CHAPTER III

A GENERALIZATION OF KOLMOGOROV THEOREM

The pupose of this chapter is to find a necessary and sufficient condition for the weak convergence of sequences of distribution functions of random sums to a limit distribution function. That is we generalize Theorem 1.4.2 to the case of random sums.

In [1] Bethmann gives a necessary and sufficient condition for the weak convergence of sequences of distribution functions of random sums to the standard normal distribution function. One of the important tools used by Bethmann is what is known as the "q-quantiles of Z_n". We shall also make use of this tool.

3.1 Definition and Properties of Q-Quantiles.

Let Z be a positive integral-valued random variable. Let $l:(0,1)\rightarrow N$ be defined by

$$l(q) = \max \{ k\in N \mid P(Z < k) \leq q \}.$$

The function l is called the q-quantile of Z.

Remark 3.1.1 Let Z be a positive integral-valued random variable. For each $q \in (0,1)$, the set $\{k\in N \mid P(Z<k) \leq q\}$ is non-empty and bounded. So the function q-quatile l of Z is well-defined.

Remark 3.1.2 For a positive integral-valued random variable Z, the function q-quatile of Z is non-decreasing.

Remark 3.1.3 For a positive integral-valued random variable Z, the function q-quatile l of Z is a Borel function.

Proof. Let J be the range of Z. Let $k_1, k_2, \ldots, k_m, \ldots$, be the sequence obtained by arranging the values in J in ascending order. Note that this sequence may be finite or infinite. For each $k_j \in J$, we define

$$q(k_j) = \sum_{k=1}^{k_j} P(Z=k)$$

and

$$q(k_o) = 0.$$

Observe that for each $k_j \in J$ and a positive number q in $[q(k_{j-1}), q(k_j))$ we have

$$l(q) = k_j.$$

So

$$l^{-1}(\{k_j\}) = [q(k_{j-1}), q(k_j))$$

for j = 2,3,4,..., and

$$l^{-1}(\{k_1\}) = (0, q(k_1)).$$

It follows that for any open set O in N

$$l^{-1}(O) = \bigcup_{k_j \in O} [q(k_{j-1}), q(k_j)) \cap (0,1)$$

which implies that $l^{-1}(O)$ is a Borel set in (0,1). Therefore l is a Borel function.

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Lemma 3.1.4 Let Z be a positive integral-valued random variable and $g: N \to \mathbb{C}$ be a Borel function. If $E[g \circ Z] < \infty$, then

$$E[goZ] = \int_0^1 g(l(q))dq.$$

Proof. Let J be the range of Z. Let $k_1, k_2, \ldots, k_m, \ldots$, be the sequence obtained by arranging the values J in ascending order. Note that this sequence may be finite or infinite. For each $k_j \in J$, we define

$$q(k_j) = \sum_{k=1}^{k_j} P(Z = k)$$

and

$$q(k_0) = 0.$$

Since for each k_j and $q \in [q(k_{j-1}), q(k_j))$ we have $l(q) = k_j$, so

$$\int_{[q(k_{j-1}), q(k_j))} g(l(q))dq = g(k_j)(q(k_j) - q(k_{j-1}))$$

for every k_j in J. Therefore,

$$\begin{aligned} E[goZ] &= \sum_{k_j \in J} g(k_j)P(Z=k_j) \\ &= \sum_{k_j \in J} g(k_j)(q(k_j) - q(k_{j-1})) \\ &= \sum_{k_j \in J} \int_{[q(k_{j-1}), q(k_j))} g(l(q))dq \\ &= \int_0^1 g(l(q))dq. \end{aligned}$$

#

In the sequel, for each n we shall denote the function q-quatile of Z_n by l_n.

Lemma 3.1.6 (Bethmann [1]) Let (a_{nj}) be a double sequence of non-negative real numbers such that for each n a sequence (a_{nj}) is non-decreasing. Let a be a non-negative real number. Then $a_{nZ_n} \xrightarrow{P} a$ if and only if for every q in (0,1), $a_{nl_n(q)} \rightarrow a$.

3.2 Generalization of Kolmogorov Theorem.

In this section we will give a necessary and sufficient condition for weak convergence of the distribution functions of random sums to a limit distribution function.

Lemma 3.2.1 Let (X_n) be a sequence of random variables. Let x be a real mumber. If $X_n \xrightarrow{P} x$, then there exists a real number $c > 0$ such that

$$P(X_n \geq c) \longrightarrow 0.$$

Proof. Let c be a positive number such that $x + 1 < c$. Observe that

$$\begin{aligned} P(X_n \geq c) &\leq P(X_n \geq x + 1) \\ &= P(X_n - x \geq 1) \\ &\leq P(|X_n - x| \geq 1). \end{aligned}$$

Since $X_n \xrightarrow{P} x$, hence

$$P(|X_n - x| \geq 1) \longrightarrow 0.$$

Therefore

$$P(X_n \geq c) \longrightarrow 0.$$

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Lemma 3.2.2 Let $X_1, X_2, \ldots, X_n$ be a complex-valued random variables. Then for every $\varepsilon > 0$, we have

$$P(|\sum_{j=1}^{n} X_j| \geq \varepsilon) \leq \sum_{j=1}^{n} P(|X_j| \geq \frac{\varepsilon}{n}).$$

Proof. Let $\omega \in \Omega$ be such that $|\sum_{j=1}^{n} X_j(\omega)| \geq \varepsilon$. So $\sum_{j=1}^{n} |X_j(\omega)| \geq \varepsilon$. Hence there exists $j \in \{1,2,3\ldots,n\}$ such that $|X_j(\omega)| \geq \frac{\varepsilon}{n}$. So

$$\{\omega | |\sum_{j=1}^{n} X_j(\omega)| \geq \varepsilon\} \subseteq \bigcup_{j=1}^{n} \{\omega \mid |X_j(\omega)| \geq \frac{\varepsilon}{n}\}$$

which imples the conclusion of the lemma.

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In the following lemmas, we study the convergence of sequence in the space $\mathfrak{M}$ of bounded, non-decreasing and right-continuous functions from R into $[0,\infty)$ which vanish at $-\infty$ and the metric L defined on $\mathfrak{M}$

Lemma 3.2.3 Let $K, K_1, K_2, \ldots,$ be elements in $\mathfrak{M}$. Assume that the following conditions are satisfied.

(a) There exist real numbers $\mu, \mu_1, \mu_2, \ldots$, such that

$$i\mu_n t + \int_{-\infty}^{\infty} f(t,x)dK_n(x) \to i\mu t + \int_{-\infty}^{\infty} f(t,x)dK(x)$$

for every real number t.

(b) $(K_n(+\infty))$ is bounded.

Then we have the following.

(i) $\mu_n \to \mu$.

(ii) There exists a subsequence of (K_n) which converges weakly to K.

(iii) For every subsequence of (K_n), it contains a subsequence which converges weakly to K.

(iv) $K_n \xrightarrow{w} K$.

Proof. By (a), for $t \neq 0$ we have

$$i\mu_n + \frac{1}{t}\int_{-\infty}^{\infty} f(t,x)dK_n(x) \longrightarrow i\mu + \frac{1}{t}\int_{-\infty}^{\infty} f(t,x)dK(x).$$

So, in order to prove (i) it suffices to show that

(1) $$\lim_{t \to 0} \frac{1}{t}\int_{-\infty}^{\infty} f(t,x)dK_n(x) = 0. \text{ uniformly, and}$$

(2) $$\lim_{t \to 0} \frac{1}{t}\int_{-\infty}^{\infty} f(t,x)dK(x) = 0.$$

Observe that

$$\left|\frac{1}{t}\int_{-\infty}^{\infty} f(t,x)dK_n(x)\right| \le \frac{1}{|t|}\int_{-\infty}^{\infty} |f(t,x)|dK_n(x)$$

$$\le \frac{1}{|t|}\int_{-\infty}^{\infty} \frac{t^2}{2}\, dK_n(x)$$

$$\le \frac{|t|}{2}K_n(+\infty).$$

From this fact together with (b), we have (1).

Equation (2) can be proved similarly to equation (1).

Since K_n is non-decreasing and $(K_n(+\infty))$ is bounded, we have (K_n) is uniformly bounded. By Corollary 1.2.11, there exist a subsequence (K_{nk}) of (K_n) and a function $\overline{K}$ in $\mathfrak{M}$ such that $K_{nk} \xrightarrow{w} \overline{K}$. Since for each arbitrary but fixed t, $|f(t,x)|$ is bounded, it follows from Theorem 1.2.12 that

$$\int_{-\infty}^{\infty} f(t,x)dK_{nk}(x) \longrightarrow \int_{-\infty}^{\infty} f(t,x)d\overline{K}(x)$$

for every real number t. By (i) and this fact we have

$$i\mu_{nk}t + \int_{-\infty}^{\infty} f(t,x)dK_{nk}(x) \longrightarrow i\mu t + \int_{-\infty}^{\infty} f(t,x)d\overline{K}(x).$$

From this fact and (a) we have

$$i\mu t + \int_{-\infty}^{\infty} f(t,x)d\overline{K}(x) = i\mu t + \int_{-\infty}^{\infty} f(t,x)dK(x).$$

By Theorem 1.3.4, we have $\overline{K} = K$. So $K_{n_k} \xrightarrow{w} K$. That is we have (ii).

To prove (iii), we use the same argument of (ii).

It follows from (iii) by using Theorem 1.2.12 that for every subsequence of (K_n), it contains a subsequence which converges to K with the metric L. This implies that (K_n) converges to K with metric L. So by Theorem 1.2.12 we have (iv).

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In order to state following theorem and lemmas, we make uses of families of random variables defined in terms of Z_n. This is done in the following remark.

<u>Proposition 3.2.4</u> For each $u \in R$, Let

$$K_{Z_n}(u) = \sum_{j=1}^{Z_n} \int_{-\infty}^{u} x^2 dF_{nj}(x + \mu_{nj})$$

Then the following hold.

(i) For each u, $K_{Z_n}(u)$ are random variables.

(ii) $K_{Z_n}(+\infty) = \sum_{j=1}^{Z_n} \sigma_{nj}^2$.

(iii) For each $\omega \in \Omega$, $K_{Z_n(\omega)}$ may be considered as a function from R to R for which its value at u is $K_{Z_n(\omega)}(u)$. With this interpretation, we see that $K_{Z_n(\omega)} \in \mathfrak{M}$.

Lemma 3.2.5 Let $(Z_n;X_{nj})$ be a random double sequence of random variables and (A_{nj}) be a sequence of real numbers. Let $\hat{\varphi}_{Z_n}$ be the random accompanying characteristic functions associated with (Z_n), (φ_{nj}) and (A_{nj}). Let φ be the characteristic function of a distribution function with finite variance σ^2. Assume that the following hold.

(a) $\hat{\varphi}_{l_n(q)}(t) \to \varphi(t)$ for every q and every real number t.

(b) $\sum_{j=1}^{Z_n} \sigma_{nj}^2 \xrightarrow{P} \sigma^2$.

Then there exists a function K in $\mathfrak{M}$ such that

(i) $K(+\infty) = \sigma^2$, and

(ii) $K_{Z_n}(u) \xrightarrow{P} K(u)$, for every continuity point u of K.

Proof. It follows from Remark 2.3.1 that $\hat{\varphi}_{l_n(q)}$ is infinitely divisible. By (a), it follows from Theorem 1.3.7. that φ is infinitely divisible. Therefore, according to Theorem 1.3.5, there exist the constant μ and the function K in $\mathfrak{M}$ such that $K(+\infty) = \sigma^2$ and

$$\mathrm{Log}\varphi(t) = i\mu t + \int_{-\infty}^{\infty} f(t,x)dK(x),$$

for every t. So we have (i).

To prove (ii), let u be any continuity point of K. For each n and j, let

$$a_{nj}(u) = \sum_{k=1}^{j} \int_{-\infty}^{u} x^2 dF_{nk}(x+\mu_{nk}).$$

Hence $a_{nZ_n}(u) = K_{Z_n}(u)$. In order to prove $K_{Z_n}(u) \xrightarrow{p} K(u)$ by using Lemma 3.1.5, it suffices to show that

$$a_{nl_n(q)}(u) \longrightarrow K(u)$$

for every $q \in (0,1)$, i.e.

$$K_{l_n(q)}(u) \longrightarrow K(u),$$

for every $q \in (0,1)$.

To do this, we will apply Lemma 3.2.3 (iv) to a sequence $(K_{l_n(q)})$. So it suffices to show that the conditions (a) and (b) of Lemma 3.2.3 are satisfied. Let t be any real number. By Theorem 1.3.1, $\varphi(t) \neq 0$. From this fact together with (a) we can choose a branch of logarithm such that

$$\log \hat{\varphi}_{l_n(q)}(t) \longrightarrow \log \varphi(t).$$

This implies that

$$\mathrm{Log}\, \hat{\varphi}_{l_n(q)}(t) \longrightarrow \mathrm{Log}\, \varphi(t),$$

i.e.,

$$i\Big(\sum_{j=1}^{l_n(q)} \mu_{nj} - A_{nl_n(q)}\Big)t + \sum_{j=1}^{l_n(q)} \int_{-\infty}^{\infty} f(t,x)dF_{nj}(x+\mu_{nj}) \rightarrow i\mu t + \int_{-\infty}^{\infty} f(t,x)dK(x)$$

which is equivalent to

$$i\Big(\sum_{j=1}^{l_n(q)} \mu_{nj} - A_{nl_n(q)}\Big)t + \int_{-\infty}^{\infty} f(t,x)d\,K_{l_n(q)}(x) \rightarrow i\mu t + \int_{-\infty}^{\infty} f(t,x)dK(x)$$

So (a) of Lemma 3.2.3 is satisfied where

$$\mu_n = \sum_{j=1}^{l_n(q)} \mu_{nj} - A_{nl_n(q)}.$$

By (b), it follows from Lemma 3.1.5 that

$$\sum_{j=1}^{l_n(q)} \sigma^2 \to \sigma^2.$$

This implies that $(K_{l_n(q)}(+\infty))$ is uniformly bounded. Therefore the condition (b) of Lemmma 3.2.3 is satisfied.

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In the following, we state and prove the main theorem of this chapter.

Theorem 3.2.6 Let $(Z_n;X_{nj})$ be a random double sequence of random variables which satisfies the condition $(\tilde{\alpha})$ and for each n, $Z_n, X_{n1}, X_{n2}, \ldots$ are independent. Then there exist a double sequence (A_{nj}) of real numbers and a distribution function F with finite variance σ^2 and the characteristic function φ such that

(i-a) the distribution functions of the random sums

$$S_{Z_n} = X_{n1} + X_{n2} + \ldots + X_{nZ_n} - A_{nZ_n}$$

converge weakly to F, and

(i-b) $\hat{\varphi}_{l_n(q)}(t) \to \varphi(t)$ for every $q \in (0,1)$ and every real number t where $\hat{\varphi}_{Z_n}$ are the random accompanying characteristic functions associated with (Z_n), (φ_{nj}) and (A_{nj}), and

(ii) $\sum_{j=1}^{Z_n} \sigma_{nj}^2 \xrightarrow{p} \sigma^2$

if and only if that there exist a function K in $\mathfrak{M}$ such that

(i') $K_{Z_n}(u) \xrightarrow{p} K(u)$ for every continuity point u of K and

(ii') $K_{Z_n}(+\infty) \xrightarrow{p} K(+\infty)$.

The real number A_{nk} may be chosen according to the formula

$$A_{nk} = \sum_{j=1}^{k} \mu_{nj} - \mu$$

where μ is any real number. The logarithm of characteristic function of the limit distribution is given by

$$\text{Log}\varphi(t) = i\mu t + \int_{-\infty}^{\infty} f(t,x)dK(x)).$$

Proof. By (i-b) and (ii), it follows from Lemma 3.3.5 that (i') holds and $K(+\infty) = \sigma^2$. According to this fact and Remark 3.3.4 (ii) we have (ii) is equivalent to (ii').

Conversely, let F be the distribution function whose the logarithm of characteristic function defined by

$$\text{Log}\varphi(t) = i\mu t + \int_{-\infty}^{\infty} f(t,x)dK(x)).$$

and $$A_{nk} = \sum_{j=1}^{k} \mu_{nj} - \mu.$$

By Theorem 1.3.5, $K(+\infty)$ is the variance of F. Hence (ii) follows from this fact and (ii').

To prove (i-b), let q be given and t be any real number. By Lemma3.1.5, it follows from (i') that

$$K_{l_n(q)} \xrightarrow{w} K.$$

By Theorem 1.2.12,

$$\int_{-\infty}^{\infty} f(t,x)dK_{l_n(q)}(x) \to \int_{-\infty}^{\infty} f(t,x)dK(x).$$

Since

$$A_{nl_n(q)} = \sum_{j=1}^{l_n(q)} \mu_{nj} - \mu,$$

we have

$$-it\, A_{nl_n(q)} + it \sum_{j=1}^{l_n(q)} \mu_{nj} + \int_{-\infty}^{\infty} f(t,x)dK_{l_n(q)}(x) \to it\mu + \int_{-\infty}^{\infty} f(t,x)dK(x).$$

So

$$\mathrm{Log}\hat{\varphi}_{l_n(q)}(t) \to \mathrm{Log}\varphi\,(t).$$

Hence,(i-b) hold.

We shall prove (i-a) by using Theorem 2.4.5. So we must to show that

(1) $(Z_n;X_{nj})$ satisfy the condition $(\tilde{\beta})$ and

(2) $E[\hat{\varphi}_{Z_n}(t)] \to \varphi(t)$ for every t.

By (ii), it follows from Lemma 3.2.1 that (1) holds.

To prove (2), let t be arbitrary but fixed. Since $|\hat{\varphi}_{1_n(q)}(t)| \leq 1$, hence

$\hat{\varphi}_{1_n(q)}$ considered as a function of q is dominated by a constant function. By (i-b), it follows from Lebegue Dominated Convergence Theorem that

$$\int_0^1 \hat{\varphi}_{1_n(q)}(t)dq \to \varphi(t).$$

By Lemma 3.1.4 we have (2).

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3.3 <u>A Speacialization</u>

In this section we specialization our main theorem to the cases where each n, Z_n has a single value, i.e., there exists a positive integer j_n such that

$$Z_n(\omega) = j_n$$

for every $\omega \in \Omega$. As a result, we obtain Theorem 1.4.2 (Kolmogorov Theorem).

Let (X_{nj}) be a double sequence of random variables with finite variance which are independent in each row and are defined on a common probability space $(\Omega, \mathcal{Q}, P)$. In this section, we again assume that (X_{nj}) is a sequence which is finite in each row, i.e., we assume that $j = 1, 2, \ldots, j_n,$ $n = 1, 2, \ldots$. Let (A_n) be a sequence of real numbers. For each n, let

$$S_n = X_{n1} + X_{n2} + \ldots + X_{nj_n} - A_n.$$

In order that S_n can be viewed as a random sums, we define Z_n and $\tilde{X}_{nj}$ as follows.

Let n be any positive integer. Define

$$Z_n(\omega) = j_n$$

for all $\omega \in \Omega$. For $j = 1, 2, \ldots, j_n$, define

$$\tilde{X}_{nj}(\omega) = X_{nj}(\omega)$$

for all $\omega \in \Omega$, and for $j > j_n$, define

$$\tilde{X}_{nj}(\omega) = 0$$

for all $\omega \in \Omega$.

It follows that $(Z_n;\tilde{X}_{nj})$ is a random double sequence of random variables which are independent in each row. We shall call $(Z_n;\tilde{X}_{nj})$, a random double sequence of random variables induced by (X_{nj}).

We shall denote the distribution function, characteristic function, mean and variance of $\tilde{X}_{nj}$ by $\tilde{F}_{nj}$, $\tilde{\varphi}_{nj}$, $\tilde{\mu}_{nj}$ and $\tilde{\sigma}^2_{nj}$ respectively. Observe that for each n and j be such that $j \le j_n$, we have

$$\tilde{F}_{nj} = F_{nj},$$
$$\tilde{\varphi}_{nj} = \varphi_{nj},$$
$$\tilde{\mu}_{nj} = \mu_{nj}$$

and

$$\tilde{\sigma}^2_{nj} = \sigma^2_{nj}.$$

Remark 3.3.1 In the following, we shall make uses of q-quatiles of the random variables Z_n defined above. For later reference, we note that

$$l_n(q) = j_n.$$

for every q in (0,1).

Lemma 3.3.2 Let (a_n) be a sequence of complex numbers. Let (j_n) be a sequence of positive integers. For each n, let Z_n be defined by

$$Z_n(\omega) = j_n$$

for every $\omega \in \Omega$. Then the following hold.

(i) $E[a_{Z_n}] = a_{j_n}$

(ii) $a_{j_n} \longrightarrow a$ if and only if $a_{Z_n} \xrightarrow{P} a$.

Proof.

Since

$$P(Z_n = j) = \begin{cases} 1 & \text{if } j = j_n, \\ 0 & \text{otherwise,} \end{cases}$$

hence we have

$$E[a_{Z_n}] = \sum_{k=1}^{\infty} P(Z_n = k) a_{j_k}$$

$$= a_{j_n},$$

i.e. (i) holds.

To prove (ii), first we assume that

$$a_{j_n} \longrightarrow a.$$

Let $\varepsilon > 0$ be given. So there exists an $n_o \in N$ such that

$$|a_{j_n} - a| < \varepsilon.$$

for all $n \geq n_o$. So, using the fact that $P(Z_n = j_n) = 1$ we have

$$P(|a_{Z_n} - a| \geq \varepsilon) = 0.$$

for all $n \geq n_o$. Hence

$$a_{Z_n} \xrightarrow{P} a.$$

Conversely, assume that

$$a_{Z_n} \xrightarrow{P} a.$$

Let $\varepsilon > 0$ be given. So

$$P(|a_{Z_n} - a| \geq \varepsilon) \to 0.$$

Hence there exists a positive integer n_o such that

$$P(|a_{Z_n} - a| \geq \varepsilon) < 1$$

for all $n \geq n_o$. Since Z_n is a single value,

$$P(|a_{Z_n} - a| \geq \varepsilon) = 0$$

for all $n \geq n_o$. So

$$P(|a_{Z_n} - a| < \varepsilon) = 1.$$

for all $n \geq n_o$. Therefore

$$|a_{j_n} - a| < \varepsilon$$

for all $n \geq n_o$. Hence

$$a_{j_n} \longrightarrow a.$$

#

<u>Proposition 3.3.3</u> Let (X_{nj}) be a double sequence of random variables. Let

$(Z_n;\widetilde{X}_{nj})$ be the random double sequence of random variables induced by (X_{nj}). If (X_{nj}) satisfies the condition (α), then $(Z_n;\widetilde{X}_{nj})$ satisfies the condition $(\widetilde{\alpha})$.

Proof. Let $\varepsilon > 0$ be given. Since (X_{nj}) satisfies the condition (α), we have

$$\sup_{1\le j\le j_n} P(|X_{nj}-\mu_{nj}| \ge \varepsilon) \to 0.$$

Let

$$a_{j_n} = \sup_{1\le j\le j_n} P(|X_{nj}-\mu_{nj}| \ge \varepsilon).$$

So

$$a_{j_n} \longrightarrow 0.$$

According to Lemma 3.3.2 (ii), we see that

$$a_{Z_n} \xrightarrow{P} 0$$

i.e.

$$\sup_{1\le j\le Z_n} P(|X_{nj}-\mu_{nj}| \ge \varepsilon) \xrightarrow{P} 0$$

This implies that $(Z_n;\widetilde{X}_{nj})$ satisfies the condition $(\widetilde{\alpha})$.

#

<u>Proposition 3.3.4</u> Let (X_{nj}) be a double sequence of random variables which are independent in each row. Let $(Z_n;\widetilde{X}_{nj})$ be a random double sequence of random variables induced by (X_{nj}). Let F be a distribution function. Then there exists a sequence (A_n) such that the distribution functions of the sums

$$S_n = X_{n1} + X_{n2} + \ldots + X_{nj_n} - A_n$$

converge weakly to F, if and only if there exists a double sequence (A_{nj}) of real numbers such that the distribution functions of the random sums

$$\widetilde{S}_{Z_n} = \widetilde{X}_{n1} + \widetilde{X}_{n2} + \ldots + \widetilde{X}_{nZ_n} - A_{nZ_n}$$

converge weakly to F.

Proof. Let φ be the characteristic function of F. According to fact that $P(Z_n = j_n) = 1$, we have the characteristic function $\widetilde{\varphi}_n$ of $\widetilde{S}_{Z_n}$ is given by

$$\widetilde{\varphi}_n(t) = E[\exp(-itA_{nZ_n})\prod_{j=1}^{Z_n} \widetilde{\varphi}_{nj}(t)]$$

$$= \sum_{k=1}^{\infty} P(Z_n = k)\exp(-itA_{nk})\prod_{j=1}^{k} \widetilde{\varphi}_{nj}(t)$$

$$= \exp(-itA_{nj_n})\prod_{j=1}^{j_n} \widetilde{\varphi}_{nj}(t)$$

$$= \exp(-itA_{nj_n})\prod_{j=1}^{j_n} \varphi_{nj}(t)$$

which is the characteristic function φ_n of S_n with $A_n = A_{nj_n}$. Hence

$$\varphi_n(t) \to \varphi(t)$$

for every real number t, if and only if

$$\widetilde{\varphi}_n(t) \to \varphi(t)$$

for every real number t. Therefore the distribution functions of S_n converge weakly to F if and only if the distribution function functions of $\tilde{S}_{Z_n}$ converge weakly to F.

#

Proposition 3.3.5 Let F be a distribution function with finite variance σ^2 and the characteristic function φ. Let (X_{nj}) be double sequence of random variables which are independent in each row and satisfy the condition (α). Let $(Z_n;\tilde{X}_{nj})$ be a random double sequence of random variables induced by (X_{nj}). Let (A_{nj}) be a sequence of real numbers. Let $\hat{\varphi}_{Z_n}$ be the random accompanying characteristic functions associated with (Z_n), $(\tilde{\varphi}_{nj})$ and (A_{nj}). Assume that

$$\sum_{j=1}^{Z_n} \tilde{\sigma}_{nj}^2 \xrightarrow{P} \sigma^2.$$

Then the distribution functions of the sums

$$\tilde{S}_{Z_n} = \tilde{X}_{n1} + \tilde{X}_{n2} + \ldots + \tilde{X}_{nZ_n} - A_{nZ_n}$$

converge weakly to F, if and only if

$$\hat{\varphi}_{l_n(q)}(t) \longrightarrow \varphi(t)$$

for every $q \in (0,1)$ and every real number t.

Proof. Since

$$\sum_{j=1}^{Z_n} \tilde{\sigma}_{nj}^2 \xrightarrow{P} \sigma^2,$$

it follows from Lemma 3.2.1 that $(Z_n,\widetilde{X}_{nj})$ satisfy the condition $(\widetilde{\beta})$. So $(Z_n;\widetilde{X}_{nj})$ satisfies both $(\widetilde{\alpha})$ and $(\widetilde{\beta})$. Therefore, by Theorem 2.4.5, the distribution functions of the sums

$$\widehat{S}_{Z_n} = \widetilde{X}_{n1} + \widetilde{X}_{n2} + \ldots + \widetilde{X}_{nZ_n} - A_{nZ_n}$$

converge weakly to F, if and only if

$$E[\widehat{\varphi}_{Z_n}(t)] \longrightarrow \varphi(t)$$

for every real number t.

Since

$$Z_n(\omega) = j_n$$

for every $\omega \in \Omega$. It follows from Lemma 3.3.2 (i) that

$$\widehat{\varphi}_{j_n}(t) \rightarrow \varphi(t)$$

for every t. Hence, by Remark 3.3.1, we have

$$\widehat{\varphi}_{l_n(q)}(t) \longrightarrow \varphi(t)$$

for every $q \in (0,1)$ and every real number t.

#

Remark 3.3.6 Let (X_{nj}) be a double sequence of random variables which are independent in each row and satisfies the condition (α). Let $(Z_n;\widetilde{X}_{nj})$ be a random double sequence of random variables induced by (X_{nj}). Then the following hold

(1) By Lemma 3.3.3, $(Z_n;\widetilde{X}_{nj})$ satisfy the condition $(\widetilde{\alpha})$. According to Theorem 3.2.6 we know that in order that there exist a double sequence (A_{nj}) of real numbers and a distribution function F with finite variance σ^2 and the characteristic function φ such that

(i-a) the distribution functions of the random sums

$$\tilde{S}_{Z_n} = \tilde{X}_{n1} + \tilde{X}_{n2} + \ldots + \tilde{X}_{nZ_n} - A_{nZ_n}$$

converge weakly to F, and

(i-b) $\hat{\hat{\varphi}}_{1_n(q)}(t) \to \varphi(t)$ for every $q \in (0,1)$ and real number t, where $\hat{\hat{\varphi}}_{Z_n}$ are the random accompanying characteristic function associated with (Z_n), (φ_{nj}) and (A_n), and

(ii) $\sum_{j=1}^{Z_n} \tilde{\sigma}_{nj}^{2} \xrightarrow{P} \sigma^2$

it necessary and sufficient that there exists a function K in $\mathfrak{M}$ such that

(i') $\tilde{K}_{Z_n}(u) \xrightarrow{P} K(u)$, for every continuity point u of K, and

(ii') $\tilde{K}_{Z_n}(+\infty) \xrightarrow{P} K(+\infty)$,

where

$$\tilde{K}_{Z_n}(u) = \sum_{j=1}^{Z_n} \int_{-\infty}^{u} x^2 d\tilde{F}_{nj}(x + \tilde{\mu}_{nj}).$$

The real number A_{nk} may be chosen according to the formula

$$A_{nk} = \sum_{j=1}^{k} \tilde{\mu}_{nj} - \mu$$

where μ is any real number. The logarithm of the characteristic function of the limit distribution function is given by

$$\text{Log}\varphi(t) = i\mu t + \int_{-\infty}^{\infty} f(t,x)dK(x).$$

(2) By Proposition 3.3.5 we see that the condition (i-a) and (i-b) of (1) are equivalent. Hence by (1), in order that there exist a double sequence (A_{nj}) of real numbers and a distribution function F with finite variance σ^2 and the characteristic function φ such that

(i) the distribution functions of the random sums

$$\tilde{S}_{Z_n} = \tilde{X}_{n1} + \tilde{X}_{n2} + \ldots + \tilde{X}_{nZ_n} - A_{nZ_n}$$

converge weakly to F, and

(ii) $\sum_{j=1}^{Z_n} \tilde{\sigma}_{nj}^{\,2} \xrightarrow{P} \sigma^2$

it is necessary and sufficient that there exists a function K in $\mathfrak{M}$ such that

(i') $\tilde{K}_{Z_n}(u) \xrightarrow{P} K(u)$, for every continuity point u of K, and

(ii') $\tilde{K}_{Z_n}(+\infty) \xrightarrow{P} K(+\infty)$.

(3) It follows from Proposition 3.3.4 that the condition (i) of (2) is equivalent to the condition (K-i).

(K-i) The distribution functions of the sums

$$S_n = X_{n1} + X_{n2} + \ldots + X_{nj_n} - A_n$$

converge weakly to F.

And, it follows from Lemma 3.3.2 (ii) that the conditions (ii) of (2) is equivalent to the condition (K-ii).

(K-ii) $\sum_{j=1}^{j_n} \sigma_{nj}^2 \longrightarrow \sigma^2$.

By using from Lemma 3.3.2 (ii), it follows that

(a) the condition (i') of (2) is equivalent to the condition (K-i').

(K-i') $K_{j_n}(u) \to K(u)$ for every continuity point u of K,

(b) the condition (ii') of (2) is equivalent to the condition (K-ii')

(ii'-K) $K_{j_n}(+\infty) \to K(+\infty)$.

Therefore Theorem 1.4.2 is a consequence of Theorem 3.2.6.

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