CHAPTER III

FIXED POINT THEOREMS FOR NONEXPANSIVE MAPPINGS ON STAR-SHAPED SUBSETS OF BANACH SPACES

In the first part of this chapter, we study about some definitions and the classical Banach-Caccioppoli fixed point theorem which is a key to many other fixed point theorems. And in the last part of this chapter, we show two fixed point theorems for nonexpansive mappings on star-shaped subsets of Banach spaces.

3.1 Definition. Let T be a mapping of a metric space (X,d) into itself. The point $x \in X$ is said to be a fixed point of T provided that $T(x) = x$.

3.2 Definition. A mapping T of a metric space (X,d) into itself is said to be a contraction mapping if there exists a number k satisfying $0 \leq k < 1$ such that

$$d(T(x),T(y)) \leq k\, d(x,y)$$

for arbitrary x and y in X.

3.3 Definition. A mapping T of a metric space (X,d) into itself is said to be a nonexpansive mapping provided that

$$d(T(x),T(y)) \leq d(x,y) \qquad \text{.... (1)}$$

holds for all x and y in X.

We note that every nonexpansive mapping is automatically continuous, since it follows from condition (1) that $T(x_n) \to T(x)$ whenever $x_n \to x$.

Clearly, nonexpansive mappings contains all contraction mappings as a proper subclass, and they form a proper subclass of the collection of all continuous mappings.

3.4 Definition. A subset C of a vector space X is said to be star-shaped provided that there is at least one $p \in C$ such that, if $x \in C$ and $0 < t < 1$, then $(1-t)p + tx \in C$.

Of course the star-shaped subsets of a vector space include the convex subsets as a proper subclass.

Now, we continue to use $\longrightarrow$ to denote strong convergence, and we use $\rightharpoonup$ to denote weak convergence.

3.5 Definition. If C is a subset of a Banach space X, then the mapping $S : C \longrightarrow C$ is said to be demiclosed provided that if $\{x_n\} \subset C$ and $x_n \rightharpoonup x \in C$ and $S(x_n) \longrightarrow y \in X$, then $S(x) = y$.

Lemma 2 of [12] has been proved that, if C is a closed and convex subset of a Hilbert space H, then for every nonexpansive mapping $T : C \longrightarrow H$, the mapping $(I-T)$ is demiclosed.

3.6 Remark. If X is a seperated locally convex topological vector space, then the points of X are seperated by the continuous linear forms on X.

For the proof of this remark see e.g. [6] on page 119.

3.7 Theorem. (Banach-Caccioppoli)

Every contraction mapping T defined on a complete metric space (X,d) has a unique fixed point.

Proof. Existence :

Given an arbitrary point $x_o \in X$. Let

$$x_1 = T(x_o),$$
$$x_2 = T(x_1) = T(T(x_o)) = T^2(x_o),$$
$$x_3 = T(x_2) = T^3(x_o),$$
$$\dots$$
$$x_n = T(x_{n-1}) = T^n(x_o).$$
$$\dots$$

Then the sequence $\{x_n\}$ is a Cauchy sequence. In fact, if $n' \geq n$, we have

$$\begin{aligned} d(x_n,x_{n'}) &= d(T^n(x_o),T^{n'}(x_o)) \\ &= d(T(T^{n-1}(x_o)),T(T^{n'-1}(x_o))) \\ &\leq k\, d(T^{n-1}(x_o),T^{n'-1}(x_o)) \\ &\leq k^n d(x_o,T^{n'-n}(x_o)) \\ &= k^n d(x_o,x_{n'-n}). \end{aligned}$$

Hence,

$$\begin{aligned} d(x_n,x_{n'}) &\leq k^n\left[d(x_o,x_1) + d(x_1,x_{n'-n})\right] \\ &= k^n\left[d(x_o,x_1) + d(x_1,x_2) + \dots + d(x_{n'-n-1},x_{n'-n})\right]. \end{aligned}$$

Since $d(x_1,x_2) = d(T(x_o),T(x_1)) \leq k\, d(x_o,x_1)$ and

$$d(x_2,x_3) = d(T(x_1),T(x_2)) \leq k\, d(x_1,x_2) \leq k^2 d(x_o,x_1)$$

$$\cdots$$

$$d(x_{n'-n-1},x_{n'-n}) \leq k^{n'-n-1} d(x_o,x_1)\ ,$$

$$d(x_n,x_{n'}) = k^n d(x_o,x_1)\left[1 + k + k^2 + \ldots + k^{n'-n-1}\right]$$

$$< k^n d(x_o,x_1)\ \frac{1}{1-k}\ .$$

Since $k < 1$, for $n \longrightarrow \infty$ we have $d(x_n,x_{n'}) \longrightarrow 0$. Then $\{x_n\}$ forms a Cauchy sequence. Since X is complete metric space, there exists $x \in X$ such that

$$\lim_{n\to\infty} x_n = x.$$

Then by the continuity of T,

$$T(x) = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = x.$$

This proves the existence of a fixed point x.

Uniqueness :

Suppose there are x and y in X such that

$$T(x) = x \text{ and } T(y) = y.$$

Since T is a contraction mapping,

$$d(T(x),T(y)) \leq k\, d(x,y).$$

Then $d(x,y) \leq k\, d(x,y)$. But then $d(x,y) = 0$, since $k < 1$, and hence $x = y$. This proves the theorem.

Our principal results are the following theorems :

3.8 Theorem. Suppose C is a compact star-shaped subset of a Banach space X, and suppose T is a nonexpansive self-mapping of C. Then T has a fixed point in C.

In this theorem, the star-shaped hypothesis can not be omitted. For example, let $C = \{0, 1\}$. Then C is a compact subset of $\mathbb{R}^1$ with usual topology. But, C is not star-shaped.

Define a mapping $T : C \longrightarrow C$ by

$$T(0) = 1 \text{ and } T(1) = 0.$$

It is clear that T is a nonexpansive self-mapping of C. But T has no fixed point.

In the next theorem, we will prove when the compactness in the above theorem is relaxed.

3.9 Theorem. Suppose C is weakly compact star-shaped subset of a Banach space X, and suppose T is a nonexpansive self-mapping of C such that (I-T) is demiclosed. Then T has a fixed point in C.

Before proving the above theorems, we establish the following lemma :

3.10 Lemma.* Banach space X with the weak topology is a Hausdorff space.

Proof. Let $a \neq b \in X$. We can assume without loss of any generality that $a = 0$.

Then, by Remark 3.6, there exists a continuous linear functional f on X such that

$$f(0) = 0 \text{ and } f(b) \neq 0.$$

Let $|f(b)| = A$. Then note that $A > 0$.

By the continuity of f at zero, we obtain that the set

$$U_f(0) = \{x : |f(x)| < A/2\}$$

is a neighborhood of zero.

And since f is continuous at b, the set

$$U_f(b) = \{x : |f(x-b)| < A/2\}$$

is a neighborhood of b. We contend that $U_f(0)$ and $U_f(b)$ are disjoint. In fact, for any $y \in U_f(b)$, we have

$$|f(y-b)| = |f(y)-f(b)| < A/2.$$

Then

$$|f(b)|-|f(y)| < A/2 ,$$

and hence

$$|f(y)| > A/2.$$

This implies that $y \notin U_f(0)$, that is $U_f(b) \cap U_f(0) = \emptyset$.

Hence the lemma is proved.

<u>Proof</u>. (of Theorem 3.8) Choose $p \in C$ such that

$$(1-t)p + tx \in C$$

for all $x \in C$ and all $t \in (0,1)$.

For each $n = 2, 3, 4, \ldots$, let $k_n = 1 - (1/n)$, and define a mapping $T_n: C \longrightarrow C$ by

$$T_n(x) = (1-k_n)p + k_nT(x)$$

for all $x \in C$.

Since T maps C into C, we observe that each T_n also maps C into C. Each T_n is clearly a contraction mapping, since

$$\begin{aligned} \|T_n(x) - T_n(y)\| &= \|(1-k_n)p + k_nT(x) - (1-k_n)p - k_nT(y)\| \\ &= \|k_nT(x) - k_nT(y)\| \\ &= k_n\|T(x) - T(y)\| \\ &\leq k_n\|x - y\| \end{aligned}$$

and $0 < k_n < 1$.

Since C is compact subset of Banach space X, C is a complete metric space. Hence each T_n has a unique fixed point x_n in C, by Theorem 3.7.

Since C is compact metric space, C is sequentially compact, i.e., there exists a subsequence $\{x_{n_j}\}$ of the sequence $\{x_n\}$ which converges to some $x \in C$.

Since T is a nonexpansive mapping, T is continuous. Then

$$\lim_{j\to\infty} T(x_{n_j}) = T(\lim_{j\to\infty} x_{n_j}) = T(x),$$

and also $k_n \longrightarrow 1$ as $n \longrightarrow \infty$. Then

$$x_{n_j} = T_{n_j}(x_{n_j}) = (1-k_{n_j})p + k_{n_j}T(x_{n_j}) \longrightarrow 0.p + T(x) = T(x)$$

as $n \longrightarrow \infty$. But $x_{n_j} \longrightarrow x$, then $T(x) = x$.

Hence the theorem is proved.

Proof. (of Theorem 3.9) As in the proof of Theorem 3.8, we choose $p \in C$ such that

$$(1-t)p + tx \in C$$

for all $x \in C$ and all $t \in (0,1)$.

For each $n = 2, 3, 4, \ldots$, let $k_n = (1 - {}^1/n)$.

Define a mapping $T_n: C \longrightarrow C$ by

$$T_n(x) = (1-k_n)p + k_n T(x)$$

for all $x \in C$. We have T_n is then a contraction mapping.

Since the weak topology is Hausdorff (by Lemma 3.10) and C is weakly compact, we have C is weakly closed, and hence strongly closed.

Thus C is a complete metric space (with the norm topology of the Banach space X), and then each T_n has a unique fixed point $x_n \in C$.

Since C is weakly compact, by Remark 2.15, C is weakly sequentially compact, and hence there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$x_{n_j} \longrightarrow x \in C.$$

By Theorem 2.23, there exists $M > 0$ such that

$$\|x_{n_j}\| \leq M$$

for all j. For each j , we have

$$(I-T)(x_{n_j}) = x_{n_j} - T(x_{n_j}).$$

Because $T_{n_j}(x_{n_j}) = (1-k_{n_j})p + k_{n_j} T(x_{n_j})$, and $T_{n_j}(x_{n_j}) = x_{n_j}$.

Then

$$k_{n_j} T(x_{n_j}) = x_{n_j} - (1-k_{n_j})p ,$$

$$T(x_{n_j}) = k_{n_j}^{-1} x_{n_j} - k_{n_j}^{-1}(1-k_{n_j})p,$$

implies that

$$(I-T)(x_{n_j}) = x_{n_j} - k_{n_j}^{-1}x_{n_j} + k_{n_j}^{-1}(1-k_{n_j})p$$

$$= x_{n_j}(1-k_{n_j}^{-1}) + (k_{n_j}^{-1}-1)p$$

$$= (k_{n_j}^{-1}-1)(p-x_{n_j}).$$

Hence,

$$\|(I-T)(x_{n_j})\| = |k_{n_j}^{-1}-1| (\|p-x_{n_j}\|)$$

$$\leq |k_{n_j}^{-1}-1| (\|p\| + \|x_{n_j}\|)$$

$$\leq (k_{n_j}^{-1}-1)(M + \|p\|).$$

Since $k_n = 1 - 1/n$, $k_n^{-1} = 1 + \frac{1}{n-1}$. Then

$$k_{n_j}^{-1} \longrightarrow 1 \text{ as } j \longrightarrow \infty .$$

Then

$$(I-T)(x_{n_j}) \longrightarrow 0 \in X.$$

Since we also have $x_{n_j} \longrightarrow x \in C$ and $(I-T)$ is demiclosed, it now follows that

$$(I-T)(x) = 0,$$

i.e., $T(x) = x$. Hence the theorem is proved.