CHAPTER IV

MATHEMATICAL MODELING

4.1 Introduction

This chapter describes the mathematical modeling of the robot. Knowing the pose of the robot is important for analysis and control. Section 4.2 presents a method to estimate wheel-ground contact angle. Section 4.3 use the Denavit-Hartenburg Notation [19] to derive forward and inverse kinematics of the robot using wheel-ground contact angle information from section 4.2.

4.2 Wheel-Ground Contact Angle Estimation

To formulate kinematics modeling of the mobile robot, the wheel-ground contact angles must be known. But it is difficult to make a direct measurement of these angles; a method for estimating these contact angles is presented in this section and described in details in appendix A.

In kinematics modeling and contact angle estimation, we introduce the following assumptions.

1) Each wheel makes contact with the ground at a single point.
2) No side slip and rolling slip between a wheel and the ground.

Consider the left bogie on uneven terrain, the bogie pitch, μ_1 is defined with respect to the horizon. The wheel center velocities v_1 and v_2 are parallel to the wheel-ground tangent plane. The distance between the wheel centers is constant and defined as L_B.

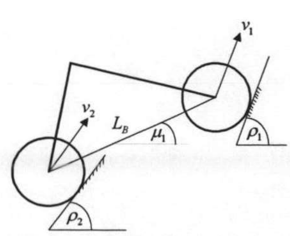

Figure 4.1: The left bogie on uneven terrain

The kinematics equations can be written as following

$$v_1 \cos(\rho_1 - \mu_1) = v_2 \cos(\rho_2 - \mu_1) \tag{4.1}$$

$$v_1 \sin(\rho_1 - \mu_1) - v_2 \sin(\rho_2 - \mu_1) = L_B \dot{\mu}_1 \tag{4.2}$$

Combining Equations (4.1) and (4.2) results in:

$$\sin[(\rho_1 - \mu_1) - (\rho_2 - \mu_1)] = \frac{L_B \dot{\mu}_1}{v_1} \cos(\rho_2 - \mu_1) \tag{4.3}$$

Define:

$$a_1 = \frac{L_B \dot{\mu}_1}{v_1} \qquad b_1 = \frac{v_2}{v_1} \qquad \delta_1 = \rho_1 - \mu_1 \qquad \varepsilon_1 = \mu_1 - \rho_2$$

then

$$\cos\delta_1 = b_1 \cos\varepsilon_1 \tag{4.4}$$

$$(\sin\delta_1 + b_1 \sin\varepsilon_1)\cos\varepsilon_1 = a_1 \cos\varepsilon_1 \tag{4.5}$$

From equation (4.4) and (4.5), we can derive contact angles of the wheel 1 and 2.

Contact angles of the wheel 1 and 2 are given by

$$\rho_1 = \mu_1 + \arcsin\left(\frac{a_1^2 - b_1^2}{2a_1}\right) \tag{4.6}$$

$$\rho_2 = \mu_1 + \arcsin\left(\frac{1 + a_1^2 - b_1^2}{2a_1}\right) \tag{4.7}$$

In order to compute the contact angle of the rear wheel, we need to know velocity of the bogie joint first:

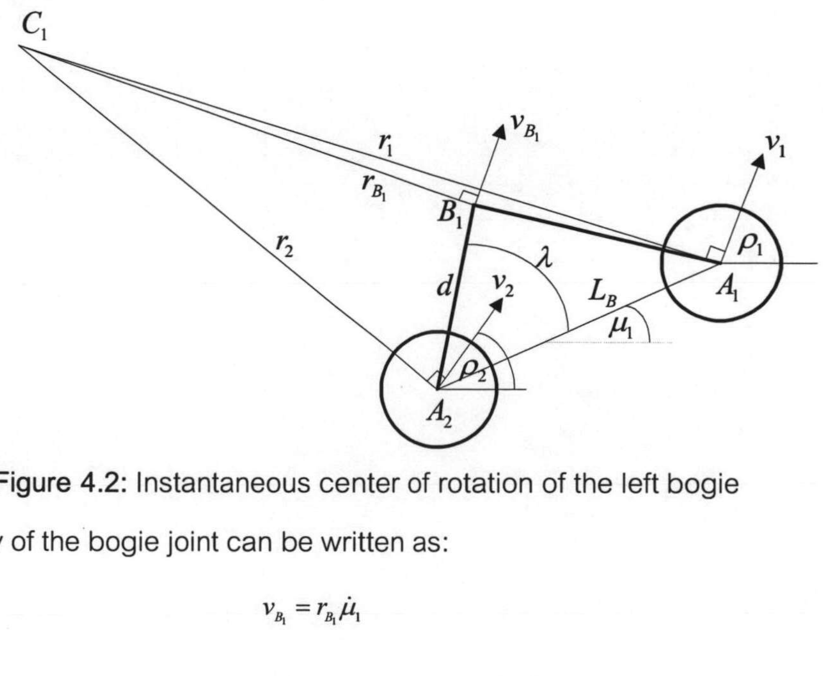

Figure 4.2: Instantaneous center of rotation of the left bogie

Velocity of the bogie joint can be written as:

$$v_{B_1} = r_{B_1} \dot{\mu}_1 \tag{4.8}$$

where

$$r_{B_1} = \sqrt{r_2^2 + d^2 - 2r_2 d \cos(90 + \rho_2 - \mu_1 - \lambda)}$$

$$r_1 = \frac{L_B \sin(90 + \rho_2 - \mu_1)}{\sin(\rho_1 - \rho_2)}$$

$$r_2 = \frac{L_B \sin(90 - \rho_1 + \mu_1)}{\sin(\rho_1 - \rho_2)}$$

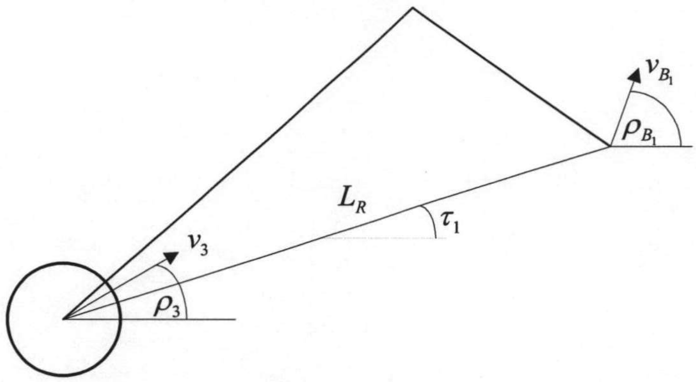

Figure 4.3: Left Rocker on uneven terrain

Consider the left rocker, the rocker pitch, τ_1, is defined with respect to the horizon. The distance between rear wheel center and bogie joint is L_R.

Contact angles of the wheel 3 is given by

$$\rho_3 = \arccos[\frac{v_{B_1}}{v_3}\cos(\rho_{B_1} - \tau_1)] \quad (4.9)$$

In the same way, we repeated these procedures with the right side:

Contact angles of the wheel 4, 5 and 6 are given by

$$\rho_4 = \mu_2 + \arcsin(\frac{a_2^2 - b_2^2}{2a_2}) \quad (4.10)$$

$$\rho_5 = \mu_2 + \arcsin(\frac{1 + a_2^2 - b_2^2}{2a_2}) \quad (4.11)$$

$$\rho_6 = \arccos[\frac{v_{B_2}}{v_6}\cos(\rho_{B_2} - \tau_2)] \quad (4.12)$$

There are special cases that the contact angles cannot be estimated [8]. First case occurs when the robot is stationary. Pitch rates of the bogie and rocker cannot be computed. Then equations (4.6) – (4.12) do not yield a solution. Since a robot in a fixed configuration has an infinite set of contact angles.

The second case occurs when the bogie is parallel to the surface and the front wheel encounter a vertical obstacle with respect to the surface. Consider left bogie, in this case $\cos\varepsilon_1$ is equal to zero. The equation (4.4) is degenerated and the system is unsolvable. However, by observation that v_2 is zero, equation (4.1) and (4.2) can be written as

$$v_1 \cos(\rho_1 - \mu_1) = 0 \tag{4.13}$$

$$v_1 \sin(\rho_1 - \mu_1) = L_B \dot{\mu}_1 \tag{4.14}$$

The variable ρ_2 is undefined since wheel 2 is stationary, and

$$\rho_1 = \mu_1 + \frac{\pi}{2}\operatorname{sgn}(\dot{\mu}_1) \tag{4.15}$$

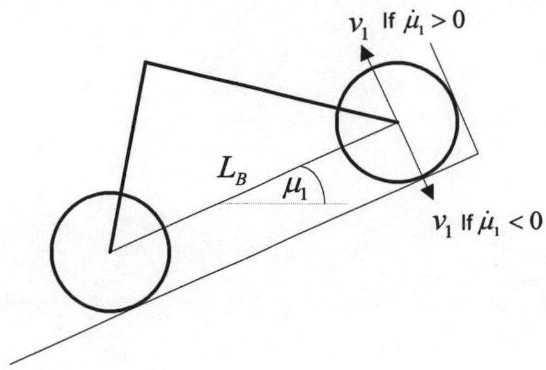

Figure 4.4: Left Bogie where $\cos \varepsilon_1 = 0$

The last case occurs when ρ_1 is equal to ρ_2. The pitch rate $\dot{\mu}_1$ is zero and ratio of v_2 and v_1 is unity. Then equations (4.6) and (4.7) have no solution. But it is easy to detect constant pitch rate from an inclinometer. If the bogie is on the flat terrain, the contact angles are equal to the pitch angle. In the case that pitch rate is zero temporary; we assume that the terrain profile varies slowly with respect to data sampling rate and use previously estimated contact angle instead.

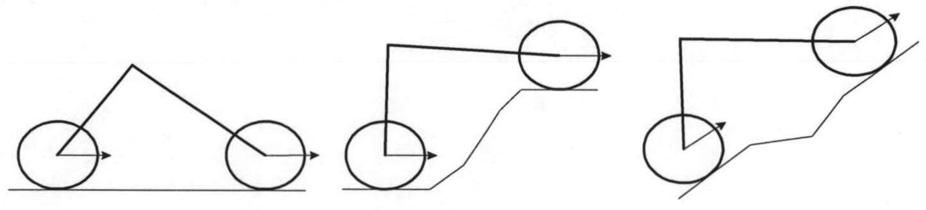

Figure 4.5: Left Bogie where $\dot{\mu}_1 = 0$ and $\frac{v_2}{v_1} = 0$

4.3 Forward Kinematics

We define coordinate frames as in figure 4.6 and 4.7. The subscripts for the coordinate frames are as follows: O: robot frame, D: Differential joint, R_i: Left and

Right Rocker ($i = 1,2$), B_i : Left and Right Bogie ($i = 1,2$), S_i : Steering of left front, left back, right front and right back wheels ($i = 1,3,4,6$) and A_i : Axle of all wheels ($i = 1-6$)

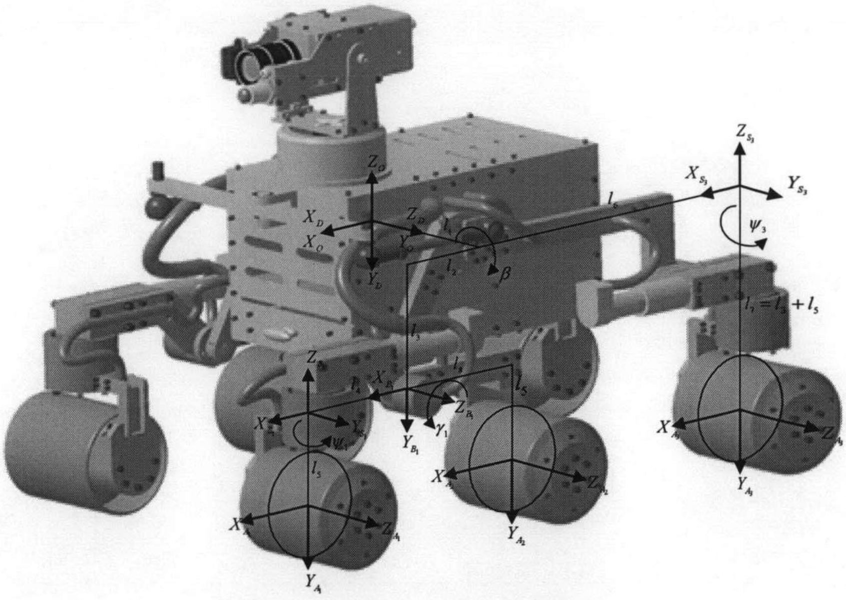

Figure 4.6: Left coordinate frames

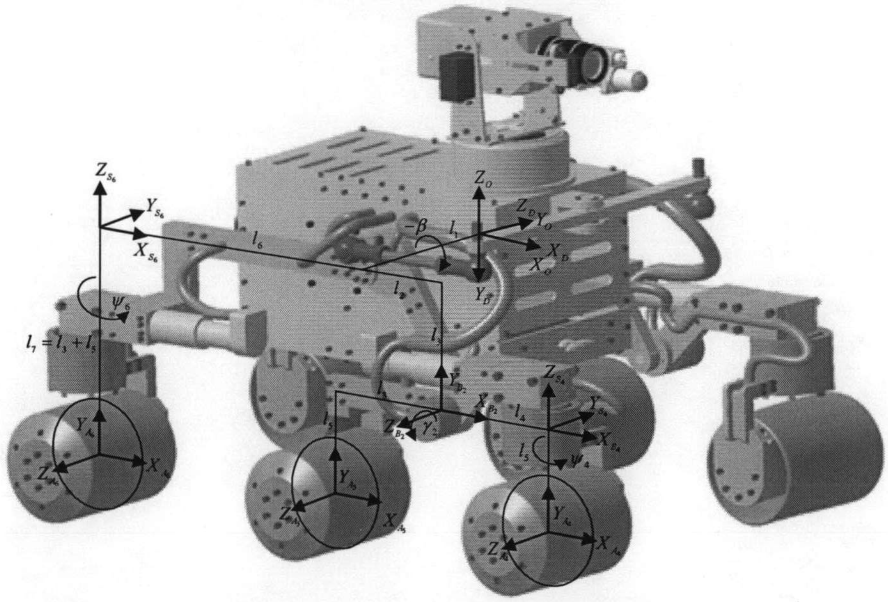

Figure 4.7: Right coordinate frames

Other quantities shown in figure 4.6 and 4.7 are steering angles ψ_i ($i = 1,3,4,6$), rocker angle β , left and right bogie angle γ_1 and γ_2

Using the Denavit-Hartenburg parameters , the transformation matrix for coordinate i to j can be written as follows:

$$\mathbf{T}_{j,i} = \begin{bmatrix} C(\eta_j) & -S(\eta_j)C(\alpha_j) & S(\eta_j)S(\alpha_j) & a_jC(\eta_j) \\ S(\eta_j) & C(\eta_j)C(\alpha_j) & -C(\eta_j)S(\alpha_j) & a_jS(\eta_j) \\ 0 & S(\alpha_j) & C(\alpha_j) & d_j \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (4.16)$$

The transformations from the robot reference frame (O) to the wheel axle frames (A_i) are obtained by cascading the individual transformations. For example, the transformation for wheel 1 is

$$\mathbf{T}_{O,A_1} = \mathbf{T}_{O,D}\mathbf{T}_{D,S_1}\mathbf{T}_{S_1,A_1} \quad (4.17)$$

In order to capture the wheel motion, we need to derive two additional coordinate frames for each wheel, contact frame and motion frame. Contact frame is obtained by rotating the wheel axle frame (A_i) about the z-axis, then followed by a 90 degree rotation about the x-axis. The z-axis of the contact frame (C_i) points away from the contact point as shown in figure 4.8.

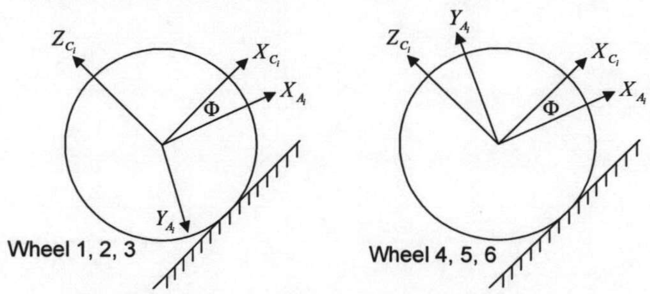

Figure 4.8 Contact Coordinate Frame

The transformation matrix for contact frame can be derived by Z-X-Y Euler angle.

$$\mathbf{T}_{A_i,C_i} = \begin{bmatrix} Cp_iCr_i - Sp_iSq_iSr_i & Cr_iSp_i + Cp_iSq_iSr_i & -Cq_iSr_i & 0 \\ -Cq_iSp_i & Cp_iCq_i & Sq_i & 0 \\ Cr_iSp_iSq_i + Cp_iCr_i & -Cp_iCr_iSq_i + Sp_iSr_i & Cq_iCr_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (4.18)$$

The wheel motion frame is obtained by translating along the negative z-axis by wheel radius (R_w) and translating along the x-axis for wheel roll ($R_w\theta_i$).

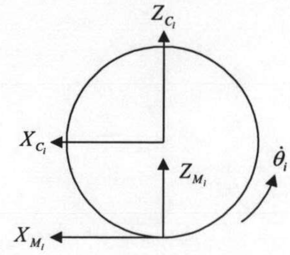

Figure 4.9 Wheel Motion Frame

The transformation matrices for all wheels can be written as

$$
\begin{aligned}
\mathbf{T}_{O,M_1} &= \mathbf{T}_{O,D}\mathbf{T}_{D,B_1}\mathbf{T}_{B_1,S_1}\mathbf{T}_{S_1,A_1}\mathbf{T}_{A_1,C_1}\mathbf{T}_{C_1,M_1} \\
\mathbf{T}_{O,M_2} &= \mathbf{T}_{O,D}\mathbf{T}_{D,B_1}\mathbf{T}_{B_1,A_2}\mathbf{T}_{A_2,C_2}\mathbf{T}_{C_2,M_2} \\
\mathbf{T}_{O,M_3} &= \mathbf{T}_{O,D}\mathbf{T}_{D,S_3}\mathbf{T}_{S_3,A_3}\mathbf{T}_{A_3,C_3}\mathbf{T}_{C_3,M_3} \\
\mathbf{T}_{O,M_4} &= \mathbf{T}_{O,D}\mathbf{T}_{D,B_2}\mathbf{T}_{B_2,S_4}\mathbf{T}_{S_4,A_4}\mathbf{T}_{A_4,C_4}\mathbf{T}_{C_4,M_4} \\
\mathbf{T}_{O,M_5} &= \mathbf{T}_{O,D}\mathbf{T}_{D,B_2}\mathbf{T}_{B_2,A_5}\mathbf{T}_{A_5,C_5}\mathbf{T}_{C_5,M_5} \\
\mathbf{T}_{O,M_6} &= \mathbf{T}_{O,D}\mathbf{T}_{D,S_6}\mathbf{T}_{S_6,A_6}\mathbf{T}_{A_6,C_6}\mathbf{T}_{C_6,M_6}
\end{aligned}
\tag{4.19}
$$

To obtain the wheel Jacobian matrices, we must express the motion of the robot to the wheel motion frame, by applying the instantaneous transformation $\dot{\mathbf{T}}_{\hat{O},\hat{M}_i}$ as follows:

$$\dot{\mathbf{T}}_{\hat{O},O} = \mathbf{T}_{\hat{O},\hat{M}_i}\dot{\mathbf{T}}_{M_i,O} \tag{4.20}$$

$\dot{\mathbf{T}}_{\hat{O},O}$ is found to have the following form:

$$\dot{\mathbf{T}}_{\hat{O},O} = \begin{bmatrix} 0 & -\dot{\phi} & \dot{p} & \dot{x} \\ \dot{\phi} & 0 & -\dot{r} & \dot{y} \\ -\dot{p} & \dot{r} & 0 & \dot{z} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.21}$$

where

ϕ = yaw angle of the robot

p = pitch angle of the robot

r = roll angle of the robot

Once the instantaneous transformations of each wheel are obtained, we can extract a set of equations relating the robot's motion in vector form $[\dot{x} \;\; \dot{y} \;\; \dot{z} \;\; \dot{\phi} \;\; \dot{p} \;\; \dot{r}]^T$ to the joint angular rates.

The results of wheel 1 (the left front wheel) and 4 (the right front wheel) are found to be:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{\phi} \\ \dot{p} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} A_i & 0 & B_i & C_i \\ D_i & 0 & E_i & F_i \\ G_i & 0 & H_i & I_i \\ 0 & 0 & 0 & J_i \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & K_i \end{bmatrix} \begin{bmatrix} \dot{\theta}_i \\ \dot{\beta} \\ \dot{\gamma}_i \\ \dot{\psi}_i \end{bmatrix} \quad i = 1,4 \tag{4.22}$$

The results of wheel 2 (the left middle wheel) and 5 (the right middle wheel) are found to be:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{\phi} \\ \dot{p} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} A_i & 0 & B_i \\ C_i & 0 & 0 \\ D_i & 0 & E_i \\ 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_i \\ \dot{\beta} \\ \dot{\gamma}_i \end{bmatrix} \quad i = 2,5 \tag{4.23}$$

The results of wheel 3 (the left back wheel) and 6 (the right back wheel) are found to be:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{\phi} \\ \dot{p} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} A_i & 0 & B_i \\ C_i & 0 & D_i \\ E_i & 0 & F_i \\ 0 & 0 & G_i \\ 0 & -1 & 0 \\ 0 & 0 & H_i \end{bmatrix} \begin{bmatrix} \dot{\theta}_i \\ \dot{\beta} \\ \dot{\psi}_i \end{bmatrix} \quad i = 3,6 \tag{4.24}$$

The parameters A_i to K_i in the matrices above can be easily derived in terms of wheel-ground contact angle $(\rho_1,\ldots,\rho_6)$ and joint angle $(\beta, \gamma \text{ and } \psi)$.

It is seen that these sets of equation are in the general form:

$$\dot{\mathbf{u}} = \mathbf{J}_i \dot{\mathbf{q}}_i \quad i = 1-6 \tag{4.25}$$

where $\mathbf{J}_i$ is the Jacobian matrix of wheel i, and $\dot{\mathbf{q}}_i$ is the joint angular rate vector.

We will see that the 5th equation (5th row) does not contribute to any unknowns. It simply states that the change in pitch is equal to the change in the bogie and rocker angles. With the inclinometer installed, $\dot{p}$ can be sensed without knowledge of the rocker and bogie angles. Since only the $\dot{p}$, in equation (4.22) to (4.24), contains $\dot{\gamma}$ and $\dot{\beta}$, we can remove these from further consideration.

4.4 Inverse Kinematics

The purpose of inverse kinematics is to determine the individual wheel angular velocities which will accomplish desired robot motion. The desired robot motion is given by forward velocity and turning rate. In this section, we will develop all six wheels angular velocities equations and use geometric approach to determine steering angle of steerable wheels.

4.4.1 Wheel Angular Velocities

Consider forward kinematics of the front wheel, define the desired forward velocity is $\dot{x}_d$ and desired heading angular rate is $\dot{\phi}_d$. From equation (4.22) the first and the fourth equation give:

$$\begin{aligned} \dot{x}_d &= A_i\dot{\theta}_i + B_i\dot{\gamma}_i + C_i\dot{\psi}_i \\ \dot{\phi}_d &= J_i\dot{\psi}_i \end{aligned} \qquad i = 1,3 \tag{4.26}$$

The angular velocities of the front wheels can be written as:

$$\dot{\theta}_i = \frac{\dot{x}_d - B_i\dot{\gamma}_i - \dfrac{C_i}{J_i}\dot{\phi}_d}{A_i} \qquad i = 1,3 \tag{4.27}$$

Similarly the angular velocities of the middle wheels can be written as:

$$\dot{\theta}_i = \frac{\dot{x}_d - B_i\dot{\gamma}_i}{A_i} \qquad i = 2,5 \tag{4.28}$$

Finally the angular velocities of the back wheels can be written as:

$$\dot{\theta}_i = \frac{\dot{x}_d - \dfrac{B_i}{G_i}\dot{\phi}_d}{A_i} \qquad i = 3,6 \tag{4.29}$$

4.4.2 Steering Angles

In this section, we estimate an instantaneous center of rotation, called turning center, based on two non-steerable middle wheels. This turning center will be used to determine the steering angles of the four corner steerable wheels.

From figure 4.6 and 4.7, we can derive coordinate of the wheel centers respect to the robot reference frame as follows:

<u>Wheel 1</u>

$$\begin{aligned} x_{C1} &= l_2\cos\beta + l_3\sin\beta + l_4\cos(\beta-\gamma_1) + l_5\sin(\beta-\gamma_1) \\ z_{C_1} &= l_2\sin\beta - l_3\cos\beta + l_4\sin(\beta-\gamma_1) - l_5\cos(\beta-\gamma_1) \end{aligned} \tag{4.30}$$

<u>Wheel 2</u>

$$\begin{aligned} x_{C2} &= l_2\cos\beta + l_3\sin\beta - l_8\cos(\beta-\gamma_1) + l_5\sin(\beta-\gamma_1) \\ z_{C2} &= l_2\sin\beta - l_3\cos\beta - l_8\sin(\beta-\gamma_1) - l_5\cos(\beta-\gamma_1) \end{aligned} \tag{4.31}$$

Wheel 3

$$x_{C3} = -l_6 \cos\beta + (l_3 + l_5)\sin\beta$$
$$z_{C3} = -l_6 \sin\beta - (l_3 + l_5)\cos\beta \tag{4.32}$$

Wheel 4

$$x_{C4} = l_2 \cos(-\beta) - l_3 \sin(-\beta) + l_4 \cos(-\beta + \gamma_2) + l_5 \sin(-\beta + \gamma_2)$$
$$z_{C4} = -l_2 \sin(-\beta) - l_3 \cos(-\beta) + l_4 \sin(-\beta + \gamma_2) - l_5 \cos(-\beta + \gamma_2) \tag{4.33}$$

Wheel 5

$$x_{C5} = l_2 \cos(-\beta) - l_3 \sin(-\beta) - l_8 \cos(-\beta + \gamma_2) + l_8 \sin(-\beta + \gamma_2)$$
$$z_{C5} = -l_2 \sin(-\beta) - l_3 \cos(-\beta) - l_8 \sin(-\beta + \gamma_2) - l_5 \cos(-\beta + \gamma_2) \tag{4.34}$$

Wheel 6

$$x_{C6} = -l_6 \cos(-\beta) - (l_3 + l_5)\sin\beta$$
$$z_{C6} = -l_6 \sin(-\beta) - (l_3 + l_5)\cos\beta \tag{4.35}$$

From figure 4.10, the instantaneous center of rotation can be estimated by average the distance in X axis of both middle wheels. The distance in Z axis is neglected because there is only 1 degree of freedom per each steering. If the wheel's axis is steered to intersect with the center of rotation on the X axis, the angle in Z direction is coupled and cannot be controlled.

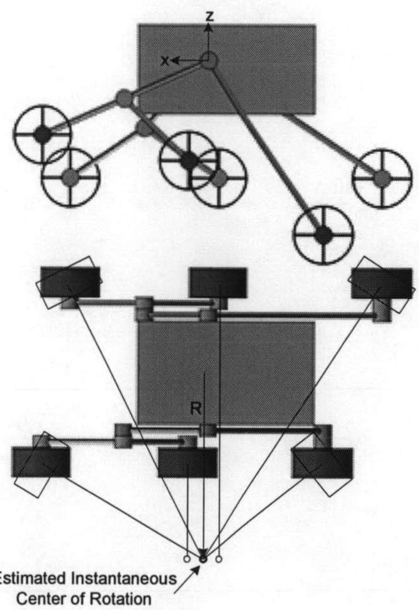

Figure 4.10: Instantaneous Center of Rotation

Using the estimated center of rotation, the desired steering angle for each steerable wheel can be determined. Define R is a turning radius, x_R is the distance in X direction of the center of rotation with respect to the robot reference frame. l_1 is the distance from the robot reference frame to steering joint in Y direction (see figure 4.6 and 4.7). The desired steering angles are:

$$\begin{aligned}
\psi_1 &= \arctan\left(\frac{x_{C1} - x_R}{R - l_1}\right) && \text{for wheel 1} \\
\psi_3 &= \arctan\left(\frac{x_{C3} - x_R}{R - l_1}\right) && \text{for wheel 3} \\
\psi_4 &= \arctan\left(\frac{x_{C4} - x_R}{R + l_1}\right) && \text{for wheel 4} \\
\psi_6 &= \arctan\left(\frac{x_{C6} - x_R}{R + l_1}\right) && \text{for wheel 6}
\end{aligned} \tag{4.36}$$