CHAPTER II

DIVISION SEMINEAR-RINGS

Definition 2.1. A division seminear-ring is a seminear-ring $(D,+,.)$ such that $(D,.)$ is a group. If D is also left distributive then D is called a division semiring.

Definition 2.2. Let $(S,+,.)$ be a seminear-ring and $T \subseteq S$. T is said to be a subseminear-ring of S iff $(T,+,.)$ is a seminear-ring. A subseminear-ring of a seminear-ring is said to be a division subseminear-ring iff it is a division seminear-ring.

Remark. A near-field is never a division seminear-ring since the additive identity has no multiplicative inverse.

Example 2.3.

(1) Q^+ and R^+ with the usual addition and multiplication are division seminear-rings. Also, if we define $x + y = \min\{x, y\}$ or $x + y = \max\{x, y\}$ and use the usual multiplication we get division seminear-rings.

(2) Let $D = \{1\}$. Then D is a division seminear-ring.

(3) Let $(D,.)$ be a group. Define + on D by $x + y = y$ for all x, y in D. Then $(D,+,.)$ is a division seminear-ring which is in fact a division semiring.

(4) Let $(D,.)$ be a group. Define + on D by $x + y = x$ for all x, y in D. Then $(D,+,.)$ is a division seminear-ring which is in fact a division semiring.

(5) If D_1 and D_2 are division seminear-rings then $D_1 \times D_2$

with the usual product structure is a division seminear-ring.

Note that Examples (1), (2), (3) and (4) are division semirings. We shall now prove a theorem which gives us a way to construct a whole family of examples of division seminear-rings which are not division semirings.

<u>Theorem 2.4</u>. Let $(D,\cdot)$ be a group such that $D = D_1 * D_2$ for some subgroups D_1, D_2 of D. Then there exists a unique binary operation $+$ on D such that

(1) $(D,+,\cdot)$ is a division seminear-ring,

(2) $x + x' = x$ for all x, x' in D_1,

(3) $y + y' = y'$ for all y, y' in D_2,

(4) $D_2 + D_1 = \{1\}$,

(5) $D = D_1 + D_2$.

(Therefore, from (2) and (3) we see that D_1 and D_2 are division subseminear-rings of D.). Furthermore,

(6) $(D,+) \cong (D_1,+) \times (D_2,+)$,

(7) $(D,+)$ is a rectangular band and

(8) The left distributive law holds iff $D = D_1 \times D_2$ as multiplicative groups (i.e. $D_1, D_2 \trianglelefteq D$).

<u>Proof</u>. To define $+$ on D, let $d, d' \in D$. By Lemma 1.13(2), there exist unique d_1, d_1' in D_1, d_2, d_2' in D_2 such that $d = d_2d_1$ and $d' = d_1'd_2'$. By Lemma 1.13(3), there exist unique $\bar{x}$ in D_1, $\bar{y}$ in D_2 such that $\bar{x}d_2' = \bar{y}d_1$. Define $d + d' = \bar{x}d_2' = \bar{y}d_1$.

For d_1 in D_1 and d_2 in D_2, $d_1 = 1d_1$ and $d_2 = 1d_2$. By Lemma 1.13(3), there exist unique $\bar{x}$ in D_1, $\bar{y}$ in D_2 such that $\bar{x}d_2 = \bar{y}d_1$. Thus $d_1 + d_2 = \bar{x}d_2 = \bar{y}d_1$.

Claim that for all d in D there exist unique d_1 in D_1, d_2 in D_2 such that $d = d_1 + d_2$. To prove existence, let $d \in D$. By Lemma 1.13(2), there exist unique c_1, d_1 in D_1, c_2, d_2 in D_2 such that $d = c_2d_1 = c_1d_2$. Thus $d = d_1 + d_2$. Hence (5) holds. To prove uniqueness, suppose that $x_1 \in D_1$, $y_1 \in D_2$ are such that $d = x_1 + y_1$. Then there exist unique $\bar{x} \in D_1$, $\bar{y} \in D_2$ such that $d = x_1 + y_1 = \bar{x}y_1 = \bar{y}x_1$. Thus $c_2d_1 = \bar{y}x_1 = \bar{x}y_1 = c_1d_2$, so by Lemma 1.13(2), $x_1 = d_1$ and $y_1 = d_2$.

For $d_1, d_1' \in D_1$ and $d_2, d_2' \in D_2$, we have that

$$(d_1 + d_2) + (d_1' + d_2') = \bar{y}d_1 + \bar{x}d_2' \quad \text{for unique } \bar{x} \in D_1,\ \bar{y} \in D_2.$$

$$= \bar{\bar{y}}d_1 = \bar{\bar{x}}d_2' \quad \text{for unique } \bar{\bar{x}} \in D_1,\ \bar{\bar{y}} \in D_2.$$

$$= d_1 + d_2'.$$

To show $(D,+,\cdot)$ is a division seminear-ring, let $d, d', d'' \in D$. Then there exist unique $x, x', x'' \in D_1$, $y, y', y'' \in D_2$ such that $d = x + y$, $d' = x' + y'$ and $d'' = x'' + y''$. Then

$$d + (d' + d'') = (x + y) + [(x' + y') + (x' + y'')]$$

$$= (x + y) + (x' + y'') = x + y'' \text{ and}$$

$$(d + d') + d'' = [(x + y) + (x' + y')] + (x'' + y'')$$

$$= (x + y') + (x'' + y'') = x + y''.$$

Thus $d + (d' + d'') = (d + d') + d''$, so $(D,+)$ is a semigroup.

Let $x, y, z \in D$. Then there exist unique $x_1, \bar{x}_1, y_1, \bar{y}_1 \in D_1$, $x_2, \bar{x}_2, y_2, \bar{y}_2 \in D_2$ such that $x = x_1x_2 = \bar{x}_2\bar{x}_1$ and $y = y_1y_2 = \bar{y}_2\bar{y}_1$. Since xz and $yz \in D$, there exist unique $a_1, a_2 \in D_1$, $b_1, b_2 \in D_2$ such that $xz = a_1 + b_1$ and $yz = a_2 + b_2$. Then there exist unique $a_3, a_4 \in D_1$, $b_3, b_4 \in D_2$ such that $a_1 + b_1 = a_3b_1 = b_3a_1$ and $a_2 + b_2 = a_4b_2 = b_4a_2$. Thus $xz = b_3a_1$ and $yz = a_4b_2$. Then $z = x^{-1}b_3a_1 = y^{-1}a_4b_2$. Then

$$x^{-1}b_3a_1 = y^{-1}a_4b_2$$

$$(\bar{x}_2\bar{x}_1)^{-1}b_3a_1 = (y_1y_2)^{-1}a_4b_2$$

$$\bar{x}_1^{-1}\bar{x}_2^{-1}b_3a_1 = y_2^{-1}y_1^{-1}a_4b_2.$$

$$\bar{x}_2^{-1} b_3 a_1 = \bar{x}_1 y_2^{-1} y_1^{-1} a_4 b_2.$$

Since $\bar{x}_1 y_2^{-1} \in D$, there exist unique $c_1 \in D_1$, $c_2 \in D_2$ such that $\bar{x}_1 y_2^{-1} = c_2 c_1$.

Then
$$\bar{x}_2^{-1} b_3 a_1 = c_2 c_1 y_1^{-1} a_4 b_2$$
$$c_2^{-1} \bar{x}_2^{-1} b_3 a_1 = c_1 y_1^{-1} a_4 b_2.$$

Since $c_1 y_1^{-1} a_4 \in D_1$ and $c_2^{-1} \bar{x}_2^{-1} b_3 \in D_2$, $a_1 + b_2 = c_1 y_1^{-1} a_4 b_2$. From $x = x_1 x_2 = \bar{x}_2 \bar{x}_1$ and $y = y_1 y_2 = \bar{y}_2 \bar{y}_1$, we have that $x = \bar{x}_1 + x_2$ and $y = \bar{y}_1 + y_2$. Then there exist unique $\bar{x} \in D_1$, $\bar{y} \in D_2$ such that $\bar{x}_1 + y_2 = \bar{x} y_2 = \overline{yx}_1$. Thus $\bar{x}_1 y_2^{-1} = \bar{y}^{-1} \bar{x}$ so $\bar{x} = c_1$. Then we have that

$$xz + yz = (a_1 + b_1) + (a_2 + b_2) = a_1 + b_2 = c_1 y_1^{-1} a_4 b_2 \text{ and}$$
$$\begin{aligned}(x + y)z &= [(\bar{x}_1 + x_2) + (\bar{y}_1 + y_2)]\, y^{-1} a_4 b_2 \\ &= (\bar{x}_1 + y_2)(y_1 y_2)^{-1} a_4 b_2 \\ &= \bar{x} y_2 y_2^{-1} y_1^{-1} a_4 b_2 = \bar{x} y_1^{-1} a_4 b_2 = c_1 y_1^{-1} a_4 b_2.\end{aligned}$$

Therefore $(x + y)z = xz + yz$ for all $x, y, z \in D$. Hence $(D,+,.)$ is a division seminear-ring. Thus (1) holds.

Claim that $x + 1 = x$ for all x in D_1. To show this, let $x \in D_1$. Then there exist unique $\bar{x} \in D_1$, $\bar{y} \in D_2$ such that $\bar{x}1 = \bar{y}x$. Thus $1\bar{x} = \bar{x}1 = \bar{y}x$, so $\bar{x} = x$. Therefore $x + 1 = \bar{x}1 = x1 = x$. So we have the claim. Let $x, x' \in D_1$. Thus $xx'^{-1} \in D_1$, so $xx'^{-1} + 1 = xx'^{-1}$. Thus $x + x' = (xx'^{-1} + 1)x' = (xx'^{-1})x' = x(x'^{-1}x') = x1 = x$. Thus (2) holds.

Claim that $1 + y = y$ for all y in D_2. To show this, let $y \in D_2$. Then there exist unique $\bar{x} \in D_1$, $\bar{y} \in D_2$ such that $\bar{x}y = \bar{y}1$. Thus $1\bar{y} = \bar{y}1 = \bar{x}y$, so $\bar{y} = y$. Therefore $1 + y = \bar{y}1 = y1 = y$. So we have the claim. Let $y, y' \in D_2$. Thus $yy'^{-1} \in D_2$, so $1 + yy'^{-1} = yy'^{-1}$. Thus $y + y' = (1 + yy'^{-1})y = (yy'^{-1})y = y'(y^{-1}y) = y'1 = y'$. Thus (3) holds.
Note that $1 + 1 = 1$ since $1 \in D_1$. Let $x \in D_1$, $y \in D_2$. Then $y + x = (1 + y) + (x + 1) = 1 + 1 = 1$. Hence $D_2 + D_1 = \{1\}$. Thus (4) holds.

To show the uniqueness of +, suppose $+'$ is a binary operation on D such that (1) - (5) are true. Claim that $1 +' d +' 1 = 1$ for all $d \in D$. To show this, let $d \in D$. Since (5) is true, there exist $d_1 \in D_1$, $d_2 \in D_2$ such that $d = d_1 +' d_2$. Then $1 +' d +' 1 = 1 +' (d_1 +' d_2) +' 1 = (1 +' d_1) +' (d_2 +' 1) = 1 +' 1 = 1$ since (1) - (5) are true. Thus $1 +' d +' 1 = 1$ for all $d \in D$. Thus we have the claim. Next we shall show that $x +' y = x + y$ for all $x \in D_1$, $y \in D_2$. To show this, let $x \in D_1$, $y \in D_2$. Thus $x +' y \in D$. Then there exist unique $d_1, \bar{d}_1 \in D_1$, $d_2, \bar{d}_2 \in D_2$ such that $x +' y = d_1 d_2 = \bar{d}_2 \bar{d}_1 = \bar{d}_1 + d_2$. From $x +' y = \bar{d}_2 \bar{d}_1$, we have that $\bar{d}_2 \bar{d}_1 x^{-1} +' 1 = (x +' y)x^{-1} +' 1 = 1 +' yx^{-1} +' 1 = 1$. Then

$$\bar{d}_2 +' x\bar{d}_1^{-1} = (\bar{d}_2 \bar{d}_1 x^{-1} +' 1)x\bar{d}_1^{-1} = 1x\bar{d}_1^{-1} = x\bar{d}_1^{-1}.$$

Since $\bar{d}_2 \in D_2, x\bar{d}_1^{-1} \in D_1$ and (4) is true, $\bar{d}_2 + x\bar{d}_1^{-1} = 1$. Thus $x\bar{d}_1^{-1} = 1$, so $x = \bar{d}_1$. From $x +' y = d_1 d_2$, we have that $1 +' d_1 d_2 y^{-1} = 1 +' (x +' y)y^{-1}$ $=1 +' xy^{-1} +' 1 = 1$. Then

$$yd_2^{-1} +' d_1 = (1 +' d_1 d_2 y^{-1})yd_2^{-1} = 1yd_2^{-1}.$$

Similarly, we have that $y = d_2$. Therefore $x +' y = \bar{d}_1 + d_2 = x + y$. Therefore $x +' y = x + y$ for all $x \in D_1$, $y \in D_2$. Now let $d, d' \in D$. Then there exist $x_1, x_2 \in D_1$, $y_1, y_2 \in D_2$ such that $d = x_1 +' y_1 = x_1 + y_1$ and $d' = x_2 +' y_2 = x_2 + y_2$. Thus

$$d +' d' = (x_1 +' y_1) +' (x_2 +' y_2) = x_1 +' (y_1 +' x_2) +' y_2 =$$
$$x_1 +' 1 +' y_2 = x_1 +' y_2 = x_1 + y_2 = d + d'.$$

Therefore $d +' d' = d + d'$ for all $d, d' \in D$. Hence $+' = +$.

To show (6), define $f: D_1 \times D_2 \rightarrow D$ by $f(d_1, d_2) = d_1 + d_2$ for all $d_1 \in D_1$, $d_2 \in D_2$. f is well-defined, one-to-one and onto since for all $d \in D$ there exist unique $d_1 \in D_1$, $d_2 \in D_2$ such that $d = d_1 + d_2$. To show f is a homomorphism, let $d_1, d_1' \in D_1$, $d_2, d_2' \in D_2$. Then $f((d_1,d_2) + (d_1',d_2')) = f(d_1 + d_1', d_2 + d_2') = f(d_1, d_2') = d_1 + d_2'$ and $f(d_1,d_2) + f(d_1',d_2') = (d_1 + d_2) + (d_1' + d_2') = d_1 + d_2'$. Thus

$f((d_1,d_2) + (d_1',d_2')) = f(d_1,d_2) + f(d_1',d_2')$. Therefore f is a homomorphism. Hence $(D,+) \cong (D_1,+) \times (D_2,+)$. Thus (6) holds.

To show (7), let $d \in D$. Since $1 + 1 = 1$, $d + d = (1 + 1)d = 1d = d$. Thus $d + d = d$ for all $d \in D$. Let $d, d' \in D$. Since $1 + dd'^{-1} + 1 = 1$, $d + d' + d = (1 + dd'^{-1} + 1)d = 1d = d$. Thus $d + d' + d = d$ for all $d, d' \in D$. Hence $(D,+)$ is a rectangular band. Thus (7) holds.

To show (8), assume that the left distributive law holds. Claim that $D_1 = \{x \in D \mid x + 1 = x\}$ and $D_2 = \{x \in D \mid x + 1 = 1\}$. From (2) and (3), $D_1 \subseteq \{x \in D \mid x + 1 = x\}$ and $D_2 \subseteq \{x \in D \mid x + 1 = 1\}$. Now let $x \in D$ be such that $x + 1 = x$. Since $x \in D$, there exist unique $\bar{x} \in D_1$, $\bar{y} \in D_2$ such that $x = \bar{x} + \bar{y}$. Thus $x = x + 1 = (\bar{x} + \bar{y}) + 1 = \bar{x} + (\bar{y} + 1) = \bar{x} + 1 = \bar{x}$, so $x = \bar{x} \in D_1$. Therefore $D_1 = \{x \in D \mid x + 1 = x\}$. Let $x \in D$ be such that $x + 1 = 1$. Since $x \in D$, there exist unique $\bar{x} \in D_1$, $\bar{y} \in D_2$ such that $x = \bar{x} + \bar{y}$. Thus $1 = x + 1 = (\bar{x} + \bar{y}) + 1 = \bar{x} + (\bar{y} + 1) = \bar{x} + 1 = \bar{x}$, so $\bar{x} = 1$. Thus $x = \bar{x} + \bar{y} = 1 + \bar{y} = \bar{y} \in D_2$. Thus $D_2 = \{x \in D \mid x + 1 = 1\}$. Thus we have the claim. To show that $D_1, D_2 \trianglelefteq D$, let $d_1 \in D_1$, $d_2 \in D_2$ and $d \in D$. Then $dd_1d^{-1} + 1 = dd_1d^{-1} + dd^{-1} = d(d_1 + 1)d^{-1} = dd_1d^{-1}$ and $dd_2d^{-1} + 1 = dd_2d^{-1} + dd^{-1} = d(d_2 + 1)d^{-1} = d1d^{-1} = 1$. By the claim, $dd_1d^{-1} \in D_1$ and $dd_2d^{-1} \in D_2$. Thus $D_1, D_2 \trianglelefteq D$. Hence $D = D_1 \times D_2$.

Conversely, assume that $D = D_1 \times D_2$. Claim that $d_1d_2 = d_2d_1$ for all $d_1 \in D_1$, $d_2 \in D_2$. To prove this, let $d_1 \in D_1$ and $d_2 \in D_2$. Then $d_1d_2d_1^{-1}d_2^{-1} \in D_1 \cap D_2 = \{1\}$, since $D_1, D_2 \trianglelefteq D$. Thus $d_1d_2d_1^{-1}d_2^{-1} = 1$, so $d_1d_2 = d_2d_1$. Thus we have the claim. To show that the left distributive law holds, let $x, y, z \in D$. Then there exist unique x_1, y_1, z_1 in D_1 and x_2, y_2, z_2 in D_2 such that $x = x_1x_2 = x_2x_1 = x_1 + x_2$, $y = y_1y_2 = y_2y_1 = y_1 + y_2$ and $z = z_1z_2 = z_2z_1 = z_1 + z_2$. Thus

$xy + xz = x_1x_2y_2y_1 + x_2x_1z_1z_2 = x_2y_2x_1y_1 + x_1z_1x_2z_2 = x_1y_1x_2z_2 =$ $x_1x_2y_1z_2$ and $x(y + z) = x((y_1 + y_2) + (z_1 + z_2)) = x(y_1 + z_2) =$ $xy_1z_2 = x_1x_2y_1z_2$. Thus $x(y + z) = xy + xz$. Therefore the left distributive law holds. Thus (8) holds.

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So we see that every Zappa-Szép product of a group which is not a direct product gives us an example of a division seminear-ring which is not a division semiring.

Definition 2.5. Let G and G' be groups and suppose that $G = G_1 * G_2$, $G' = G'_1 * G'_2$ for some subgroups G_1, G_2 of G and G'_1, G'_2 of G'. Let $f: G \to G'$ be an isomorphism. Then f is said to preserve the Zappa-Szép products iff $f(G_1) = G'_1$ and $f(G_2) = G'_2$.

Theorem 2.6. Let G and G' be groups such that $G = G_1 * G_2$ and $G' = G'_1 * G'_2$ for some subgroups G_1, G_2 of G and G'_1, G'_2 of G'. Let $f: G \to G'$ be a map. Then f is a division seminear-ring isomorphism with respect to the addition given in Theorem 2.4 iff f is a group isomorphism preserving the Zappa-Szép products.

Proof. By Theorem 2.4 and its proof, there exist unique $+$ and $+'$ on G and G' respectively such that $G_1 = \{x \in G \mid x + 1 = x\}$, $G_2 = \{x \in G \mid x + 1 = 1\}$, $G'_1 = \{x \in G' \mid x +' 1' = x\}$ and $G'_2 = \{x \in G' \mid x +' 1' = 1'\}$ where 1 and $1'$ are the identities of G and G' respectively.

Assume that f is a division seminear-ring isomorphism. Then f is a group isomorphism from $(G, .)$ to $(G', .')$. To show that $f(G_1) \subseteq G'_1$ and $f(G_2) \subseteq G'_2$, let $g_1 \in G_1$ and $g_2 \in G_2$. Then

$$f(g_1) +' 1' = f(g_1) +' f(1) = f(g_1 + 1) = f(g_1) \quad \text{and}$$
$$f(g_2) +' 1' = f(g_2) +' f(1) = f(g_2 + 1) = f(1) = 1'.$$

Thus $f(g_1) \in G_1'$ and $f(g_2) \in G_2'$, so $f(G_1) \subseteq G_1'$ and $f(G_2) \subseteq G_2'$. To show that $G_1' \subseteq f(G_1)$ and $G_2' \subseteq f(G_2)$, let $g_1' \in G_1'$, $g_2' \in G_2'$. Since f is onto, there exist $x \in G$, $y \in G$ such that $f(x) = g_1'$ and $f(y) = g_2'$. Then

$$f(x + 1) = f(x) \,\dot{+}\, f(1) = g_1' \,\dot{+}\, 1' = g_1' = f(x) \quad \text{and}$$
$$f(y + 1) = f(y) \,\dot{+}\, f(1) = g_2' \,\dot{+}\, 1' = 1' = f(1).$$

Since f is one-to-one, $x + 1 = x$ and $y + 1 = 1$. Thus $x \in G_1$ and $y \in G_2$. Thus $g_1' \in f(G_1)$ and $g_2' \in f(G_2)$. So $G_1' \subseteq f(G_1)$ and $G_2' \subseteq f(G_2)$. Therefore $f(G_1) = G_1'$ and $f(G_2) = G_2'$. Hence f is a group isomorphism preserving the Zappa-Szép products.

Conversely, assume that f is a group isomorphism preserving the Zappa-Szép products. We must show that $f(g + h) = f(g) \,\dot{+}\, f(h)$ for all $g, h \in G$. To prove this, let $g, h \in G$. Then there exist g_1, h_1, $\bar{g}_1$, $\bar{h}_1 \in G_1$, g_2, $\bar{g}_2$, h_2, $\bar{h}_2 \in G_2$ such that $g = g_1g_2 = \bar{g}_2\bar{g}_1$ and $h = h_1h_2 = \bar{h}_2\bar{h}_1$. Let $x_1 \in G_1$, $x_2 \in G_2$ be such that $\bar{g}_1 + h_2 = x_1h_2 = x_2\bar{g}_1$ From $x_1h_2 = x_2\bar{g}_1$ we have that $f(x_1)f(h_2) = f(x_1h_2) = f(x_2\bar{g}_1) = f(x_2)f(\bar{g}_1) = f(\bar{g}_1) \,\dot{+}\, f(h_2)$. Similarly, we have that $f(g) = f(\bar{g}_1) \,\dot{+}\, f(g_2)$ and $f(h) = f(\bar{h}_1) \,\dot{+}\, f(h_2)$. Therefore

$$f(g + h) = f(\bar{g}_1 + h_2) = f(x_1h_2) \quad \text{and}$$
$$f(g) \,\dot{+}\, f(h) = (f(\bar{g}_1) \,\dot{+}\, f(g_2)) \,\dot{+}\, (f(\bar{h}_1) \,\dot{+}\, f(h_2))$$
$$= f(\bar{g}_1) \,\dot{+}\, f(h_2) = f(x_1)f(h_2) = f(x_1h_2).$$

Thus $f(g + h) = f(g) \,\dot{+}\, f(h)$ for all $g, h \in G$. Therefore f is division seminear-ring isomorphism.

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We shall now show that in the finite case the family of division seminear-rings constructed in Theorem 2.4 is the only type of example that exists.

Lemma 2.7. If $(D,+,.)$ is a finite division seminear-ring, then $(D,+)$ is a band.

Proof. Let $(D,+,.)$ be a finite division seminear-ring. Then $(D,+)$ is a finite semigroup. By Theorem 1.15, there exists a d in D such that $d + d = d$. Let $x \in D$. Then $x + x = (d + d)d^{-1}x = dd^{-1}x = x$. Thus $x + x = x$ for all $x \in D$. Therefore $(D,+)$ is a band.#

Theorem 2.8. Let D be a finite division seminear-ring. Then there exist unique division subseminear-rings $D_1, D_2 \subseteq D$ such that

(1) $x + y = x$ for all $x, y \in D_1$,

(2) $x + y = y$ for all $x, y \in D_2$,

(3) $(D,.) = (D_1,.) * (D_2,.)$ $(= (D_2,.) * (D_1,.))$.

Furthermore,

(4) $(D,+) \cong (D_1,+) \times (D_2,+)$,

(5) $D_2 + D_1 = \{1\}$,

(6) $(D,+)$ is a rectangular band,

(7) D is left distributive iff $D_1, D_2 \trianglelefteq D$ as multiplicative groups.

Proof. Let $D_1 = \{x \in D \mid x + 1 = x\}$ and $D_2 = \{x \in D \mid x + 1 = 1\}$ By Lemma 2.7, $1 + 1 = 1$. Thus $1 \in D_1 \cap D_2$, so $D_1 \neq \emptyset$ and $D_2 \neq \emptyset$. To show that D_1 is a division subseminear-ring of D and (1) holds, let $x, y \in D_1$. Then $xy + 1 = (x + 1)y + 1 = xy + (y + 1) = xy + y = (x + 1)y = xy$ and $(x + y) + 1 = x + (y + 1) = x + y$, so $xy, x + y \in D_1$. Since $(D,.)$ is a finite group, there exists an $n \in \mathbb{Z}^+$ such that $x^n = 1$. So $x^{-1} = x^{n-1} \in D_1$. Hence D_1 is a division subseminear-ring of D. Since $xy^{-1} \in D_1$, $xy^{-1} + 1 = xy^{-1}$. Thus $x + y = (xy^{-1} + 1)y = xy^{-1}y = x$.

To show that D_2 is a division subseminear-ring of D and (2) holds, let $x, y \in D_2$. Then $xy + 1 = xy + y + 1 = (x + 1)y + 1 = 1y + 1 = y + 1 = 1$ and $(x + y) + 1 = x + (y + 1) = x + 1 = 1$, so xy, $x + y$ are in D_2. Similarly, $x^{-1} \in D_2$. Thus D_2 is a division subseminear-ring of D. Since $xy^{-1} \in D_2$, $xy^{-1} + 1 = 1$. Thus $x + y = (xy^{-1} + 1)y = 1y = y$.

To show (3), we show that $(D,.) = (D_2,.) * (D_1,.)$. Let $d \in D$. Then $d + 1 \in D_1$ and $d(d + 1)^{-1} \in D_2$ since $(d + 1) + 1 = d + (1 + 1) = d + 1$ and

$$
\begin{aligned}
d(d + 1)^{-1} + 1 &= d(d + 1)^{-1} + (d + 1)(d + 1)^{-1} \\
&= (d + (d + 1))(d + 1)^{-1} \\
&= ((d + d) + 1)(d + 1)^{-1} \\
&= (d + 1)(d + 1)^{-1} \quad \text{(By Lemma 2.7, } d + d = d) \\
&= 1.
\end{aligned}
$$

Thus $d = d(d + 1)^{-1}(d + 1) \in D_2D_1$. Therefore $D = D_2D_1$. Let $x \in D_1 \cap D_2$. Then $x = x + 1 = 1$. Hence $D_1 \cap D_2 = \{1\}$. Thus $(D,.) = (D_2,.) * (D_1,.)$. By Lemma 1.13(1), $(D,.) = (D_1,.) * (D_2,.)$.

Claim that

(a) $d + d' + d = d$ for all $d, d' \in D$. (Hence $(D,+)$ is a rectangular band. Thus (6) holds.)

(b) For all $d \in D$ there exist unique $d_1 \in D_1$, $d_2 \in D_2$ such that $d = d_1 + d_2$.

To show (a), let $d, d' \in D$. Since $dd'^{-1} + 1$ and $1 \in D_1$, $1 + (dd'^{-1} + 1) = 1$. Thus $d + d' + d = (1 + dd'^{-1} + 1)d = 1d = d$.

To show (b), let $d \in D$. By (a), $(1 + d) + 1 = 1$, so $1 + d \in D_2$. Again by (a), $d = d + 1 + d = d + (1 + 1) + d = (d + 1) + (1 + d)$ which is in $D_1 + D_2$, so we get the existence part of (b). To show uniqueness, let $d_1 \in D_1$, $d_2 \in D_2$ be such that $d = d_1 + d_2$. Then $d_1 = d_1 + 1 = d_1 + (d_2 + 1) = (d_1 + d_2) + 1 = d + 1$ and $d_2 = 1 + d_2 = (1 + d_1) + d_2 = 1 + (d_1 + d_2) = 1 + d$, so we get the uniqueness part. Hence we get (b).

To show (4), define $f: D_1 \times D_2 \to D$ by $f(d_1,d_2) = d_1 + d_2$ for all $d_1 \in D_1$, $d_2 \in D_2$. By (b), f is a bijection. To show f is a homomorphism, let $d_1, d_1' \in D_1$, $d_2, d_2' \in D_2$. Then

$$f((d_1,d_2) + (d_1',d_2')) = f(d_1 + d_1', d_2 + d_2')$$
$$= f(d_1,d_2') = d_1 + d_2' \quad \text{and}$$
$$f(d_1,d_2) + f(d_1',d_2') = (d_1 + d_2) + (d_1' + d_2')$$
$$= d_1 + (1 + d_2) + (d_1' + 1) + d_2'$$
$$= d_1 + (1 + (d_2 + d_1') + 1) + d_2'$$
$$= d_1 + 1 + d_2' = d_1 + d_2'$$

Therefore $f((d_1,d_2) + (d_1',d_2')) = f(d_1,d_2) + f(d_1',d_2')$. Thus f is a homomorphism. Hence $(D,+) \cong (D_1,+) \times (D_2,+)$. Thus (4) holds.

To show (5), let $d_1 \in D_1$, $d_2 \in D_2$. Then

$$d_2 + d_1 = (1 + d_2) + (d_1 + 1) = 1 + (d_2 + d_1) + 1 = 1.$$

Therefore $D_2 + D_1 = \{1\}$. Thus (5) holds.

To show (7), assume that D is left distributive. Let $d_1 \in D_1$, $d_2 \in D_2$ and $d \in D$. Then

$$dd_1d^{-1} + 1 = dd_1d^{-1} + dd^{-1} = d(d_1 + 1)d^{-1} = dd_1d^{-1} \quad \text{and}$$
$$dd_2d^{-1} + 1 = dd_2d^{-1} + dd^{-1} = d(d_2 + 1)d^{-1} = d1d^{-1} = 1.$$

Thus $dd_1d^{-1} \in D_1$ and $dd_2d^{-1} \in D_2$, so $D_1 \trianglelefteq D$ and $D_2 \trianglelefteq D$.

The converse can be proved as in Theorem 2.4.

To show the uniqueness of D_1 and D_2, suppose that D_1', D_2' are division subseminear-rings of D such that (1) - (3) hold. Let $d_1' \in D_1'$, $d_2' \in D_2'$. Since (1) and (2) hold, $d_1' + 1 = d_1'$ and $d_2' + 1 = 1$. Thus $d_1' \in D_1$ and $d_2' \in D_2$, so $D_1' \subseteq D_1$ and $D_2' \subseteq D_2$. Let $d_1 \in D_1$, $d_2 \in D_2$. Since (3) is true, there exist unique $d_1', c_1' \in D_1'$, $d_2', c_2' \in D_2'$ such that $d_1 = d_1'd_2'$ and $d_2 = c_2'c_1'$. Now $d_1' \in D_1' \subseteq D_1$ and $d_2' \in D_2' \subseteq D_2$, so $d_1 1 = d_1'd_2'$ in D_1D_2. Thus $d_1 = d_1' \in D_1'$. Similarly, $d_2 \in D_2'$. Thus $D_1 \subseteq D_1'$ and $D_2 \subseteq D_2'$. Hence $D_1' = D_1$ and $D_2' = D_2$. #

Remark. Since properties (1) - (5) are true, the given addition in a finite division seminear-ring must be the same as the addition which comes from Theorem 2.4.

Theorem 2.9. Let D and D' be finite division seminear-rings. Then $D \cong D'$ as division seminear-rings iff there exists a multiplicative group isomorphism $f: D \to D'$ preserving the Zappa-Szép products of the division subseminear-rings given in Theorem 2.8.

Proof. Since D and D' are finite division seminear-rings, there exist unique division subseminear-rings $D_1, D_2 \subseteq D$, $D'_1, D'_2 \subseteq D'$ such that $D = D_1 * D_2$ and $D' = D'_1 * D'_2$ as in Theorem 2.8. Let 1 and $1'$ be the identities of D and D' respectively.

Assume that $f: D \to D'$ is a division seminear-ring isomorphism. Then f is a group isomorphism. Claim that f preserves the Zappa-Szép products. We must prove that $f(D_1) = D'_1$ and $f(D_2) = D'_2$. Let $d_1 \in D_1$, $d_2 \in D_2$. Then $f(d_1) +' 1' = f(d_1) +' f(1) = f(d_1 + 1) = f(d_1)$ and $f(d_2) +' 1' = f(d_2) +' f(1) = f(d_2 + 1) = f(1) = 1'$, so $f(d_1) \in D'_1$ and $f(d_2) \in D'_2$. Thus $f(D_1) \subseteq D'_1$ and $f(D_2) \subseteq D'_2$. Now let $d'_1 \in D'_1$ and $d'_2 \in D'_2$. Since f is onto, there exist $x, y \in D$ such that $f(x) = d'_1$ and $f(y) = d'_2$. Then $f(x + 1) = f(x) +' f(1) = d'_1 +' 1' = d'_1 = f(x)$ and $f(y + 1) = f(y) +' f(1) = d'_2 +' 1' = 1' = f(1)$. Since f is one-to-one, $x + 1 = x$ and $y + 1 = 1$. Thus $x \in D_1$ and $y \in D_2$, so $d'_1 \in f(D_1)$ and $d'_2 \in f(D_2)$. Thus $D'_1 \subseteq f(D_1)$ and $D'_2 \subseteq f(D_2)$. Therefore $f(D_1) = D'_1$ and $f(D_2) = D'_2$.

Conversely, assume that $f: D \to D'$ is a group isomorphism preserving the Zappa-Szép products. We must show that $f(c + d) = f(c) +' f(d)$ for all $c, d \in D$. Let $c, d \in D$. Then there exist unique $c_1, \bar{c}_1, d_1, \bar{d}_1 \in D_1$, $c_2, \bar{c}_2, d_2, \bar{d}_2 \in D_2$ such that $c = c_1c_2 = \bar{c}_2\bar{c}_1$ and $d = d_1d_2 = \bar{d}_2\bar{d}_1$. There exist unique $\bar{x} \in D_1$, $\bar{y} \in D_2$ such that

$\bar{c}_1 + d_2 = \bar{x}d_2 = \bar{y}\bar{c}_1$. There exist unique $\bar{\bar{x}} \in D_1'$, $\bar{\bar{y}} \in D_2'$ such that $f(\bar{c}_1) +' f(d_2) = \bar{\bar{x}}f(d_2) = \bar{\bar{y}}f(\bar{c}_1)$. Claim that $\bar{\bar{x}} = f(\bar{x})$. Note that $f(x) \in D_1'$ and $f(y) \in D_2'$ for all $x \in D_1$ and $y \in D_2$ since f preserves the Zappa-Szép products. From $\bar{x}d_2 = \bar{y}\bar{c}_1$, $f(\bar{x})f(d_2) = f(\bar{y})f(\bar{c}_1)$. Thus $f(d_2)f(\bar{c}_1)^{-1} = f(\bar{x})^{-1}f(\bar{y})$. From $\bar{\bar{x}}f(d_2) = \bar{\bar{y}}f(\bar{c}_1)$, we have that $f(d_2)f(\bar{c}_1)^{-1} = \bar{\bar{x}}^{-1}\bar{\bar{y}}$. Thus $f(\bar{x}) = \bar{\bar{x}}$. Therefore

$$f(c + d) = f(\bar{c}_1 + c_2 + \bar{d}_1 + d_2) = f(\bar{c}_1 + d_2) = f(\bar{x}d_2) \text{ and}$$
$$f(c) +' f(d) = f(\bar{c}_2)f(\bar{c}_1) + f(d_1)f(d_2)$$
$$= \bar{\bar{x}}f(d_2) = f(\bar{x})f(d_2) = f(\bar{x}d_2).$$

Thus $f(c + d) = f(c) +' f(d)$ for all $c, d \in D$. Therefore f is a division seminear-ring isomorphism.#

In Example 2.3(2), the binary operations of + and . are equal. This can only happen when the order is one as the following theorem shows.

Theorem 2.10. Let $(D,+,.)$ be a division seminear-ring such that + and . are equal. Then $\|D\| = 1$.

Proof. Suppose $\|D\| > 1$. Let $x \in D \setminus \{1\}$. Then

$$x^2 = x.x = x + x = (1 + 1).x = (1.1).x = 1.x = x.$$

Thus $x^2 = x$, so $x = 1$, a contradiction. Hence $\|D\| = 1$.#

Definition 2.11. Let $(S,+,.)$ be a seminear-ring and $a \in S$. Then a is an additive zero iff $a + x = x + a = a$ for all $x \in S$ and a multiplicative zero iff $a.x = x.a = a$ for all $x \in S$.

Clearly, a division seminear-ring can have a multiplicative zero a iff it has order one since $1 = aa^{-1} = a$ so for all $x \in D$ $x = xa = a = 1$.

In a division seminear-ring of order one, we see that 1 is both an additive zero and an additive identity but in a division seminear-ring of order > 1 we cannot have this as the following theorems show.

<u>Theorem 2.12</u>. Let D be a division seminear-ring of order > 1. Then D cannot contain any additive identity.

<u>Proof</u>. Suppose D has an additive identity e. Thus $e + x = x + e = x$ for all $x \in D$, so $1 + xe^{-1} = xe^{-1} + 1 = xe^{-1}$ for all $x \in D$. Let $C = \{xe^{-1} \mid x \in D\}$. Since $(D,.)$ is a group, $C = D$. Therefore $1 + z = z + 1 = z$ for all $z \in D$, so $e = 1 + e = 1$. Let $x \in D \setminus \{1\}$. Then $1 + x = x$, so $x^{-1} + 1 = 1$. Since $x^{-1} + 1 = x^{-1}$, so $x^{-1} = 1$. Thus $x = 1$, a contradiction.#

<u>Theorem 2.13</u>. Let D be a division seminear-ring of order > 1. Then D cannot contain any additive zero.

<u>Proof</u>. Suppose D contains an additive zero a. Then $a + x = x + a = a$ for all $x \in D$, so $1 + xa^{-1} = xa^{-1} + 1 = 1$ for all $x \in D$. Let $C = \{xa^{-1} \mid x \in D\}$. Since $(D,.)$ is a group, $C = D$. Thus $1 + d = d + 1 = 1$ for all $d \in D$, so $a = a + 1 = 1$. Let $x \in D \setminus \{1\}$. Thus $1 + x = 1$, so $x^{-1} + 1 = x^{-1}$. Since $x^{-1} + 1 = 1$, so $x^{-1} = 1$. Hence $x = 1$, a contradiction.#

<u>Definition 2.14</u>. Let D be a division seminear-ring with 1 as its multiplicative identity. Then the <u>prime division seminear-ring</u> of D is the smallest division seminear-ring contained in D i.e. the prime division seminear-ring of D is the intersection of all division subseminear-rings of D. Note that this intersection cannot be empty

since every division subseminear-ring of D must contain 1 (because 1 is the only idempotent in (D,.)).

<u>Proposition 2.15</u>. Let D be a finite division seminear-ring (with multiplicative identity 1). Then the prime division seminear-ring of D is $\{1\}$.

<u>Proof</u>. Since (D,+) is a finite semigroup, by lemma 2.7 we get that 1 + 1 = 1. So $\{1\}$ is close with respect to + and . hence it is a division subseminear-ring of D.#

<u>Definition 2.16</u>. Let S be a seminear-ring and $x \in S$. Then x is said to be <u>left additively cancellative</u> iff for all y, $z \in S$ $x + y = x + z$ implies $y = z$. <u>Right additive cancellativity</u> is similarly defined. x is said to be <u>additively cancellative</u> iff it is left and right additively cancellative.

<u>Example 2.17</u>. If S has an additive identity 0, then 0 is additively cancellative.

<u>Definition 2.18</u>. Let S be a seminear-ring and $x \in S$. Then x is said to be <u>left multiplicatively cancellative</u> iff for all y, $z \in S$ $xy = xz$ implies $y = z$. <u>Right multiplicative cancellativity</u> is similarly defined. x is said to be <u>multiplicatively cancellative</u> iff it is left and right multiplicatively cancellative.

<u>Example 2.19</u>. If S has a multiplicative identity 1, then 1 is multiplicatively cancellative.

<u>Remark</u>. If D is a finite division seminear-ring of order > 1, then D cannot be additively cancellative by Theorem 2.8. So D can never be

embedded in a near-field.

Theorem 2.20. Let $(D,+,.)$ be a division seminear-ring. Then

(1) If one element of D is left additively cancellative, then every element of D is left additively cancellative.

(2) If one element of D is right additively cancellative, then every element of D is right additively cancellative.

(Hence if one element of D is additively cancellative, then every element of D is additively cancellative.)

Proof. (1) Let $d \in D$ be left additively cancellative. Let $x \in D$ and $y, z \in D$ be such that $x + y = x + z$. Then

$$d + yx^{-1}d = (x + y)x^{-1}d = (x + z)x^{-1}d = d + zx^{-1}d.$$

Since d is left additively cancellative, $yx^{-1}d = zx^{-1}d$. Thus $y = z$ since $(D,.)$ is a group.

(2) Let $d \in D$ be right additively cancellative. Let $x \in D$ and $y, z \in D$ be such that $y + x = z + x$. Then

$$yx^{-1}d + d = (y + x)x^{-1}d = (z + x)x^{-1}d = zx^{-1}d + d.$$

Since d is right additively cancellative, $yx^{-1}d = zx^{-1}d$. Thus $y = z$ since $(D,.)$ is a group. #

Proposition 2.21. Let S be a finite seminear-ring which is multiplicatively cancellative. Then S is a division seminear-ring.

Proof. Since $(S,.)$ is a finite cancellative semigroup, by theorem 1.18, $(S,.)$ is a group. Hence S is a division seminear-ring. #