CHAPTER II

RANDOM SUMS OF RANDOM VARIABLES AND THEIR ACCONPANYING DISTRIBUTION FUNCTIONS.

In this chapter we generalize a necessary and sufficient condition for convergence of sums of independent random variables as stated in Theorem 1.4.1 to the case in which the number of term in the sums are random. This is done by using the concepts of random infinitesimal and accompanying distribution functions of random sums.

2.1 Random Sums of Random Variables

Let (Z_n) be a sequence of positive integral-valued random vaiables. Let (X_{nj}) be a double sequence of complex-valued random variables. Here our double sequence is infinite in both directions, i.e., n = 1, 2, 3,..., and j = 1, 2, 3,....For each n, a value $Z_n(\omega)$ of Z_n determines a finite sequence of values

$$X_{n1}(\omega), X_{n2}(\omega),..., X_{n\,Z_n(\omega)}(\omega)$$

of $X_{n1}, X_{n2},..., X_{n\,Z_n(\omega)}$. It can be seen that for each n, Z_n and (X_{nj}) together define a random experiment in which each outcome gives rise to a finite sequence of complex numbers. However, the length of this finite sequence is random. We shall call the system $(Z_n;X_{nj})$, a <u>random double sequence of complex-valued random variables.</u>

Let $(Z_n;X_{nj})$ be a random double sequence of complex-valued random variables. For each n we define

$$\sum_{j=1}^{Z_n} X_{nj}, \quad \prod_{j=1}^{Z_n} X_{nj}, \quad \text{and} \quad X_{nZ_n}$$

to be functions from Ω to $\mathbb{C}$ given by the following formulas

$$\left(\sum_{j=1}^{Z_n} X_{nj}\right)(\omega) = \left(\sum_{j=1}^{Z_n(\omega)} X_{nj}\right)(\omega)$$

$$\left(\prod_{j=1}^{Z_n} X_{nj}\right)(\omega) = \left(\prod_{j=1}^{Z_n(\omega)} X_{nj}\right)(\omega)$$

and

$$(X_{nZ_n})(\omega) = (X_{nZ_n(\omega)})(\omega)$$

respectively.

In case X_{nj}'s are real-valued random variables we define

$$\sup_{1\le j\le Z_n} X_{nj}$$

to be the function from Ω to $\mathbb{R}$ given by

$$\left(\sup_{1\le j\le Z_n} X_{nj}\right)(\omega) = \left(\sup_{1\le j\le Z_n(\omega)} X_{nj}\right)(\omega).$$

It will be shown that $\sum_{j=1}^{Z_n} X_{nj}$, $\prod_{j=1}^{Z_n} X_{nj}$, and X_{nZ_n} are complex-valued random variables and $\sup_{1\le j\le Z_n} X_{nj}$ is a real-valued random variable. These facts are special cases of a more general result that follows.

Proposition 2.1.1 Let (Y_k) be a sequence of complex-valued random variables. Let Z be a positive intgral-valued random variable. Let Y_Z denote a function from Ω to $\mathbb{C}$ defined by

$$Y_Z(\omega) = (Y_{Z(\omega)})(\omega)$$

for all $\omega \in \Omega$. Then Y_Z is a complex-valued random variable.

Proof. Let B be a Borel subset of $\mathbb{C}$. By a straight forword verifcation, it can be shown that

$$Y_Z^{-1}[B] = \bigcup_{k \in N} (Y_k^{-1}[B] \cap Z^{-1}[\{k\}]),$$

which can be seen to be a measurable set. Hence Y_Z is a complex-valued random variable.

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Theorem 2.1.2 Let $(Z_n; X_{nj})$ be a random double sequence of complex-valued random variables. For each n, $\sum_{j=1}^{Z_n} X_{nj}$, $\prod_{j=1}^{Z_n} X_{nj}$, and X_{nZ_n} are complex-valued random variables. Furthermore, in case where the X_{nj}'s are real-valued random variables $\sup_{1 \le j \le Z_n} X_{nj}$ is a real-valued random variable.

Proof. The assertions of the theorem follow from Proposition 2.1.1 by defining Y_k to be $\sum_{j=1}^{k} X_{nj}$, $\prod_{j=1}^{k} X_{nj}$, X_{nk} and $\sup_{1 \le j \le k} X_{nj}$ respectively.

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For each complex number A, we shall associate a complex-valued random variable whose value is A for every sample point $\omega \in \Omega$. We shall denote such a complex-valued random variable by A. By using this interpretation, any sequence or double sequence of complex numbers may be considered as sequence or double sequence of complex-valued random variables.

In the sequel, we shall consider sums of the form

$$S_{Z_n} = X_{n1} + X_{n2} + \ldots + X_{nZ_n} - A_{nZ_n}$$

where $(Z_n;X_{nj})$ is a random double sequence of random variables and (A_{nj}) is a double sequence of real numbers. We shall refer to them as <u>random sums.</u>

Let (Z_n) be a sequence of positive integral-valued random variables and (φ_{nj}) be a double sequence of characteristic functions. Let (A_{nj}) be a double sequence of real numbers. We define a function

$$\varphi_{Z_n} : \Omega \times \mathbb{R} \to \mathbb{C}$$

by

$$\varphi_{Z_n}(\omega,t) = \exp(-it\, A_{nZ_n(\omega)}) \prod_{j=1}^{Z_n(\omega)} \varphi_{nj}(t).$$

We will denote $\varphi_{Z_n}(\omega,t)$ by $\varphi_{Z_n(\omega)}(t)$.

<u>Proposition 2.1.3</u> Let φ_{Z_n} be defined as above. Then the following hold.

(i) For each n and t, $\varphi_{Z_n}(t)$ is a complex-valued random variable.

(ii) For each n and ω, $\varphi_{Z_n(\omega)}$ is a characteristic function.

(iii) For each n, the function φ_n given by

$$\varphi_n(t) = E[\varphi_{Z_n}(t)]$$

is a characteristic function.

Proof.

(i) follows from Theorem 2.1.2

(ii) follows from the fact that $\exp(-it\, A_{nZ_n(\omega)})$ and φ_{nj} are characteristic functions together with Proposition 1.2.4 (i).

(iii) follows from Proposition 1.2.6 and the fact that

$$E[\varphi_{Z_n}(t)] = \sum_{k=1}^{\infty} P(Z_n=k)\exp(-itA_{nk}) \prod_{j=1}^{k} \varphi_{nj}(t).$$

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We shall call such φ_{Z_n}, the random characteristic function associated with (Z_n),(φ_{nj}) and(A_{nj}).

W shall frequently make uses of the distribution function and characteristic function of X_{nj}. For convenience, we shall denote them by F_{nj} and φ_{nj} respectively.

Theorem 2.1.4 Let (Z_n;X_{nj}) be a random double sequence of complex-valued random variables such that for each n, Z_n,X_{n1},X_{n2},...are independent. Let (A_{nj}) be a double sequence of real numbers. Then the characteristic functions φ_n of the random sums

$$S_{Z_n} = X_{n1} + X_{n2} + \ldots + X_{nZ_n} - A_{nZ_n}$$

are given by

$$\psi_n(t) = E[\varphi_{Z_n}(t)]$$

where φ_{Z_n} are the random characteristic function associated with (Z_n), (φ_{nj}) and (A_{nj}).

Proof. Fix $n \in N$. Let F_n be the distribution function of S_{Z_n}. For each j, let F_n^j and φ_n^j be the distribution function and the characteristic function of $S_n^j = X_{n1}+X_{n2}+\ldots+X_{nj}-A_{nj}$ respectively. Observe that

$$F_n(x) = P(S_{Z_n} \le x)$$

$$= \sum_{j=1}^{\infty} P(S_n^j \le x \wedge Z_n = j)$$

$$= \sum_{j=1}^{\infty} P(S_n^j \le x)P(Z_n = j)$$

$$= \sum_{j=1}^{\infty} P(Z_n = j)F_n^j(x).$$

By Proposition 1.2.6, we have

$$\psi_n(t) = \sum_{j=1}^{\infty} P(Z_n=j)\varphi_n^j(t).$$

Since $$\varphi_n^j(t) = \exp(-it\, A_{nj}) \prod_{k=1}^{j} \varphi_{nk}(t),$$

we have

$$\psi_n(t) = \sum_{j=1}^{\infty} P(Z_n=j)\exp(-it\, A_{nj}) \prod_{k=1}^{j} \varphi_{nk}(t),$$

$$= E[\varphi_{Z_n}(t)].$$

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2.2 Random Infinitesimal.

We shall say that (X_{nj}) is random infinitesimal with respect to (Z_n) if for every $\varepsilon > 0$, we have

$$\sup_{1 \le j \le Z_n} P(|X_{nj}| \ge \varepsilon) \xrightarrow{P} 0.$$

Lemma 2.2.1 Let (X_{nj}) be a double sequence of random variables. Then, for any n, k and $\varepsilon > 0$, the following hold.

(i) $$\sup_{1 \le j \le k} \int_{\infty}^{\infty} \frac{x^2}{1+x^2} dF_{nj}(x) \le \frac{\varepsilon}{2} + \sup_{1 \le j \le k} P(|X_{nj}| \ge \sqrt{\frac{\varepsilon}{2}}).$$

(ii) For any real number $t \neq 0$, we have

$$\sup_{1 \le j \le k} |\varphi_{nj}(t) - 1| \le \frac{\varepsilon}{2} + 2 \sup_{1 \le j \le k} P(|X_{nj}| \ge \frac{\varepsilon}{2|t|}).$$

Proof. Observe that

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^2} dF_{nj}(x) = \int_{|x| < \sqrt{\frac{\varepsilon}{2}}} \frac{x^2}{1+x^2} dF_{nj}(x) + \int_{|x| \ge \sqrt{\frac{\varepsilon}{2}}} \frac{x^2}{1+x^2} dF_{nj}(x)$$

$$\le \int_{|x| < \sqrt{\frac{\varepsilon}{2}}} \frac{\varepsilon}{2} dF_{nj}(x) + \int_{|x| \ge \sqrt{\frac{\varepsilon}{2}}} dF_{nj}(x)$$

$$\leq \quad \frac{\varepsilon}{2} + P(|X_{nj}| \geq \sqrt{\frac{\varepsilon}{2}}).$$

Therefore, we have (i).

To prove (ii) observe that

$$|\varphi_{nj}(t)-1| = \left| \int_{-\infty}^{\infty} (e^{itx}-1)dF_{nj}(x) \right|$$

$$\leq \int_{|x|<\frac{\varepsilon}{2|t|}} |e^{itx}-1|\, dF_{nj}(x) + \int_{|x|\geq\frac{\varepsilon}{2|t|}} |e^{itx}-1| dF_{nj}(x)$$

$$\leq \int_{|x|<\frac{\varepsilon}{2|t|}} |tx| dF_{nj}(x) + \int_{|x|\geq\frac{\varepsilon}{2|t|}} (|e^{itx}|+1) dF_{nj}(x)$$

$$\leq \quad \frac{\varepsilon}{2} + 2 \sup_{1\leq j\leq k} P(|X_{nj}| \geq \frac{\varepsilon}{2|t|}).$$

Therefore (ii) follows.

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<u>Theorem 2.2.2</u> Let $(Z_n; X_{nj})$ be a random double sequence of complex-valued random variables. Then the following statements are equivalent .

(i) (X_{nj}) is random infinitesimal with respect to (Z_n) ,

(ii) for any $\varepsilon > 0$, $\sup_{1\leq j\leq Z_n} P(|X_{nj}|\geq\varepsilon) \xrightarrow{m} 0$,

(iii) for any $\varepsilon > 0$, $E[\sup_{1\leq j\leq Z_n} P(|X_{nj}|\geq\varepsilon)] \longrightarrow 0$,

(iv) $$E[\sup_{1\le j\le Z_n} \int_{-\infty}^{\infty} \frac{x^2}{1+x^2} dF_{nj}(x)] \longrightarrow 0,$$

(v) $$\sup_{1\le j\le Z_n} \int_{-\infty}^{\infty} \frac{x^2}{1+x^2} dF_{nj}(x) \xrightarrow{m} 0.$$

Proof.

The equivalence of (i) and (ii) follows from Theorem 1.1.5.

The equivalence of (ii) and (iii) follows from the fact that

$$E[|\sup_{1\le j\le Z_n} P(|X_{nj}|\ge\varepsilon) - 0|] = E[\sup_{1\le j\le Z_n} P(|X_{nj}|\ge\varepsilon)].$$

The equivalence of (iv) and (v) follows from the fact that

$$E[|\sup_{1\le j\le Z_n} \int_{-\infty}^{\infty} \frac{x^2}{1+x^2} dF_{nj}(x) - 0|] = E[\sup_{1\le j\le Z_n} \int_{-\infty}^{\infty} \frac{x^2}{1+x^2} dF_{nj}(x)].$$

So it remains to show that (iii) is equivalent to (iv) .

First we assume (iii) holds. By Lemma 2.2.1(i), we have that for any $\varepsilon > 0$,

$$E[\sup_{1\le j\le Z_n} \int_{-\infty}^{\infty} \frac{x^2}{1+x^2} dF_{nj}(x)] \le \frac{\varepsilon}{2} + E[\sup_{1\le j\le Z_n} P(|X_{nj}|\ge\sqrt{\frac{\varepsilon}{2}})]$$

Using this fact together with (iii) , it can be shown that

$$E[\sup_{1\le j\le Z_n}\int_{-\infty}^{\infty}\frac{x^2}{1+x^2}dF_{nj}(x)]\longrightarrow 0.$$

Conversely, we assume (iv) holds. Let $\varepsilon > 0$ be given. From the fact that for $|x| \ge \varepsilon$,

$$\frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2} \ge 1 - \frac{1}{1+\varepsilon^2} = \frac{\varepsilon^2}{1+\varepsilon^2}$$

we have

$$E[\sup_{1\le j\le Z_n} P(|X_{nj}| \ge \varepsilon)] = E[\sup_{1\le j\le Z_n}\int_{|x|\ge\varepsilon} 1\, dF_{nj}(x)]$$

$$\le E[\sup_{1\le j\le Z_n}\int_{|x|\ge\varepsilon}(\frac{x^2}{1+x^2})(\frac{1+\varepsilon^2}{\varepsilon^2})dF_{nj}(x)]$$

$$= \frac{1+\varepsilon^2}{\varepsilon^2}E[\sup_{1\le j\le Z_n}\int_{|x|\ge\varepsilon}(\frac{x^2}{1+x^2})dF_{nj}(x)]$$

which converges to 0 by (iv).

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<u>Theorem 2.2.3</u> Let (Z_n, X_{nj}) be a random double sequence of random variables. If (X_{nj}) is random infinitesimal with respect to (Z_n), then

$$\sup_{1\le j\le Z_n}|\varphi_{nj}(t)-1|\xrightarrow{P} 0$$

for every t.

Proof. Clearly $\sup_{1\le j\le Z_n}|\varphi(t)-1|\xrightarrow{P} 0$ holds for $t = 0$. Assume that $t \ne 0$. Let

$\varepsilon > 0$ be given. By Lemma2.2.1(ii), we have that

(1) $$E[\sup_{1\le j\le Z_n} |\varphi_{nj}(t)-1|] \le \frac{\varepsilon}{2} + 2E[\sup_{1\le j\le Z_n} P(|X_{nj}|\ge\frac{\varepsilon}{2|t|})].$$

Since (X_{nj}) is random infinitesimal with respect to (Z_n), by Theorem2.2.2 we have

$$\lim_{n\to\infty} E[\sup_{1\le j\le Z_n} P(|X_{nj}|\ge\frac{\varepsilon}{2|t|})] \to 0.$$

Using this fact together with (1), it can be shown that

$$E[\sup_{1\le j\le Z_n} |\varphi_{nj}(t)-1|] \to 0.$$

That is $$\sup_{1\le j\le Z_n} |\varphi_{nj}(t)-1| \xrightarrow{m} 0.$$

By Theorem 1.1.5(i) we see that

$$\sup_{1\le j\le Z_n} |\varphi_{nj}(t)-1| \xrightarrow{P} 0.$$

For the remainder of this work we assume futher that for each n and j, X_{nj} has finite variance. We shall denote its mean and variance by μ_{nj} and σ^2_{nj} respectively.

2.3 Accompanying Distribution Functions of Random Sums.

In this section, we let (Z_n, X_{nj}) be a random double sequence of random variables. Let

$$S_{Z_n} = X_{n1} + X_{n2} + \ldots + X_{nZ_n} - A_{nZ_n}$$

where (A_{nj}) is a double sequence of real numbers. For each n and k, let φ_n^k be the characteristic function of the accompanying distribution function of

$$X_{n1} + X_{n2} + \ldots + X_{nk} - A_{nk}.$$

Therefore,

$$\text{Log}\varphi_n^k(t) = -itA_{nk} + it\sum_{j=1}^{k}\mu_{nj} + \sum_{j=1}^{k}\int_{-\infty}^{\infty}(e^{itx}-1)dF_{nj}(x+\mu_{nj}).$$

For each n, let

$$\hat{\varphi}_{Z_n}: \Omega \times \mathbb{R} \longrightarrow \mathbb{C}$$

be defined by

$$\hat{\varphi}_{Z_n}(\omega,t) = (\varphi_n^{Z_n(\omega)})(t).$$

We shall call such $\hat{\varphi}_{Z_n}$, the random accompanying characteristic function associated with (Z_n), (φ_{nj}) and (A_{nj}). We will denote $\hat{\varphi}_{Z_n}(\omega,t)$ by $\hat{\varphi}_{Z_n(\omega)}$ (t).

Remark 2.3.1. For each n and ω, $\hat{\varphi}_{Z_n(\omega)}$ is infinitely divisible.

The accompanying distribution function of S_{Z_n} is the distribution function whose characteristic function is given by

$$\hat{\varphi}_n(t) = E[\hat{\varphi}_{Z_n}(t)]$$

where $\hat{\varphi}_{Z_n}$ is the random accompanying characteristic function associated with (Z_n), (φ_{nj}) and (A_{nj}).

Note that the accompanying distribution function of S_{Z_n} may not be infintiely divisible.

2.4 A Necessary and Sufficient Condition for Convergence.

Let $(Z_n;X_{nj})$ be a random double sequence of random variables. Let

$$S_{Z_n} = X_{n1} + X_{n2} + \ldots + X_{nZ_n} - A_{nZ_n}$$

where (A_{nj}) is a double sequence of real numbers. In this section, we generalize Kolmogorov's result on accompanying distribution functions (Theorem 1.4.1) to the case of random sums. To do this we shall generalize conditions (α) and (β) used there. Our generalized conditions are the following.

$(\tilde{\alpha})$ $(X_{nj}-\mu_{nj})$ is random infinitesimal with respect to (Z_n).

$(\tilde{\beta})$ There exists a constant c>0 such that $P(\sum_{j=1}^{Z_n} \sigma_{nj}^2 \geq c) \to 0$.

In Lemma 2.4.1 - Lemma 2.4.3, we assume that for every real number $t, \varphi_{nj}(t)$ is non-zero.

Lemma 2.4.1 Let $(Z_n;X_{nj})$ be a random double sequence of random variables. Then for every $\varepsilon > 0$ and $\gamma \in (0,\frac{1}{2})$ we have

$$P\Big(\sum_{j=1}^{Z_n} |\operatorname{Log}\varphi_{nj}(t)-\varphi_{nj}(t)+1| \geq \varepsilon\Big) \leq P\Big(\sup_{1\leq j\leq Z_n} |\varphi_{nj}(t)-1| \geq \gamma\Big)$$

$$+ P\Big(\sum_{j=1}^{Z_n} |\varphi_{nj}(t)-1|^2 \geq \varepsilon \wedge \sup_{1\leq j\leq Z_n} |\varphi_{nj}(t)-1| < \gamma\Big)$$

for every t.

Proof. Let t be any real number. Let

$$A_\varepsilon = \{\omega \in \Omega | \sum_{j=1}^{Z_n(\omega)} |\mathrm{Log}\varphi_{nj}(t) - \varphi_{nj}(t) + 1| \geq \varepsilon \}$$

and $$B_\gamma = \{\omega \in \Omega | \sup_{1 \leq j \leq Z_n(\omega)} |\varphi_{nj}(t) - 1| \geq \gamma \}$$

Let ϖ be any element of Ω-B_γ. Then we have

$$\sum_{j=1}^{Z_n(\varpi)} |\mathrm{Log}\varphi_{nj}(t) - \varphi_{nj}(t) + 1| = \sum_{j=1}^{Z_n(\varpi)} |\mathrm{Log}(1 + \varphi_{nj}(t) - 1) - (\varphi_{nj}(t) - 1)|$$

$$\leq \sum_{j=1}^{Z_n(\varpi)} |\varphi_{nj}(t) - 1|^2.$$

The last inequality follows from the fact that $|\mathrm{Log}(1+z) - z| \leq |z|^2$ for all z such that

$|z| < \gamma < \frac{1}{2}$. Hence for $\omega \in A_\varepsilon \cap (\Omega - B\gamma)$,

$$\sum_{j=1}^{Z_n(\omega)} |\varphi_{nj}(t) - 1|^2 \geq \varepsilon$$

which imples that

$$(1)\ A_\varepsilon \cap (\Omega - B\gamma) \subseteq \{\omega \in \Omega | \sum_{j=1}^{Z_n(\omega)} |\varphi_{nj}(t) - 1|^2 \geq \varepsilon \wedge \sup_{1 \leq j \leq Z_n} |\varphi_{nj}(t) - 1| < \gamma \}.$$

From the fact that

$$A_\varepsilon = (A_\varepsilon \cap B\gamma) \cup (A_\varepsilon \cap (\Omega - B\gamma))$$

and (1) we have that

$$P(A_\varepsilon) \le P(A_\varepsilon \cap B\gamma) + P(\{\omega \in \Omega | \sum_{j=1}^{Z_n(\omega)} |\varphi_{nj}(t)-1|^2 \ge \varepsilon \wedge \sup_{1 \le j \le Z_n} |\varphi_{nj}(t)-1| < \gamma\})$$

$$\le P(B\gamma) + P(\{\omega \in \Omega | \sum_{j=1}^{Z_n(\omega)} |\varphi_{nj}(t)-1|^2 \ge \varepsilon \wedge \sup_{1 \le j \le Z_n} |\varphi_{nj}(t)-1| < \gamma\})$$

So we have the conclusion of the lemma.

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Note that, in Lemma 2.4.1 the finiteness of variances of X_{nj}'s are not assumed.

<u>Lemma 2.4.2</u> Let $(Z_n;X_{nj})$ be a random double sequence of random variables. Assume that for each n and j, X_{nj} has zero mean. Then for every $\varepsilon > 0$ and $\gamma > 0$ we have

$$P(\sum_{j=1}^{Z_n} |\varphi_{nj}(t)-1|^2 \ge \varepsilon \wedge \sup_{1 \le j \le Z_n} |\varphi_{nj}(t)-1| < \gamma) \le P(\sum_{j=1}^{Z_n} \sigma_{nj}^2 \ge \frac{2\varepsilon}{\gamma t^2})$$

for every $t \ne 0$.

Proof.

$$P(\sum_{j=1}^{Z_n} |\varphi_{nj}(t)-1|^2 \ge \varepsilon \wedge \sup_{1 \le j \le Z_n} |\varphi_{nj}(t)-1| < \gamma)$$

$$\le P(\gamma \sum_{j=1}^{Z_n} |\varphi_{nj}(t)-1| \ge \varepsilon)$$

$$= P\left(\sum_{j=1}^{Z_n} |\varphi_{nj}(t)-1| \geq \frac{\varepsilon}{\gamma}\right)$$

$$= P\left(\sum_{j=1}^{Z_n} \left|\int_{-\infty}^{\infty}(e^{itx}-1)dF_{nj}(x)\right| \geq \frac{\varepsilon}{\gamma}\right)$$

$$= P\left(\sum_{j=1}^{Z_n} \left|\int_{-\infty}^{\infty}(e^{itx}-1-itx)dF_{nj}(x)\right| \geq \frac{\varepsilon}{\gamma}\right)$$

$$\leq P\left(\sum_{j=1}^{Z_n} \int_{-\infty}^{\infty}|(e^{itx}-1-itx)|dF_{nj}(x) \geq \frac{\varepsilon}{\gamma}\right)$$

$$\leq P\left(\sum_{j=1}^{Z_n} \int_{-\infty}^{\infty}\frac{t^2x^2}{2}dF_{nj}(x) \geq \frac{\varepsilon}{\gamma}\right)$$

$$= P\left(\sum_{j=1}^{Z_n} \int_{-\infty}^{\infty}x^2dF_{nj}(x) \geq \frac{2\varepsilon}{\gamma t^2}\right)$$

$$= P\left(\sum_{j=1}^{Z_n} \sigma_{nj}^2 \geq \frac{2\varepsilon}{\gamma t^2}\right)$$

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Lemma 2.4.3 Let $(Z_n;X_{nj})$ be a random double sequence of random variables satisfying the conditions $(\tilde{\alpha})$ and $(\tilde{\beta})$. Assume that for every n and j, X_{nj} has zero mean. Then

$$\sum_{j=1}^{Z_n} |\mathrm{Log}\varphi_{nj}(t)-\varphi_{nj}(t)-1| \xrightarrow{P} 0$$

for every t.

Proof. If t=0 then done. Suppose that $t \neq 0$. Let $\varepsilon > 0$ be given. By condition (β) there exists $c > 0$ such that

(1) $$P\left(\sum_{j=1}^{Z_n} \sigma_{nj}^2 \geq c\right) \to 0.$$

Let γ be a positive real number such that $\gamma < \min\{\frac{1}{2}, \frac{2\varepsilon}{ct^2}\}$. By Lemma 2.4.1 and Lemma 2.4.2 we have

$$P\left(\sum_{j=1}^{Z_n} |\mathrm{Log}\varphi_{nj}(t)-\varphi_{nj}(t)+1| \geq \varepsilon\right) \leq P\left(\sup_{1\leq j\leq Z_n} |\varphi_{nj}(t)-1| \geq \gamma\right) + P\left(\sum_{j=1}^{Z_n} \sigma_{nj}^2 \geq \frac{2\varepsilon}{\gamma t^2}\right).$$

Since $\gamma \leq \frac{2\varepsilon}{ct^2}$, we have $P\left(\sum_{j=1}^{Z_n} \sigma_{nj}^2 \geq \frac{2\varepsilon}{\gamma t^2}\right) \leq P\left(\sum_{j=1}^{Z_n} \sigma_{nj}^2 \geq c\right)$.

Hence

(2) $$P\left(\sum_{j=1}^{Z_n} |\mathrm{Log}\varphi_{nj}(t)-\varphi_{nj}(t)+1| \geq \varepsilon\right) \leq P\left(\sup_{1\leq j\leq Z_n} |\varphi_{nj}(t)-1| \geq \gamma\right) + P\left(\sum_{j=1}^{Z_n} \sigma_{nj}^2 \geq c\right).$$

Since (X_{nj}) satisfy the condition ($\tilde{\alpha}$), it follows from Theorem 2.2.3 that

(3) $$P\left(\sup_{1\leq j\leq Z_n} |\varphi_{nj}(t)-1| \geq \gamma\right) \to 0.$$

From (1), (2) and (3) we have

$$\sum_{j=1}^{Z_n} |\mathrm{Log}\varphi_{nj}(t)-\varphi_{nj}(t)-1| \xrightarrow{P} 0$$

#

<u>Theorem 2.4.4</u>. Let $(Z_n;X_{nj})$ be a random double sequence of random

variables satisfying the conditions $(\tilde{\alpha})$ and $(\tilde{\beta})$. Let

$$S_{Z_n} = X_{n1} + X_{n2} + \ldots + X_{nZ_n} - A_{nZ_n}$$

where (A_{nj}) be a double sequence of real numbers. Let φ_{Z_n} be the random characteristic function associated with (Z_n),(φ_{nj}) and (A_{nj}) and $\hat{\varphi}_{Z_n}$ be the random accompanying characteristic function associated with (Z_n),(φ_{nj}) and (A_{nj}). Then

$$|\varphi_{Z_n}(t) - \hat{\varphi}_{Z_n}(t)| \xrightarrow{m} 0.$$

for every real number t.

Proof. Let t be given. Since $(Z_n;X_{nj})$ satisfies condition $(\tilde{\alpha})$, by Theorem 2.2.3 we have

$$P\Big(\sup_{1 \le j \le Z_n} |\varphi_{nj}(t) - 1| \ge \frac{1}{2} \Big) \to 0.$$

Since

$$P\Big(\prod_{j=1}^{Z_n} \varphi_{nj}(t) = 0 \Big) \le P\Big(\sup_{1 \le j \le Z_n} |\varphi_{nj}(t) - 1| \ge \frac{1}{2} \Big),$$

we have $P(\varphi_{Z_n}(t) = 0) \to 0$. Hence we can assume that $\varphi_{Z_n(\omega)}(t) \ne 0$ for every n and ω, i.e. $\varphi_{nj}(t) \ne 0$ for every n and j.

For each n and j, Let $\mathring{X}_{nj} = X_{nj} - \mu_{nj}$ and $\mathring{\varphi}_{nj}$ be the characteristic function of X_{nj}. Therefore $\mathring{\varphi}_{nj}(t) = e^{-it\mu_{nj}}\varphi_{nj}(t)$.

Observe that

$$\left| -itA_{nZ_n} + \sum_{j=1}^{Z_n} \mathrm{Log}\varphi_{nj}(t) + itA_{nZ_n} - it\sum_{j=1}^{Z_n} \mu_{nj} - \sum_{j=1}^{Z_n} \int_{-\infty}^{\infty} (e^{itx}-1)dF_{nj}(x+\mu_{nj}) \right|$$

$$= \left| \sum_{j=1}^{Z_n} \mathrm{Log}\varphi_{nj}(t) - it \sum_{j=1}^{Z_n} \mu_{nj} - \sum_{j=1}^{Z_n} \int_{-\infty}^{\infty} (e^{itx}-1)dF_{nj}(x+\mu_{nj}) \right|$$

$$= \left| \sum_{j=1}^{Z_n} \mathrm{Log}(e^{it\mu_{nj}} \hat{\varphi}_{nj}(t)) - it \sum_{j=1}^{Z_n} \mu_{nj} - \sum_{j=1}^{Z_n} \int_{-\infty}^{\infty} (e^{itx}-1)dF_{nj}(x+\mu_{nj}) \right|$$

$$= \left| \sum_{j=1}^{Z_n} \mathrm{Log}\mathring{\varphi}_{nj}(t) - \sum_{j=1}^{Z_n} \int_{-\infty}^{\infty} (e^{itx}-1)dF_{nj}(x+\mu_{nj}) \right|$$

$$= \left| \sum_{j=1}^{Z_n} \mathrm{Log}\mathring{\varphi}_{nj}(t) - \sum_{j=1}^{Z_n} \left(\int_{-\infty}^{\infty} e^{itx}dF_{nj}(x+\mu_{nj}) - 1 \right) \right|$$

$$= \left| \sum_{j=1}^{Z_n} \mathrm{Log}\mathring{\varphi}_{nj}(t) - \sum_{j=1}^{Z_n} \left(e^{-it\mu_{nj}} \varphi_{nj}(t) - 1 \right) \right|$$

$$= \left| \sum_{j=1}^{Z_n} \mathrm{Log}\mathring{\varphi}_{nj}(t) - \sum_{j=1}^{Z_n} (\mathring{\varphi}_{nj}(t) - 1) \right|$$

$$\leq \sum_{j=1}^{Z_n} \left| \mathrm{Log}\mathring{\varphi}_{nj}(t) - \mathring{\varphi}_{nj}(t) + 1 \right|$$

which converges in probability to 0 by Lemma 2.4.3.

So

$$-itA_{nZ_n} + \sum_{j=1}^{Z_n} \mathrm{Log}\varphi_{nj}(t) + itA_{nZ_n} - it \sum_{j=1}^{Z_n} \mu_{nj} - \sum_{j=1}^{Z_n} \int_{-\infty}^{\infty} (e^{itx}-1)dF_{nj}(x+\mu_{nj})$$

converge in probability to 0. By Theorem 1.1.2(ii) we have

$$\exp\{-itA_{nZ_n} + \sum_{j=1}^{Z_n} \mathrm{Log}\varphi_{nj}(t) + itA_{nZ_n} - it\sum_{j=1}^{Z_n}\mu_{nj} - \sum_{j=1}^{Z_n}\int_{-\infty}^{\infty}(e^{itx}-1)dF_{nj}(x+\mu_{nj})\}$$

converge in probability to 1, i.e.,

$$\frac{\exp\{-itA_{nZ_n} + \sum_{j=1}^{Z_n}\mathrm{Log}\varphi_{nj}(t)\}}{\exp\{-itA_{nZ_n} + it\sum_{j=1}^{Z_n}\mu_{nj} + \sum_{j=1}^{Z_n}\int_{-\infty}^{\infty}(e^{itx}-1)dF_{nj}(x+\mu_{nj})\}} \xrightarrow{P} 1,$$

i.e.,

$$\frac{\varphi_{Z_n}(t)}{\hat{\varphi}_{Z_n}(t)} \xrightarrow{P} 1.$$

Since $|\hat{\varphi}_{Z_n}(t)| \leq 1$, we have

$$|\varphi_{Z_n}(t) - \hat{\varphi}_{Z_n}(t)| \xrightarrow{P} 0$$

By Theorem 1.1.5(ii) we have

$$|\varphi_{Z_n}(t) - \hat{\varphi}_{Z_n}(t)| \xrightarrow{m} 0.$$

#

Theorem 2.4.5 Let $(Z_n;X_{nj})$ be a random double sequence of random variables which satisfies the conditions $(\tilde{\alpha})$and$(\tilde{\beta})$ and for each n, $Z_n,X_{n1},X_{n2},\ldots$are independent. Let (A_{nj}) be a double sequence of real numbers. Then the distribution functions of random sums

$$S_{Z_n} = X_{n1} + X_{n2} + \ldots + X_{nZ_n} - A_{nZ_n}$$

converge weakly to a limit distribution function if and only if their accompanying distribution functions converge weakly to the same limit distribution function.

Proof. Let φ_{Z_n} be the random characteristic function associated with (Z_n), (φ_{nj}) and (A_{nj}) and $\hat{\varphi}_{Z_n}$ be the random accompanying characteristic function associated with (Z_n), (φ_{nj}) and (A_{nj}). Let φ be any characteristic function. By Theorem 2.4.4 we see that the following ststements are equivalent.

(1) for every t, $E[\varphi_{Z_n}(t)] \to \varphi(t)$

(2) for every t, $E[\hat{\varphi}_{Z_n}(t)] \to \varphi(t)$.

From this fact together with the fact that the characteristic functions of S_{Z_n} are given by

$$\psi_n(t) = E[\varphi_{Z_n}(t)]$$

and the characteristic functions of the accompanying distribution function of S_{Z_n} is given by

$$\hat{\psi}_n(t) = E[\hat{\varphi}_{Z_n}(t)],$$

we have the conclusion of the theorem.

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