CHAPTER II

PRELIMINARIES

In this thesis, we assume a basic knowledge of topological vector space. However, this chapter will give some definitions and theorems which will be a basic tool for our investigation.

The conjugate space

2.1 Definition. Let T be a topological space, inparticular a metric space. Then by a real function on T we mean a mapping of T into the space $\mathbb{R}^1$ (the real line). Suppose T is a function space, i.e., a space whose elements are functions. Then a real function on T is called a functional.

2.2 Definition. Let f and g be two functionals defined on a topological vector space E, and let α be any number. Then by the sum of f and g denoted by (f+g), is meant the functional whose value at every point $x \in E$ is the sum of the values of f and g at x, while by the product of α and f, denoted by (αf), is meant the functional whose value at every point $x \in E$ is the product of α and the value of f at x. More coincisely,

$$(f+g)(x) = f(x) + g(x) ,$$
$$(\alpha f)(x) = \alpha(f(x)) , \quad (x \in E)$$

Clearly, if f and g are linear functionals, then so are (f+g) and (αf), since

$$
\begin{aligned}
(f+g)(ax+by) &= f(ax+by) + g(ax+by) \\
&= a(f(x)) + b(f(y)) + a(g(x)) + b(g(y)) \\
&= a(f(x) + g(x)) + b(f(y) + g(y)) \\
&= a((f+g)(x)) + b((f+g)(y)) ,
\end{aligned}
$$

and

$$
\begin{aligned}
(\alpha f)(ax+by) &= \alpha(af(x) + bf(y)) \\
&= a(\alpha f(x)) + b(\alpha f(y)) \\
&= a(\alpha f)(x) + b(\alpha f)(y)
\end{aligned}
$$

for all $x,y \in E$ and arbitrary numbers a,b. Moreover, if f and g are bounded (and hence continuous), so are $(f+g)$ and (αf).

2.3 Definition. Let E be a topological vector space. The space E^*, called the conjugate space of E, is the set of all continuous linear functionals on E.

It is clear that the space E^* is itself a vector space, when equipped with the operations of addition of functionals and multiplication of functionals by numbers.

Next, we shall introduce a topology in E^*. We first consider the particularly simple case where the original space E is a normed vector space.

Let f be a continuous linear functional on a normed vector space E. The norm of f equal to

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|}$$

or equivalently

$$\|f\| = \sup_{\|x\|\leq 1} |f(x)|.$$

Hence the space E* conjugate to E can be made into a normed vector space by simply equipping each functional $f \in E^*$ with its norm $\|f\|$. The corresponding topology in E* is called the strong topology in E*.

2.4 Theorem. If E is a normed vector space, then the conjugate space E* is complete.

Proof. Let $\{f_n\}$ be any Cauchy sequence of functionals in E*. Then for any given $\varepsilon > 0$, there is an integer N such that for all n, n' > N implies

$$\|f_n - f_{n'}\| < \varepsilon .$$

Since $\|f_n - f_{n'}\| = \sup_{x\neq 0} \frac{|f_n(x)-f_{n'}(x)|}{\|x\|}$,

$$|f_n(x)-f_{n'}(x)| \leq \|f_n-f_{n'}\| \cdot \|x\| < \varepsilon \|x\|$$

for every $x \in E$. Therefore the sequence $\{f_n(x)\}$ is Cauchy and hence convergent for every $x \in E$. Let

$$\lim_{n\to\infty} f_n(x) = f(x).$$

Then f is linear, since

$$\begin{aligned} f(ax+by) &= \lim_{n\to\infty} f_n(ax+by) \\ &= \lim_{n\to\infty} (af_n(x) + bf_n(y)) \\ &= af(x) + bf(y) . \end{aligned}$$

Moreover, choosing n so large that $\|f_n - f_{n+p}\| < 1$ for all $p \geqslant 0$, we have $\|f_{n+p}\| < \|f_n\| + 1$ for all $p \geqslant 0$, and hence

$$|f_{n+p}(x)| \leqslant (\|f_n\| + 1)\|x\| .$$

It follows that

$$\lim_{p \to \infty} |f_{n+p}(x)| = |f(x)| \leqslant (\|f_n\| + 1)\|x\|,$$

so that f is bounded and hence continuous.

To complete this proof, we now show that the functional f is the limit of the sequence $\{f_n\}$, i.e., that

$$\lim_{n \to \infty} \|f_n - f\| = 0. \qquad \ldots\ldots\ldots (1)$$

Given any $\varepsilon > 0$, let n be so large such that

$$\|f_n - f_{n+p}\| < \varepsilon/3 \qquad \ldots\ldots\ldots (2)$$

for all $p \geqslant 0$. By the definition of norm in E*, there is a nonzero element $x_{n,\varepsilon} \in E$ such that

$$\|f_n - f\| \leqslant \frac{|f_n(x_{n,\varepsilon}) - f(x_{n,\varepsilon})|}{\|x_{n,\varepsilon}\|} + \varepsilon/3$$

$$= |f_n(u) - f(u)| + \varepsilon/3$$

where $u = \dfrac{x_{n,\varepsilon}}{\|x_{n,\varepsilon}\|}$.

Therefore $\|f_n - f\| \leqslant |f_n(u) - f_{n+p}(u)| + |f_{n+p}(u) - f(u)| + \varepsilon/3$

$$\leqslant \|f_n - f_{n+p}\|\,\|u\| + |f_{n+p}(u) - f(u)| + \varepsilon/3$$

$$= \|f_n - f_{n+p}\| + |f_{n+p}(u) - f(u)| + \varepsilon/3 .$$

Then $\|f_n - f\| \leqslant |f_{n+p}(u) - f(u)| + 2\mathcal{E}/3$ .....(3)

after using (2) and the fact that $\|u\| = 1$. But

$$\lim_{p \to \infty} f_{n+p}(u) = f(u).$$

Hence, by taking the limit as $p \to \infty$ in (3), we get

$$\|f_n - f\| < \mathcal{E}$$

which implies (1), since $\mathcal{E}$ is arbitrary. Thus the theorem is proved.

<u>The strong topology in the conjugate space</u>

Let E be a normed vector space. Then we have seen that, the conjugate space E* is itself a normed vector space, and a neighborhood of zero in E* means the set of all continuous linear functional on E satisfying the condition $\|f\| < \mathcal{E}$ for some $\mathcal{E} > 0$. In other words, for a neighborhood base at zero in E* we can take the set of all functionals in E* such that $|f(x)| < \mathcal{E}$ when x ranges over the closed unit sphere $\|x\| \leqslant 1$ in the space E.

Suppose E is a topological vector space, but not a normed vector space. Then in defining the topology in E* it seem natural to start from an arbitrary bounded set $A \subset E$, since there is no longer a "unit sphere."

2.5 <u>Definition.</u> Let E be a topological vector space, with conjugate space E*. Then by the <u>strong</u> <u>topology</u> in E* is meant the topology generated by the neighborhood base at zero consisting of all sets of the form

$$U_{A,\mathcal{E}} = \{ f : |f(x)| < \mathcal{E} \text{ for all } x \in A \}$$

for some number $\mathcal{E} > 0$ and bounded set $A \subset E$.

The second conjugate space

Let E be a topological vector space, and E* be the set of all continuous linear functionals on E. Since E* itself a topological vector space, we can also talk about the "second conjugate space" E** = (E*)*, i.e., set of all continuous linear functionals on E*.

2.6 Theorem. Given a topological vector space E with conjugate space E*, let x be any fixed element of E. Then

$$F_x(f) = f(x)$$

for $f \in E^*$, is continuous linear functional on E*.

Before the proof of this theorem, we need the following definition and lemma :

Let E and F be normed vector spaces, and T a map from E to F, then the norm of T is defined by

$$\|T\| = \sup_{x \neq 0} \frac{\|T(x)\|}{\|x\|} .$$

2.7 Definition. The mapping T is said to be bounded if there exists a number $B > 0$ such that $\|T(x)\| \leq B\|x\|$ for all $x \in E$.

2.8 Lemma. Let E and F be normed vector spaces, and if T is a linear map from E to F. Then the following statements are equivalent:

(i) T is continuous at some point $x \in E$

(ii) T is continuous through out E

(iii) T is bounded on E.

Proof. (i) $\Longrightarrow$ (ii) Suppose that T is continuous at $x_o \in E$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\| T(x_o) - T(y) \| < \varepsilon$$

whenever $\| x_o - y \| < \delta$. Let x be any point in E. Then for all $y \in E$ with $\| x - y \| < \delta$, we have $\| x_o - (y - x + x_o) \| = \| x - y \| < \delta$. Then

$$\| T(x_o) - T(y - x + x_o) \| < \varepsilon \text{ ,and so } \| T(x) - T(y) \| < \varepsilon .$$

Hence, T is continuous at x, and then on E, since x is arbitrary.

(ii) $\Longrightarrow$ (iii) Suppose by contrary that T is not bounded on E. Then for any $n = 1, 2, 3, \ldots$, there exists x_n such that

$$\| T(x_n) \| > n . \| x_n \| .$$

Set $y_n = (1/n) \dfrac{x_n}{\| x_n \|}$. Then $\| y_n \| = 1/n$ and $\{ y_n \}$ converges to 0, while

$$\| T(y_n) \| = \frac{\| T(x_n) \|}{n . \| x_n \|} > \frac{n \| x_n \|}{n \| x_n \|} = 1 .$$

Then $\{ T(y_n) \}$ dose not converge to 0. Consequently, T is not continuous at 0, a contradiction.

(iii) $\Longrightarrow$ (i) Suppose T is bounded on E. Then there exists a number $B > 0$ such that

$$\| T(x) \| \leqslant B . \| x \| .$$

Then T is continuous at 0. In fact, for any given $\varepsilon > 0$, we choose $\delta = \varepsilon / B$. Then whenever $\| x \| < \delta$, we have $\| T(x) \| < \varepsilon$.

This proves the lemma.

Proof. (of Theorem 2.6) The linearity is obvious, since

$$\begin{aligned} F_x(af + bg) &= (af + bg)(x) \\ &= af(x) + bg(x) \\ &= aF_x(f) + bF_x(g) \quad , \end{aligned}$$

for all $f,g \in E^*$ and arbitrary numbers a,b.

Next, to show the continuity, given $\varepsilon > 0$, let A be a bounded subset of E containing x, and let $U_{A,\varepsilon}$ be the neighborhood defined as definition 2.5 . Then

$$|F_x(f)| = |f(x)| < \varepsilon \quad \text{if } f \in U_{A,\varepsilon} \; .$$

Then the functional F_x is continuous at 0, and hence by Lemma 2.8 , F_x is continuous on E. Thus the theorem is proved.

From the above theorem, we have the mapping

$$\pi(x) = F_x(f) = f(x) \; ,$$

called the natural mapping of E into E**, is the mapping of the whole space E onto some subset $\pi(E)$ of the second conjugate space E**.

Clearly π is linear, in the sense that

$$\begin{aligned} \pi(ax + by) &= F_{ax+by}(f) \\ &= f(ax + by) \\ &= af(x) + bf(y) \\ &= aF_x(f) + bF_y(f) \\ &= a\pi(x) + b\pi(y) \end{aligned}$$

for all $x,y \in E$ and arbitrary numbers a,b.

2.9 Definition. If $\pi(E) = E^{**}$, the space E is said to be semireflexive. If E is semireflexive and if π is continuous, the space E is said to be reflexive and π then establishes a homeomorphism between the space E and E^{**}.

In the case of reflexive, each element $x \in E$ can be identified with the corresponding element $\pi(x) \in E^{**}$, and hence it is convenient to denote the value of a functional $f \in E^*$ at the point $x \in E$ by the more symmetric notation

$$f(x) = (f,x)$$

Thus (f,x) can be regarded as a functional on E for each fixed $f \in E^*$, and as a functional on E^* for each fixed $x \in E$ (in the latter case, x also acts like an element of E^{**}).

2.10 Theorem. If E is a normed vector space (so that in particular E^* and E^{**} are also normed vector spaces), then the natural mapping of E into E^{**} is an isometry.

Proof. Given an element $x \in E$. Let $\|x\|$ denote the norm of x in E, and $\|x\|^{**}$ denote the norm of its image in E^{**}. We want to show that

$$\|x\| = \|x\|^{**} .$$

Let f be any element of E^*. Since

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} ,$$

$$|f(x)| = |(f,x)| \leqslant \|f\| \cdot \|x\| ,$$

i.e.,

$$\|x\| \geqslant \frac{|(f,x)|}{\|f\|} \qquad (f \neq 0),$$

and since the left-hand side is independent of f,

$$\|x\| \geq \sup_{f \in E^*} \frac{|(f,x)|}{\|f\|} = \|x\|^{**}. \qquad \ldots (1)$$

On the other hand, by the Hahn-Banach Theorem, for every $x_o \in E$ there is a linear functional f_o such that

$$|(f_o,x_o)| = \|f_o\|.\|x_o\|. \qquad \ldots (2)$$

In fact, to construct such a functional, we need only set $f_o(x) = \lambda$ for all $x \in E_o$ where

$$E_o = \{ x : x = \lambda x_o \}.$$

Then

$$\|f_o\|_{E_o} = \sup_{\lambda} \frac{|f(x)|}{\|x\|} = \sup_{\lambda} \frac{|\lambda|}{\|\lambda x_o\|}$$

$$= \sup_{\lambda} \frac{1}{\|x_o\|} = \frac{1}{\|x_o\|},$$

and then extend f_o to a functional on the whole space E (without changing its norm) , i.e.,

$$\|f_o\| = \frac{1}{\|x_o\|},$$

$$\|f_o\|.\|x_o\| = 1 = |f_o(x_o)| = |(f_o,x_o)|, \text{ implies (2).}$$

It follows from (2) that

$$\|x\|^{**} = \sup_{f \in E^*} \frac{|(f,x)|}{\|f\|} \geq \|x\|. \qquad \ldots (3)$$

Comparing (1) and (3), we get

$$\|x\| = \|x\|^{**},$$

and proves the theorem.

2.11 Theorem. Every reflexive normed vector space is complete.

Proof. If E is reflexive normed vector space, then $E = E^{**}$. But $E^{**} = (E^*)^*$ is complete , by Theorem 2.4. This proves the theorem.

The weak topology

Let E be a topological vector space, with conjugate space E^*. Given any $\varepsilon > 0$ and any finite set of continuous linear functionals $f_1, f_2, \ldots, f_n$ in E^*, the set

$$U = U_{f_1, f_2, \ldots, f_n; \varepsilon} = \{x : |f_1(x)| < \varepsilon, \ldots, |f_n(x)| < \varepsilon\} = \bigcap_{i=1}^{n} \{x : |f_i(x)| < \varepsilon\} \quad \ldots (1)$$

is open in E and contains the point zero, i.e., U is a neighborhood of zero.

Let $\mathcal{L}_0 = \{U_\alpha\}$, be the system of all sets of the form (1). Then $\mathcal{L}_0$ is a neighborhood base at zero , generating a topology in E which is again the topology of a topological vector space. This topology is called the weak topology in E.

We note that every subset of E which is open in the weak topology is also open in the original topology of E. (In fact, if O is any open set in the weak topology, then O can be represented as a union of sets in $\mathcal{L}_0$ which is open in the original topology.) But the converse may not be true, i.e., $\mathcal{L}_0$ may not be neighborhood base at zero for the original topology in E. In other words, the weak topology is weaker than the original topology, and the weak topology in E is the weakest topology τ with the property that every linear functional continuous with respect to the original topology is also continuous with respect to $\mathcal{T}$

Weak convergence and weak compactnees

2.12 Definition. Let E be a topological vector space. A sequence $\{x_n\}$ in E is said to be weakly convergent if there is an x in E with $\lim_{n\to\infty} f(x_n) = f(x)$ for every $f \in E^*$. The point x is called a weak limit of the sequence $\{x_n\}$, and the sequence $\{x_n\}$ is said to converge weakly to x, denoted by $x_n \rightharpoonup x$.

Clearly, convergence implies weak convergence, since if $\{x_n\}$ is a sequence which converges to x, i.e., $\lim_{n\to\infty} x_n = x$. Then for every $f \in E^*$; we have $f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = f(x)$, and hence $x_n \rightharpoonup x$. But the converse may not be true.

2.13 Example. Consider the Hilbert space l_2 of square, summable sequences. The sequence $\{x^1, x^2, \ldots\} = \{(1,0,0,\ldots), (0,1,0,\ldots), \ldots\}$ converges weakly to 0 because if we identify functionals f in l_2^* with points in l_2 of the form $(f_1, f_2, \ldots)$, we must have that $\lim_{n\to\infty} f_n = 0$. (Otherwise $\sum_{i=1}^{\infty} f_i^2$ would diverge.)

Thus for each $f \in l_2^*$, $f(x^n) = [f, x^n]$, where $[f, x^n]$ is the inner product of f and x^n. Thus

$$\lim_{n\to\infty} f(x^n) = \lim_{n\to\infty} f_n = 0 = f(0)$$

for all $f \in l_2^*$, and consequently $x^n \rightharpoonup 0$.

However, we have $\|x^n\| = 1$ for all n. This shows that the weak convergence does not imply convergence.

2.14 Definition. A set $A \subset E$ is said to be weakly sequentially compact if every sequence $\{x_n\}$ in A contains a subsequence which converges weakly to a point in E.

2.15 Remark. The Ebelein-Šmulian Theorem in [5] on page 430 has shown that, A is a weakly sequentially compact if and only if $\bar{A}$, the closure of A is weakly compact.

2.16 Theorem. Bounded subsets of a reflexive Banach space E are weakly compact.

To prove this theorem we need the following lemmas:

2.17 Lemma. Let M be a subspace of the normed linear space E, and x_o a point of E not in the closure of M. Then there exists a point $f \in E^*$ such that $f(x_o) = 1$ and $f = 0$ on M.

Proof. Let M_1 denote the linear hull of $M \cup \{x_o\}$. An arbitrary element z is uniquely represented as $y + tx_o$, where $y \in M$, $t \in \mathbb{R}^1$. Set $f'(z) = t$, clearly, f' is linear on M_1. In fact,

$$\begin{aligned} f'(az_1 + bz_2) &= f'[a(y_1 + t_1x_o) + b(y_2 + t_2x_o)] \\ &= f'[(ay_1 + by_2) + (at_1 + bt_2)x_o] \\ &= at_1 + bt_2 \\ &= af'(z_1) + bf'(z_2) \end{aligned}$$

for all $z_1, z_2 \in M_1$ with $z_1 = y_1 + t_1x_o$, $z_2 = y_2 + t_2x_o$, and arbitrary numbers a, b.

It is also bounded, for if $t \neq 0$,

$$\|z\| = \|y + tx_o\| = |t|.\|\frac{y}{t} + x_o\| \geq |t|.d .$$

(If $-(\frac{y}{t}) = y' \in M$, then $\|y' - x_o\| = \|-\frac{y}{t} - x_o\| = \|\frac{y}{t} + x_o\|$, and hence $\|\frac{y}{t} + x_o\| \geq \|y'\| - \|x_o\| = d$.)

Thus $|f'(z)| = |t| \leq \frac{\|z\|}{d}$. Therefore f' is bounded and hence continuous. Hence $f' \in M_1^*$.

That is there exists $f' \in M_1^*$ such that $f'(x_o) = 1$ and $f' = 0$ on M. To complete the proof , we extend f' to f on E*, by using Hahn-Banach Theorem.

2.18 Lemma. If the conjugate E* of a normed linear space E is seperable, so is E.

2.19 Definition. A subset S of a metric space M is said to be dense or dense in M if the closure of S (written $\overline{S}$) is M. A metric space is seperable if it contains a countable dense subset.

Proof. (of Lemma 2.18)

Let $\{f_n\}$ be dense subset of E* .

Choose $x_n \in E$ so that $\|x_n\| \leq 1$ and $f_n(x_n) \geq \|f_n\|/2$.

Let M be the set of all finite linear combination of elements out of $\{x_n\}$ with rational coefficients. Then M is coutable, and the closure $\overline{M}$ is a subspace.

Suppose that $\overline{M}$ is not dense in E, there exists $x_o \in E$ such that

$$\inf \{\|x_o - x\| : x \in \overline{M} \} > 0.$$

By Lemma 2.17, there exists $f \in E^*$ such that

$$f(x_o) = 1 \quad \text{and} \quad f(M) = 0.$$

Since $\{f_n\}$ is dense in E^*, take a sequence $\{f_{n_i}\}$ converging to f. Then

$$\begin{aligned} \| f_{n_i} - f \| &= \sup_{\|x_{n_i}\|=1} |(f_{n_i} - f)(x_{n_i})| \\ &\geqslant |(f_{n_i} - f)(x_{n_i})| \\ &= |f_{n_i}(x_{n_i})| \\ &\geqslant \frac{\|f_{n_i}\|}{2}. \end{aligned}$$

Since $\{f_{n_i}\} \longrightarrow f$, $\|f_{n_i} - f\| \longrightarrow 0$ as $i \longrightarrow \infty$. Then $\|f_{n_i}\| \longrightarrow 0$ as $i \longrightarrow \infty$,i.e., $f = 0$, a contradiction.

2.20 <u>Lemma</u>. (Banach - Steinhaus)

Let E be a normed linear space and F a Banach space. Let $\{A_n\}$ be a sequence in $B(E,F)$, the set of bounded linear mappings from E to F. Assume that for all n, $\|A_n\| \leqslant M$, and $\lim_{n \to \infty} \{A_n x\}$ exists for all x in any set that is dense in E. Then there exists $A \in B(E,F)$ such that $\{A_n x\} \longrightarrow Ax$ for all $x \in E$.

<u>Proof</u>. Suppose D is dense in E. Given an arbitrary $x \in E$, choose $x' \in D$ such that $\|x - x'\| < \varepsilon$.

For sufficiently large m and n, we have

$\|A_m x' - A_n x'\| \leqslant \|A_m x' - X\| + \|X - A_n x'\|$, when X is the limit of $\{A_n x\}$. Hence

$$\|A_m x - A_n x\| \leqslant \|A_m x' - A_n x'\| + \|A_m x - A_m x'\| + \|A_n x - A_n x'\|$$
$$< \varepsilon + (\|A_m\| + \|A_n\|)\,\|x - x'\|$$
$$< \varepsilon + 2M\varepsilon = (2M + 1)\varepsilon .$$

Then $\{A_m x\}$ is a Cauchy sequence in F.

Since F is complete, there exists $Ax \in F$ such that $\{A_n x\} \longrightarrow Ax$.

To prove that $A \in B(E,F)$.

The linearity is obvious, since

$$\begin{aligned} A(ax + by) &= \lim_{n\to\infty} A_n(ax + by) \\ &= a \lim_{n\to\infty} A_n x + b \lim_{n\to\infty} A_n y \\ &= a(Ax) + b(Ay). \end{aligned}$$

Moreover,

$$\|Ax\| = \lim_{n\to\infty} \|A_n x\| \leqslant \lim_{n\to\infty} \|A_n\|\,\|x\| \leqslant M.\|x\|,$$

i.e., A is bounded. Then the lemma is proved.

2.21 <u>Lemma</u>. A closed linear subspace in a reflexive Banach space is a reflexive Banach space.

<u>Proof</u>. Let M denote the subspace, let M* and M** be the conjugate of M and M*, respectively. We take the following for typical elements in the various space we shall discuss : $x \in E$, $f \in E^*$, $g \in E^{**}$, $x' \in M$, $f' \in M^*$, and $g' \in M^{**}$. We write $f(x) = (f,x)$, $g(f) = (g,f)$, etc.

Let ξ be the map that sends an element f in E* to its restriction f' in M*. Specifically, take $\xi(f)$ so that $(\xi(f),x') = (f,x')$ for all $x' \in M$. Since $\|\xi(f)\| \leqslant \|f\|$, $\xi(f)$ belongs to M*. Moreover $\xi(.)$ is linear on E*.

Since ξ is linear on E^* and g' is linear on M^*, the composition $g'\xi$ defined by $(g', \xi(f)) = g'\xi(f)$ for all $f \in E^*$ is linear on E^*. Moreover, $\|g'\xi\| \leq \|g'\|$ so that $g'\xi \in E^{**}$.

Set $\eta(g') = g'\xi$. Thus, if $f \in E^*$,

$$(g', \xi(f)) = g'\xi(f) = (\eta(g'), f).$$

Clearly, $\eta : M^{**} \longrightarrow E^{**}$.

Now take g_0' arbitrary in M^{**} and set $g_0 = \eta(g_0')$. Given $f' \in M^*$, let f be any extension of f' to E^*, so that $f' = \xi(f)$.
For some $x_0 \in E$ and all $f' \in M^*$, we have

$$(g_0' , f') = (g_0' , \xi(f)) = (\eta(g') , f) = (C(x_0), f) = (f, x_0) \ \ldots(*)$$

where C is the natural mapping.

If $x_0 \in M$, then $(f, x_0) = (f', x_0)$, and thus

$$(g_0' , f') = (f', x_0),$$

and we are done, since g_0' was arbitrary on M^{**}.

Suppose , then, that $x_0 \notin M$. By Lemma 2.17, there exists $f \in E^*$ such that $(f, x_0) \neq 0$ and $(f, x') = 0$ for all $x' \in M$. Since

$$(\xi(f), x') = (f, x') = 0 ,$$

$\xi(f) = 0$. Then

$$0 = (g_0' , \xi(f)) \overset{(*)}{=} (f, x_0) \text{ , as above.}$$

This contradiction establishes that $x_0 \in M$.

Hence the lemma is proved.

Proof. (of Theorem 2.16)

Let S be a bounded subset of E, and let $\{x(j)\}$ be a sequence in S. Let M denote the linear hull of $\{x(j)\}$, namely, the set of all possible finite linear combinations of $\{x(j)\}$. Then $\bar{M}$, the closure of M, is closed subspace of E.

Let M_o denote the set of all finite linear combinations of $\{x(j)\}$ with rational coefficients. Then M_o is a countable dense subset of $\bar{M}$. Hence $\bar{M}$ is seperable.

Since $\bar{M}$ is closed subspace of E and by Lemma 2.21, $\bar{M}$ is also reflexive Banach space.

Since $\bar{M}$ is seperable and reflexive, $\bar{M}^{**}$, the second conjugate space of $\bar{M}$, is also seperable.

Then by Lemma 2.18, $\bar{M}^*$ is seperable.

Let $\{f_1', f_2', \ldots\}$ be a dense subset of $\bar{M}^*$. Choose a subsequence $\{x(j(i,1))\}$ such that $\{(f_1', x(j(i,1)))\}$ converges.

Let $\{x(j(i,2))\}$ be a subsequence of $\{x(j(i,1))\}$ such that $\{(f_2', x(j(i,2)))\}$ converges ; and let $\{x(j(i,n))\}$ be a subsequence of $\{x(j(i,n-1))\}$ such that $\{(f_n', x(j(i,n)))\}$ converges.

The sequence $\{x_i\} = \{x(j(i,i))\}$ has a property that $\{(f_n', x_i)\}$ converges for every $n = 1, 2, \ldots$, because for $i \geqslant n$, we have $\{x(j(i,i))\}$ is a subsequence of $\{x(j(i,n))\}$. Then

$$\lim_{i\to\infty} (f_n', x_i) = \lim_{i\to\infty} (f_n', x(j(i,n))) = \lim_{i\to\infty} (g_i', f_n')$$

exists for each n., where $g_i' \in M^{**}$ for all i.

Since $\{f'_n : n = 1, 2, \ldots\}$ is dense in $\overline{M}^*$ and because g'_i is uniformly bounded, by Lemma 2.20, there exists $g \in \overline{M}^{**}$ such that

$$\lim_{i\to\infty} (g'_i, f') = \lim_{i\to\infty} (f', x_i)$$
$$= (g, f')$$

for all f' in $\overline{M}^*$.

Then the limit of (f', x_i) exists for all f' in $\overline{M}^*$, and hence a Cauchy sequence. Since $\overline{M}$ is reflexive , by Theorem 1.11, $\overline{M}$ is complete. Then there is a point $x' \in M$ such that

$$\lim_{i\to\infty} (f', x_i) = (f', x')$$

for all f' in $\overline{M}^*$.

Finally, because $x_i \in M$, then for every $f \in E^*$ there corresponds an $f' \in \overline{M}^*$ such that $(f, x_i) = (f', x_i)$ and $(f, x') = (f', x')$. Then

$$\lim_{i\to\infty} (f, x_i) = (f, x')$$

for all $f \in E^*$. Hence by definition of weakly convergence, $\{x_i\} \to x'$, which show that S is weakly sequentially compact, and then S is weakly compact. This proves the theorem.

The nested spheres theorem

A sequence of closed spheres

$$S[x_1, r_1] , \quad S[x_2, r_2], \ldots , S[x_n, r_n], \ldots$$

in a metric space R is said to be nested (or decreasing) if,

$$S[x_1, r_1] \supset S[x_2, r_2] \supset \cdots \supset S[x_n, r_n] \supset \cdots$$

2.22 Theorem. A metric space R is complete if and only if every nested sequence $\{S_n\} = \{S[x_n, r_n]\}$ of closed spheres in R such that $r_n \to 0$ as $n \to \infty$ has a nonempty intersection

$$\bigcap_{n=1}^{\infty} S_n .$$

Proof. Assume that R is complete.

Let $\{S_n\}$ be a sequence of nested closed spheres in R such that $r_n \to 0$ as $n \to \infty$. To prove that $\bigcap_{n=1}^{\infty} S_n \neq \emptyset$.

Consider the sequence $\{x_n\}$ of centers of the spheres S_n. For any given $\varepsilon > 0$, there exists N such that

$$2.r_N < \varepsilon .$$

Since $S[x_N, r_N]$ contains all centers x_i for all $i \geq N$, for all $n, n' > N$ we have

$$d(x_n, x_{n'}) < 2.r_N < \varepsilon .$$

Then $\{x_n\}$ is a Cauchy sequence in R. Since R is complete, there exists $x \in R$ such that

$$\lim_{n \to \infty} d(x_n, x) = 0$$

or

$$\lim_{n \to \infty} x_n = x .$$

Thus $x \in \bigcap_{n=1}^{\infty} S_n$. In fact, S_n contains every point of the sequence $\{x_n\}$ except possibly the points $x_1, x_2, \ldots, x_{n-1}$. Hence x is a limit point of every sphere S_n. But S_n is closed, and hence $x \in S_n$ for all n.

Conversely, suppose every nested sequence of closed spheres in R with radii converging to zero has a nonempty intersection, and let $\{x_n\}$ be any Cauchy sequence in R.

We choose a term x_{n_1} of $\{x_n\}$ such that

$$d(x_n, x_{n_1}) < 1/2$$

for all $n \geq n_1$.

Let $S_1 = S[x_{n_1}, 1]$. Next, we choose a term x_{n_2} of $\{x_n\}$ such that $n_2 > n_1$ and that

$$d(x_n, x_{n_2}) < 1/2^2$$

for all $n \geq n_2$.

Let $S_2 = S[x_{n_2}, 1/2]$.

Continue this construction indefinitely, i.e., once having chosen terms $x_{n_1}, x_{n_2}, x_{n_3}, \ldots, x_{n_k}$ $(n_1 < n_2 < n_3 < \cdots < n_k)$, choose a term $x_{n_{k+1}}$ such that $n_{k+1} > n_k$ and

$$d(x_n, x_{n_{k+1}}) < 1/2^{k+1}$$

for all $n \geq n_{k+1}$, and let $S_{k+1} = S[x_{n_{k+1}}, 1/2^k]$, and so on.

This gives a nested sequence $\{S_n\}$ of closed spheres with radii converging to zero. By hypothesis, these spheres have a nonempty intersection, i.e., there is a point x in all the spheres.

We claim that

$$\lim_{n \to \infty} d(x_n, x) = 0 .$$

Given $\varepsilon > 0$, let N be such that

$$1/2^{N-1} < \varepsilon/2 .$$

Then for all $n > n_N$, we have

$$d(x,x_n) \leqslant d(x,x_{n_N}) + d(x_{n_N}, x_n) .$$

Since $x \in S_N = S\left[x_{n_N}, 1/2^{N-1}\right]$, $d(x,x_{n_N}) < 1/2^{N-1} < \varepsilon/2$.

Since $n > n_N$, $d(x_{n_N}, x_n) < 1/2^N < 1/2^{N-1} < \varepsilon/2$.

Then $$d(x,x_n) < \varepsilon/2 + \varepsilon/2 = \varepsilon .$$

Hence $\lim_{n\to\infty} d(x, x_n) = 0$, i.e., $\lim_{n\to\infty} x_n = x$.

Thus the theorem is proved.

2.23 Theorem. Let $\{x_n\}$ be a weakly convergent sequence of elements in a normed vector space E. Then $\{x_n\}$ is bounded, i.e., there is a constant C such that

$$\|x_n\| \leqslant C \qquad (n = 1,2,\ldots).$$

Proof. Suppose that $\{x_n\}$ is unbounded. Then $\{x_n\}$ is unbounded on every closed sphere

$$S\left[f_o, \varepsilon\right] = \left\{f: \|f - f_o\| \leqslant \varepsilon\right\}$$

in E*, in the sense that the set of numbers

$$\left\{(f, x_n) : f \in S\left[f_o,\varepsilon\right] , n = 1,2,\ldots\right\}$$

is unbounded for every $S\left[f_o, \varepsilon\right] \subset E^*$. In fact, if the sequence $\{x_n\}$ is bounded on $S\left[f_o, \varepsilon\right]$, then it is also bounded on the sphere $S\left[0, \varepsilon\right] = \left\{ g : \|g\| \leqslant \varepsilon\right\}$, since if $g \in S\left[0, \varepsilon\right]$, then $\|g\| \leqslant \varepsilon$, and so

$$\| f_o + g - f_o \| \leqslant \varepsilon .$$

Thus $f_o + g \in S\left[f_o, \varepsilon\right]$, and

$$(g, x_n) \quad = \quad (f_o + g, x_n) - (f_o, x_n)$$

where the number (f_o, x_n) are bounded, by the weak convergence of $\{x_n\}$. Since $f_o + g \in S\,[f_o, \varepsilon]$, $(f_o + g, x_n)$ are bounded, and then (g, x_n) are bounded. But if $|(g, x_n)| \leqslant C$ for all $g \in S\,[0, \varepsilon]$, then by the isometry of the natural mapping of E into E** (Theorem 2.10)

$$\|x_n\| \quad = \sup_{\|g\| \leqslant 1} |(g, x_n)| \quad = \quad \frac{1}{\varepsilon} \cdot \sup_{\|g\| \leqslant \varepsilon} |(g, x_n)| \leqslant \frac{C}{\varepsilon}$$

$(n = 1, 2, \ldots)$.

Then $\{x_n\}$ is bounded, contrary to assumption. It follows that if $\{x_n\}$ is unbounded, then $\{x_n\}$ is unbounded on every closed sphere in E*.

Next, choosing any closed sphere $S_o \subset E^*$, we find an integer n_1 and an element $f \in S_o$ such that

$$|(f, x_{n_1})| \quad > \quad 1\,.$$

Since (f, x) depends continuously on x, there exists $S_1 \subset S_o$ such that

$$|(f, x_{n_1})| \; > \; 1$$

for all $f \in S_1$.

By repeating this argument, we find an integer n_2 and the closed sphere $S_2 \subset S_1$ such that

$$|(f, x_{n_2})| \; > \; 2$$

for all $f \in S_2$, and so on, where in general there is an integer n_k and a closed sphere $S_k \subset S_{k-1}$ such that

$$|(f, x_{n_k})| > k$$

for all $f \in S_k$. We can see that the radius of the sphere S_k approaches zero as $k \longrightarrow \infty$.

By theorem 1.4, E* is complete, it follows from the nested sphere theorem that there is an element $\bar{f} \in \bigcap_{k=1}^{\infty} S_k$. But then

$$|(\bar{f}, x_{n_k})| > k , \quad (k = 1, 2, \ldots)$$

contrary to the assumed weak convergence of the sequence $\{x_n\}$.