}} \cup P^{-1}_{D_{\alpha_o}}$. By Proposition 2.8(7), $D_{\alpha_o}$ is totally ordered.

Hence we have the claim.

If $|D_\alpha| = 1$ for all $\alpha \varepsilon I$ then $D_\alpha = \{1_\alpha\}$ for all $\alpha \varepsilon I$. Assume that there exists an $\alpha_o \varepsilon I$ such that $|D_{\alpha_o}| > 1$. By the claim, $D_{\alpha_o}$ is totally ordered. This implies that $P_{D_{\alpha_o}} \neq \{1_{\alpha_o}\}$.

Next, we shall show that $|D_\alpha| = 1$ for all $\alpha \varepsilon I \setminus \{\alpha_o\}$. Suppose not. Then there exists a $\beta \varepsilon I \setminus \{\alpha_o\}$ such that $|D_\beta| > 1$. By the claim, $D_\beta$ is totally ordered which implies that $P_{D_\beta} \neq \{1_\beta\}$. Let $x_{\alpha_o} \varepsilon P_{D_{\alpha_o}} \setminus \{1_{\alpha_o}\}$ and $y_\beta \varepsilon P_{D_\beta} \setminus \{1_\beta\}$. Let $x_\alpha = 1_\alpha$ for all $\alpha \varepsilon I \setminus \{\alpha_o\}$ and $y_\alpha = 1_\alpha$ for all $\alpha \varepsilon I \setminus \{\beta\}$. Thus $(x_\alpha)_{\alpha\varepsilon I} \leqslant (y_\alpha)_{\alpha\varepsilon I}$ or $(y_\alpha)_{\alpha\varepsilon I} \leqslant (x_\alpha)_{\alpha\varepsilon I}$, so we have that $x_{\alpha_o} \leqslant 1_{\alpha_o}$ or $y_\beta \leqslant 1_\beta$, a contradiction. Hence $|D_\alpha| = 1$ for all $\alpha \varepsilon I \setminus \{\alpha_o\}$.

The converse is obvious. #

Next, we shall characterize those distributive seminear-rings which can be the positive cones of a partially ordered distributive ratio seminear-ring

<u>Theorem 2.29</u>. Let P be a distributive seminear-ring with multiplicative identity 1. Then there exists a partially ordered distributive

ratio seminear-ring having P as its positive cone if and only if P satisfies the following properties:

(i) P is multiplicatively cancellative.

(ii) $Pa = aP$ for all $a \in P$.

(iii) For any $a, b \in P$, $ab = 1$ implies $a = b = 1$.

(iv) $a + cb \in P(a+b)$ and $cb + a \in P(b+a)$ for all $a,b,c \in P$.

Moreover, if P satisfies properties (i) - (iv) then there exist a partially ordered distributive ratio seminear-ring D and a monomorphism $i: P \to D$ such that

(1) $i(P)$ is the positive cone of D and

(2) if $D'$ is a partially ordered distributive ratio seminear-ring and $j: P \to D'$ is a monomorphism such that $j(P)$ is the positive cone of $D'$ then there exists a unique order monomorphism $f: D \to D'$ such that $f \circ i = j$, that is, D is the smallest partially ordered distributive ratio seminear-ring having P as its positive cone up to isomorphism.

Furthermore, D is directed and upper additive.

Proof: Since the positive cone of a partially ordered distributive ratio seminear-ring D has properties (i) - (iv), so if P is isomorphic to the positive cone of D then P also has properties (i) - (iv).

Conversely, assume that P satisfies properties (i) - (iv). By properties (i) and (ii) of P, we get that for any $a, x \in P$ there exists a unique $x_a \in P$ such that $xa = ax_a$. Using the same proof as in Theorem 1.21 we get that

$$(1) \quad (xy)_a = x_a y_a \quad \text{and}$$

$$(2) \quad (x_a)_b = x_{ab}$$

for all $a,b,x,y \in P$. From $a(x+y)_a = (x+y)a = xa + ya = ax_a + ay_a = a(x_a + y_a)$ for all $a,x,y \in P$, we have that

$$(3) \quad (x+y)_a = x_a + y_a$$

for all $a,x,y \in P$.

Define a ralation $\sim$ on $P \times P$ as follows: For $a,b,c,d \in P$, $(a,b) \sim (c,d)$ if and only if $ad_b = cb$. Using the same proof as in Theorem 1.21 we get that $\sim$ is an equivalence relation. Let $D = \frac{P \times P}{\sim}$. Define operations + and $\cdot$ on D by

$$[(a,b)]\cdot[(c,d)] = [(ac_b, db)] \quad \text{and}$$

$$[(a,b)]+[(c,d)] = [(ad + cb_d, bd)]$$

for all $a,b,c,d \in P$. Using the same proof as in Theorem 1.21 we get that $\cdot$ is well-defined and $(D,\cdot)$ is a group with $[(1,1)]$ as the identity and $[(b,a)]$ as the inverse of $[(a,b)]$ for all $a,b \in P$.

Now, we shall show that + is well-defined. Let $v,w,x,y \in P$ be such that $(v,w) \in [(a,b)]$ and $(x,y) \in [(c,d)]$. Then $(a,b) \sim (v,w)$ and $(c,d) \sim (x,y)$, so $aw_b = vb$ and $cy_d = xd$ .................(*).

From

$$\begin{aligned} (ad+cb_d)(wy)_{bd} &= ad(wy)_{bd} + cb_d(wy)_{bd} \\ &= ad((wy)_b)_d + c(b(wy)_b)_d \quad \text{(by (1) and (2))} \\ &= a(wy)_b d + c((wy)b)_d \\ &= aw_b y_b d + c((yw_y)b)_d \quad \text{(by (1))} \\ &= vby_b d + cy_d(w_y b)_d \quad \text{(by (*) and (1))} \end{aligned}$$

$$= vybd + xd(w_y b)_d \qquad \text{(by (*))}$$

$$= vybd + xw_y bd$$

$$= (vy + xw_y)bd,$$

we have that $(ad+cb_d, bd) \sim (vy+xw_y, wy)$. It follows that

$[(a,b)]+[(c,d)] = [(ad+cb_d, bd)] = [(vy+xw_y, wy)] = [(v,w)]+[(x,y)]$.

Hence + is well - defined.

To show that + is associative, let a,b,c,d,x,y ε P. Then

$$(cb_d)y + x(db_d)_y = c(b_d y) + x(d_y(b_d)_y) \qquad \text{(by (1))}$$

$$= c(y(b_d)_y)+(xd_y)b_{dy} \qquad \text{(by (2))}$$

$$= (cy)b_{dy}+ (xd_y)b_{dy} \qquad \text{(by (2))}$$

$$= (cy+xd_y)b_{dy} \qquad \text{.......(**)}.$$

Hence

$$([(a,b)]+[(c,d)])+[(x,y)] = [(ad+cb_d, bd)]+[(x,y)]$$

$$= [((ad+cb_d)y + x(bd)_y, (bd)y)]$$

$$= [((ad)y + (cb_d)y + x(db_d)_y, b(dy))]$$

$$= [(a(dy)+(cy+xd_y)b_{dy}, b(dy))] \qquad \text{(by (**))}$$

$$= [(a,b)]+[(cy+xd_y, dy)]$$

$$= [(a,b)]+([(c,d)]+[(x,y)]).$$

Therefore + is associative.

To show that • is distributive over + in D, let a,b,c,d, x,y ε P. Then

$$(\mathrm{I})\begin{cases}(ad)x_{bd} = a(d(x_b)_d) = a(x_b d) = (ax_b)d, \\ (cb_d)x_{bd} = c(b_d(x_b)_d) = c(bx_b)_d = c(xb)_d = cx_d b_d, \\ ((yb)(dy_d))_{ybd} = ((ybd)y_d)_{ybd} \\ \qquad = (ybd)_{ybd}(y_d)_{ybd} \quad \text{(by (1))} \\ \qquad = ybd(y_d)_{ybd} \\ \qquad = y_d(ybd) \quad \text{and}\end{cases}$$

$$(\mathrm{II}) \quad b_d y_d = (by)_d = (yb_y)_d = (y_y b_y)_d = (yb)_{yd}.$$

Hence

$$((ad+cb_d)x_{bd})((yb)(yd))_{y(bd)}$$

$$= ((ad)x_{bd} + (cb_d)x_{bd})((yb)(dy_d))_{ybd}$$

$$= ((ax_b)d + (cx_d)b_d)y_d(ybd) \quad \text{(by I)}$$

$$= ((ax_b)dy_d + (cx_d)b_d y_d)y(bd)$$

$$= ((ax_b)(yd) + (cx_d)(yb)_{yd})y(bd) \quad \text{(by II)}.$$

It follows that

$$((ad+cb_d)x_{bd}, y(bd)) \sim ((ax_b)(yd)+(cx_d)(yb)_{yd}, (yb)(yd)) \quad \text{.......(III)}.$$

Therefore

$$([(a,b)]+[(c,d)])[(x,y)] = [(ad+cb_d, bd)][(x,y)]$$

$$= [((ad+cb_d)x_{bd}, y(bd))]$$

$$= [((ax_b)(yd)+(cx_d)(yb)_{yd}, (yb)(yd))] \quad \text{(by (III))}$$

$$= [(ax_b, yb)]+[(cx_d, yd)]$$

$$= [(a,b)][(x,y)]+[(c,d)][(x,y)].$$

From

$$((by)(dy))_{(bd)y} = (by(dy))_{bdy}$$

$$= (bdy\ y_{dy})_{bdy}$$

$$= (bdy)_{bdy}(y_{dy})_{bdy}$$

$$= bdy(y_{dy})_{bdy}$$

$$= y_{dy}(bdy) \quad \ldots\ldots\ldots\ldots\ldots\ldots(IV)$$

and $d_y y_{dy} = (d_d)_y y_{dy} = d_{dy} y_{dy} = (dy)_{dy} = dy$ ...........(V),

we have that

$$x(ad+cb_d)_y((by)(dy))_{(bd)y}$$

$$= x((ad)_y + (cb_d)_y)y_{dy}(bdy) \quad \text{(by (3) and (IV))}$$

$$= (x(a_y d_y) + x(c_y(b_d)_y))y_{dy}(bd)y$$

$$= ((xa_y)d_y y_{dy} + (xc_y)b_{dy} y_{dy})(bd)y$$

$$= ((xa_y)dy + (xc_y)(by)_{dy})(bd)y \quad \text{(by (V) and (1))}.$$

It follows that

$$(x(ad+cb_d)_y,\ (bd)y) \sim ((xa_y)dy + (xc_y)(by)_{dy},\ (by)(dy)) \quad \ldots\ldots\ldots\ldots(VI).$$

Hence

$$[(x,y)]([(a,b)]+[(c,d)]) = [(x,y)][(ad+cb_d,\ bd)]$$

$$= [(x(ad+cb_d)_y,\ (bd)y)]$$

$$= [((xa_y)dy + (xc_y)(by)_{dy},\ (by)(dy))]$$

(by (VI))

$$= [(xa_y,\ by)]+[(xc_y,\ dy)]$$

$$= [(x,y)][(a,b)]+[(x,y)][(c,d)].$$

Therefore $(D,+,\cdot)$ is a distributive ratio seminear-ring.

Define $i: P \to D$ by $i(a) = [(a,1)]$ for all $a \in P$. Using the same proof as in Theorem 1.21 we get that $i$ is injective and $i(ab) = i(a)i(b)$ for all $a,b \in P$. For any $a,b \in P$, $i(a)+i(b) = [(a,1)]+[(b,1)] = [(a+b,1)] = i(a+b)$. Thus $i$ is a homomorphism. Therefore $i$ is a monomorphism.

Now, we shall show that $i(P)$ is an O-set of $D$. Using the same proof as in Theorem 1.21, we get that $i(P)\cap i(P)^{-1} = \{[(1,1)]\}$, $i(P)^2 \subseteq i(P)$ and $\alpha i(P)\alpha^{-1} \subseteq i(P)$ for all $\alpha \in D$. Let $a,b,c \in P$. By property (iv) of $P$, $a + cb \in P(a+b)$ and $cb + a \in P(b+a)$. Then $a + cb = x(a+b)$ and $cb + a = y(b+a)$ for some $x,y \in P$. Also, we have that $b(a+cb)_{a+b} = b(x(a+b))_{a+b} = bx_{a+b}(a+b)_{a+b} = bx_{a+b}(a+b)$ and $b(cb+a)_{b+a} = b(y(b+a))_{b+a} = by_{b+a}(b+a)_{b+a} = by_{b+a}(b+a)$. Since $bP = Pb$, $bx_{a+b} = ub$ and $by_{b+a} = vb$ for some $u,v \in P$. It follows that $b(a+cb)_{a+b} = ub(a+b)$ and $b(cb+a)_{b+a} = vb(b+a)$.

Hence

$$(b(a+cb)_{a+b},\ b(a+b)) \sim (u,1) \qquad \text{............(VII)}$$

and

$$(b(cb+a)_{b+a},\ b(b+a)) \sim (v,1) \qquad \text{............(VIII)}.$$

Thus

$$\begin{aligned}([(a,b)]+[(1,1)])^{-1}([(a,b)]+i(c)) &= [(a+b,b)]^{-1}([(a,b)]+[(c,1)]) \\ &= [(b,a+b)][(a+cb,b)] \\ &= [(b(a+cb)_{a+b},\ b(a+b))] \\ &= [(u,1)] \qquad \text{(by (VII))} \\ &= i(u)\end{aligned}$$

and

$$([(1,1)]+[(a,b)])^{-1}(i(c)+[(a,b)]) = [(b+a,b)]^{-1}([(c,1)]+[(a,b)])$$
$$= [(b,b+a)][(cb+a,b)]$$
$$= [(b(cb+a)_{b+a}, b(b+a))]$$
$$= [(v,1)] \qquad \text{(by (VIII))}$$
$$= i(v).$$

Hence $(\alpha+[(1,1)])^{-1}(\alpha+\beta)$, $([(1,1)]+\alpha)^{-1}(\beta+\alpha) \in i(P)$ for all $\alpha \in D$, $\beta \in i(P)$.

Therefore $i(P)$ is an O-set of D. By Theorem 2.11, there exists a unique compatible partial order on D such that $i(P)$ is the positive cone of D. Since $[(1,1)]+[(1,1)] = [(1+1,1)] = i(1+1) \in P_D$, by Proposition 2.8(1), D is upper additive. Since for any $a,b \in P$, $[(a,b)] = [(a,1)][(1,b)] = [(a,1)][(b,1)]^{-1} = i(a)i(b)^{-1}$, $i(P) = P_D$ generates $(D,\cdot)$. By Proposition 2.8(4), D is directed.

We shall now show that D is the smallest partially ordered distributive ratio seminear-ring having P as its positive cone up to isomorphism. Assume that $D'$ is a partially ordered distributive ratio seminear-ring and $j: P \to D'$ is a monomorphism such that $j(P) = P_{D'}$. Define $f: D \to D'$ by $f([(a,b)]) = j(a)j(b)^{-1}$ for all $a,b \in P$. Using the same prove as in Remark 1.22 we get that f is well-defined, injective, $f(P_D) = P_{f(D)}$ and $f(\alpha\beta) = f(\alpha)f(\beta)$ for all $\alpha,\beta \in D$. Let $a,b,c,d \in P$. Since $db_d = bd$, so $j(d)j(b_d) = j(b)j(d)$. Hence

$$j(b_d) = j(d)^{-1}j(b)j(d) \qquad \ldots\ldots\ldots\ldots\ldots(***).$$

Thus

$$
\begin{aligned}
f([(a,b)]+[(c,d)]) &= f([(ad+cb_d, bd)] \\
&= j(ad+cb_d)j(bd)^{-1} \\
&= (j(a)j(d)+j(c)j(b_d))j(d)^{-1}j(b)^{-1} \\
&= j(a)j(b)^{-1}+ j(c)j(d)^{-1} \quad \text{(by (***))} \\
&= f([(a,b)]) + f([(c,d)]).
\end{aligned}
$$

Hence f is a homomorphism. Therefore f is an order monomorphism. Using the same proof as in Remark 1.22 we get that f is the unique order monomorphism such that $f \circ i = j$. #

Let D be an ordered distributive ratio seminear-ring such that $1 + 1 = 1$. Then for any $x,y,z \in D$, if $x,y \in LI_D(1)$ and $x \leqslant z \leqslant y$ then $z \in LI_D(1)$. This statement is also true for $RI_D(1)$.

We shall now classify all complete ordered distributive ratio seminear-rings such that $1 + 1 = 1$. First, we shall need some lemmas.

<u>Lemma 2.30</u>. Let D be a complete ordered distributive ratio seminear-ring such that $1 + 1 = 1$. Assume that $LI_D(1)$ is a proper subset of D. Then the following statements hold:

(1) If $LI_D(1) \cap P_D \neq \{1\}$ then $LI_D(1) = P_D$.

(2) If $LI_D(1) \cap P_D^{-1} \neq \{1\}$ then $LI_D(1) = P_D^{-1}$.

If $RI_D(1)$ is a proper subset of D then the statements(1) and (2) are also true for $RI_D(1)$.

<u>Proof</u>: (1) Assume that $LI_D(1) \cap P_D \neq \{1\}$. Let $x \in LI_D(1)$ be such that $x > 1$. To show that $P_D \subseteq LI_D(1)$, let $y \in P_D$. Then $y \geqslant 1$. If $y \leqslant x$ then $y \in LI_D(1)$, so we are done. Assume that $x < y$. Since D is complete, by Proposition 1.14, $(D,\cdot)$ is Archimedean. Hence there exists an $n \in \mathbb{Z}$ such that $y < x^n$. Since $y > 1$, $n \neq 0$. If $n \in \mathbb{Z}^-$, it follows from $1 < x$ that $x^n < 1$, so $y < 1$, a contradiction. Hence $n \in \mathbb{Z}^+$. By Remark 1.29(2), $x^n \in LI_D(1)$. Since $1 \leqslant y < x^n$, $y \in LI_D(1)$. Therefore $P_D \subseteq LI_D(1)$. Suppose that $P_D \subset LI_D(1)$. Let $z \in LI_D(1) \setminus P_D$. To show that $P_D^{-1} \subseteq LI_D(1)$, let $w \in P_D^{-1}$. Then $w \leqslant 1$. If $w \geqslant z$ then $w \in LI_D(1)$, so we are done. Assume that $w < z$. Since $(D,\cdot)$ is Archimedean, there exists an $n \in \mathbb{Z}$ such that $z^n < w$. Since $w \leqslant 1$, $n \neq 0$. If $n \in \mathbb{Z}^-$, it follows from $z < 1$ that $1 < z^n$, so $1 < w$, a contradiction. Hence $n \in \mathbb{Z}^+$. By Remark 1.29 (2), $z^n \in LI_D(1)$. Since $z^n < w \leqslant 1$, $w \in LI_D(1)$. Hence $P_D^{-1} \subseteq LI_D(1)$. Thus $P_D \cup P_D^{-1} \subseteq LI_D(1)$. Since D is totally ordered, by Proposition 2.8(7), $D = P_D \cup P_D^{-1}$. This implies that $LI_D(1) = D$ which contradicts the hypothesis. Therefore $P_D = LI_D(1)$.

(2) The proof is similar to the proof of (1). #

<u>Lemma 2.31</u>. Let D be a complete ordered distributive ratio seminear-ring such that $1 + 1 = 1$ and $|D| > 1$. Then exactly one of the following statements hold:

(1) $x + y = \min\{x,y\}$ for all $x, y \in D$.

(2) $x + y = \max\{x,y\}$ for all $x, y \in D$.

(3) $x + y = x$ for all $x, y \in D$.

(4) $x + y = y$ for all $x, y \in D$.

Proof: Case 1: $LI_D(1) = \{1\}$. Let $x \in D$. Then $1 + (1+x) = (1+1) + x = 1 + x$, so $1 \in LI_D(1+x)$. By Remark 1.29(1), $LI_D(1+x) = (1+x)LI_D(1) = \{1+x\}$, so $1 = 1 + x$. Thus $x \in RI_D(1)$. Hence $D \subseteq RI_D(1)$. Therefore $RI_D(1) = D$. Let $x,y \in D$. Then $yx^{-1} \in RI_D(1)$, so $1 + yx^{-1} = 1$ which implies that $x + y = x$. Hence $x + y = x$ for all $x,y \in D$.

Case 2: $LI_D(1) = D$. Let $x,y \in D$. Then $xy^{-1} + 1 = 1$, so $x + y = y$. Hence $x + y = y$ for all $x,y \in D$.

Case 3: $\{1\} \subset LI_D(1) \subset D$. If $RI_D(1) = \{1\}$ then $LI_D(1) = D$ by using a proof similar to the proof of Case 1 which is a contradiction. If $RI_D(1) = D$ then for each $x \in D$, $1 + x^{-1} = 1$, so $x + 1 = x$ for all $x \in D$ which implies that $LI_D(1) = \{1\}$, a contradiction. Hence $\{1\} \subset RI_D(1) \subset D$. Let $x \in LI_D(1) \setminus \{1\}$.

Subcase 3.1: $x > 1$. Then $x \in LI_D(1) \cap P_D$. By Lemma 2.30(1), $LI_D(1) = P_D$. Let $y \in RI_D(1) \setminus \{1\}$. We shall show that $y > 1$. Suppose that $y < 1$. Then $y \in RI_D(1) \cap P_D^{-1}$. By Lemma 2.30, $RI_D(1) = P_D^{-1}$. Let $a,b \in D$ be such that $a < b$. Then $ba^{-1} \in P_D$, so $ba^{-1} \in LI_D(1)$. Hence $ba^{-1} + 1 = 1$, so $b + a = a$. But $ab^{-1} \in P_D^{-1}$, so $ab^{-1} \in RI_D(1)$. Thus $1 + ab^{-1} = 1$, so $b + a = b$. This is a contradiction since $a \neq b$. Therefore $y > 1$. Then we have that

$y \in RI_D(1) \cap P_D$. By Lemma 2.30, $RI_D(1) = P_D$. Hence $LI_D(1) = RI_D(1) = P_D$.

Let $x,y \in D$. Without loss of generality, assume that $x \leqslant y$. Then $yx^{-1} \in P_D$, so $yx^{-1} + 1 = 1 + yx^{-1} = 1$. Thus $y + x = x + y = x = \min\{x,y\}$. Therefore $x + y = \min\{x,y\}$ for all $x,y \in D$.

<u>Subcase 3.2</u>: $x < 1$. This proof is similar to the proof of Subcase 3.1 and shows that $x + y = \max\{x,y\}$ for all $x,y \in D$. #

<u>Theorem 2.32</u>. Let $(D,+,\cdot,\leqslant)$ be a complete ordered distributive ratio seminear-ring such that $1 + 1 = 1$. Then $(D,+,\cdot,\leqslant)$ is order isomorphic to exactly one of the following:

(1) $(\{1\},+,\cdot,\leqslant)$.

(2) $(\mathbb{R}^+,\min,\cdot,\leqslant)$.

(3) $(\mathbb{R}^+,\max,\cdot,\leqslant)$.

(4) $(\mathbb{R}^+,+_\ell,\cdot,\leqslant)$ where $x +_\ell y = x$.

(5) $(\mathbb{R}^+,+_r,\cdot,\leqslant)$ where $x +_r y = y$.

(6) $(\{2^n \mid n \in \mathbb{Z}\}, \min,\cdot,\leqslant)$.

(7) $(\{2^n \mid n \in \mathbb{Z}\}, \max,\cdot,\leqslant)$.

(8) $(\{2^n \mid n \in \mathbb{Z}\}, +_\ell,\cdot,\leqslant)$.

(9) $(\{2^n \mid n \in \mathbb{Z}\}, +_r,\cdot,\leqslant)$.

<u>Proof</u>: If $|D| = 1$ then $D$ is order isomorphic to (1). Assume that $|D| > 1$. Since $(D,\cdot,\leqslant)$ is a complete totally ordered group, by Theorem 1.15, $(D,\cdot,\leqslant)$ ia order isomorphic to either $(\mathbb{R}^+,\cdot,\leqslant)$ or $(\{2^n \mid n \in \mathbb{Z}\},\cdot,\leqslant)$.

Case 1: $(D,\cdot,\leqslant) \simeq (\mathbb{R}^+,\cdot,\leqslant)$. Then by Lemma 2.31, $(D,+,\cdot,\leqslant)$ is order isomorphic to either (2), (3), (4) or (5).

Case 2: $(D,\cdot,\leqslant) \simeq (\{2^n \mid n \in \mathbb{Z}\},\cdot,\leqslant)$. Then by Lemma 2.31, $(D,+,\cdot,\leqslant)$ is order isomorphic to either (6), (7), (8) or (9).

Finally, we shall show that (1) to (9) are not order isomorphic to each other. Clearly, (1) is not order isomorphic to any of the others and $\mathbb{R}^+$ is not isomorphic to $\{2^n \mid n \in \mathbb{Z}\}$. Since (4) and (5) are not additively commutative, so (4) and (5) are not order isomorphic to (2) and (3).

To show that (2) is not order isomorphic to (3), suppose not. Let $f: (\mathbb{R}^+,\min,\cdot,\leqslant) \to (\mathbb{R}^+,\max,\cdot,\leqslant)$ be an order isomorphism. Since $1 < 2$, so $f(1) < f(2)$, hence $f(2) = f(1) + f(2) = f(1+2) = f(1)$, a contradiction. Therefore (2) is not order isomorphic to (3).

To show that (4) is not order isomorphic to (5), suppose not. Let $f: (\mathbb{R}^+,+_\ell,\cdot,\leqslant) \to (\mathbb{R}^+,+_r,\cdot,\leqslant)$ be an order isomorphism Since $f(1) = f(2) +_r f(1) = f(2+_\ell 1) = f(2)$, a contradiction. Hence (4) is not order isomorphic to (5).

Similarly, (6), (7), (8) and (9) are not order isomorphic to each other. #

Let D be a distributive ratio seminear-ring. For each $n \in \mathbb{Z}^+$, we shall denote $1 + 1 + \ldots + 1$ (n times) by n.

<u>Definition 2.33</u>. Let D be an ordered distributive ratio seminear-ring such that $1 + 1 \neq 1$. D is called <u>Archimedean</u> if for any $x,y \in D$, $x < y$ implies that either

a) there exists an $n \in \mathbb{Z}^+$ such that $y < nx$ or

b) there exists an $n \in Z^{+}_{-}$ such that $ny < x$.

Remark 2.34. ([3]). Let D be an ordered distributive ratio seminear-ring and P the prime distributive ratio seminear-ring of D. Then

(i) a) in Definition 2.33 holds if P is order isomorphic to $(Q^{+},+,\cdot,\leqslant)$.

(ii) b) in Definition 2.33 holds if P is order isomorphic to $(Q^{+},+,\cdot,\leqslant_{opp})$.

Let D be an ordered distributive ratio seminear-ring, P the prime distributive ratio seminear-ring of D and $x \in D$. Then we shall use the following notations: $A_x = \{y \in P \mid y < x\}$ and $B_x = \{y \in P \mid x < y\}$.

We need the following lemmas to classify all complete ordered distributive ratio seminear-ring which has property that $1 + 1 \neq 1$. The first, second and third lemmas have been proven in [3], pages 33 - 35 and 37.

Lemma 2.35 ([3]). If D is a complete ordered distributive ratio seminear-ring such that $1 + 1 \neq 1$ then D is Archimedian.

Lemma 2.36 ([3]). Let D be a complete ordered distributive ratio seminear-ring such that the prime distributive ratio seminear-ring of D is order isomorphic to $(Q^{+},+,\cdot,\leqslant)$. Then the following statements hold:

(1) $1 = \inf \{1+n^{-1} \mid n \in Z^{+}\}$

(2) For any $x,y \in D$, $x < y$ implies that $nx + 1 < ny$ for some $n \in \mathbb{Z}^+$.

(3) For any $x \in D$ there exists an $n_o \in \mathbb{Z}^+$ such that for each $n \in \mathbb{Z}^+$, $n_o \leqslant n$ implies $n^{-1} < x$.

Lemma 2.37 ([3]). Let D be a complete ordered distributive ratio seminear-ring and P the prime distributive ratio seminear-ring of D which is order isomorphic to $(\mathbb{Q}^+,+,\cdot,\leqslant)$. Then the following statements hold:

(1) $\sup A_x = \inf B_x = x$ and $A_{x+y} = A_x + A_y$ for all $x,y \in D$.

(2) If $f: D \to D$ is isotone such that $f(x) = x$ for all $x \in P$ then f is the identity map of D.

Lemma 2.38. Let D be a complete ordered distributive ratio seminear-ring such that P, the prime distributive ratio seminear-ring of D, is order isomorphic to $(\mathbb{Q}^+,+,\cdot,\leqslant)$. Then P is the strongly dense in D.

Proof: Let $x,y \in D$ be such that $x < y$. By Lemma 2.36(2), $m_ox + 1 < m_oy$ for some $m_o \in \mathbb{Z}^+$. Claim that for any $k \in \mathbb{Z}^+$, $m_o \leqslant k$ implies that $kx + 1 < ky$. Let $k \in \mathbb{Z}^+$ be such that $m_o \leqslant k$. If $k = m_o$ then we are done. Assume that $m_o < k$. Then $k = \ell + m_o$ for some $\ell \in \mathbb{Z}^+$. Since $x < y$, so $kx + 1 = (\ell+m_o)x + 1 = \ell x+(m_ox+1) \leqslant \ell y + m_oy = (\ell+m_o)y = ky$. Suppose that $kx + 1 = ky$. Then $x + k^{-1} = y$. Since $m_o < k$, $k^{-1} < m_o^{-1}$. It follows that $y = x + k^{-1} \leqslant x + m_o^{-1}$, and hence $m_oy \leqslant m_ox + 1$, a contradiction. Therefore $kx + 1 < ky$. Hence we have the claim. By Lemma 2.36(3), there exists an $n_o \in \mathbb{Z}^+$ such

that $n_o^{-1} < x$, so $1 < n_o x$. Let $\ell \in \mathbb{Z}^+$ be such that $m_o < \ell n_o$. By the claim, $(\ell n_o)x + 1 < (\ell n_o)y$ ..............(*).
Since D is complete, by Lemma 2.35, D is Archimedian. Since $1 \leqslant \ell < \ell(n_o x)$, so there exists an $r \in \mathbb{Z}^+$ such that $(\ell n_o)x < r \cdot 1 = r$. Let $r_o = \min \{r \in \mathbb{Z}^+ \mid (\ell n_o)x < r\}$. Then $r_o - 1 \leqslant (\ell n_o)x < r_o$. From (*), we have that $r_o \leqslant (\ell n_o)x + 1 < (\ell n_o)y$. Thus $(\ell n_o)x < r_o < (\ell n_o)y$, so $x < (\ell n_o)^{-1} r_o < y$. Hence P is strongly dense in D. #

<u>Theorem 2.39</u> ([3]). Let $(D,+,\cdot,\leqslant)$ be a complete ordered distributive ratio seminear-ring such that $1 + 1 \neq 1$. Then $(D,+,\cdot,\leqslant)$ is either order isomorphic to $(\mathbb{R}^+,+,\cdot,\leqslant)$ or $(\mathbb{R}^+,+,\cdot,\leqslant_{opp})$.