CHAPTER IV

THE FUNCTIONAL EQUATION :

$f(x,y) + f(x + y, z) = f(y, z) + f(x, y + z)$

ON TOPOLOGICAL GROUP

In this chapter we will discuss the functional equation :

(A) $f(x, y) + f(x + y, z) = f(y, z) + f(x, y + z)$

where $f : G \times G \longrightarrow \mathbb{R}^{(k)}$ is a symmetric continuous on an abelian topological group $G \times G$ into $\mathbb{R}^{(k)}$, a Euclidean k - space.

Our purpose is to give conditions which guarantee the existence of a continuous function $g : G \longrightarrow \mathbb{R}^{(k)}$ such that

(B) $f(x, y) = g(x) + g(y) - g(x + y)$

for all x, y in G .

As in chapter III we shall assume that

$$f(e, e) = \underline{O} ,$$

where e and $\underline{O}$ are the identities of G and $\mathbb{R}^{(k)}$ respectively.

<u>Definition</u> 4.1 Let n be a positive integer larger than 1. A group (G, +) is said to be an n - <u>divisible group</u>, if for all y in G there exists y' in G such that $ny' = y$.

Definition 4.2 Let G be an n - divisible topological group, γ be an ordinal. If there exists a γ - sequence $\{x_\alpha\}(\alpha < \gamma)$ in G such that

(1) $nx_{\alpha+1} = x_\alpha$ for all $\alpha < \gamma$,

(2) the union of all subgroups S_α ($\alpha < \gamma$) which each S_α is defined by

$$S_\alpha = \overline{\langle \{ x_\beta / \beta < \alpha \} \rangle},$$

is dense in G, then we say that G is (γ , n) divisible.

Lemma 4.3 Let H be a subgroup of an abelian topological group (G, +). Let $f : G \times G \longrightarrow \mathbb{R}$ be symmetric and continuous and satisfy

(A) $f(x, y) + f(x + y, z) = f(y, z) + f(x, y + z)$

for all x, y, z, in G. Let $g : H \longrightarrow \mathbb{R}$ be such that

(B) $f(x, y) = g(x) + g(y) - g(x + y)$

for all x, y, in H.

For a given $\varepsilon > 0$ and a compact subset K of H such that there exists a neighborhood N^* of e such that $N^* \subset K$, there exists a neighborhood N of e such that for all x, y, in H with $y - x \in N$ and x, y, in K, we have $|g(y) - g(x)| < \varepsilon$.

Proof. Let $L = \limsup |g(x)|$ as $x \rightarrow e$ in H.

It follows from (B) that for any $x \in H$ we have

$$f(x, x) = g(x) + g(x) - g(2x).$$

This implies

$$2g(x) = g(2x) + f(x, x).$$

Therefore,

$$2 \limsup_{x \to e} |g(x)| \leq \limsup_{x \to e} |g(2x)| + \limsup_{x \to e} |f(x,x)|$$

Hence $$2L \leq L + \limsup_{x \to e} f(x,x) .$$

Since f is continuous and $f(e,e) = 0$, hence

$$\limsup_{x \to e} |f(x, x)| = \lim_{x \to e} |f(x, x)|,$$

$$= 0 .$$

Therefore $$2L \leq L .$$

From the definition of L, we see that $L \geq 0$, hence $L = 0$, i.e. $\limsup_{x \to e} |g(x)| = 0$. Therefore $\lim_{\substack{x \to e \\ x \in H}} g(x) = 0$

Choose a neighborhood N_1 of e such that

(4.3.1) $$|g(z)| < \varepsilon/2 ,$$

for $z \in N_1$.

Since f is continuous and $K \times K$ is compact, hence f is uniformly continuous on $K \times K$. Choose a neighborhood N_2 of e such that

(4.3.2) $\qquad |f(x, z) - f(x', z')| < \varepsilon/2$

whenever $(x, z) - (x', z') \in N_2 \times N_2$ and $(x, z), (x', z') \in K \times K$.

Let $N = N_1 \cap N_2 \cap N^*$.

Let $x, y \in K$ be such that $y - x \in N$.

Since $y - x \in N \subseteq N_2$ and $y - x \in N^* \subseteq K$, hence we have

$$|f(x,y - x) - f(x,e)| < \varepsilon/2 .$$

Therefore

(4.3.3) $\qquad |f(x,y - x)| < \varepsilon/2$.

Since $x, y - x \subseteq H$ and (B) holds on H, hence we have

$$f(x, y - x) = g(x) + g(y - x) - g(y) .$$

Therefore

$$|g(x) - g(y)| \leq |f(x, y - x)| + |g(y - x)| .$$

Since $y - x \in N \subseteq N_1$, it follows from (4.3.1) that

$$|g(y - x)| < \varepsilon/2 .$$

Hence

$$|g(y) - g(x)| \leq |f(x, y-x)| + |g(y - x)| < \varepsilon .$$

<u>Lemma</u> 4.4 Let G be an abelian $\mathcal{C}^*$ - compact Hausdorff topological group which has the property (Γ N) for some limit ordinal Γ. Let H, f and g be as in lemma 4.3. Then g can be extended to be continuous on $\overline{H}$, the closure of H, and f and g satisfy

(B) $\quad f(x, y) = g(x) + g(y) - g(x + y)$

for all x, y, in $\overline{H}$.

<u>Proof.</u> Let x be in $\overline{H}$. Since G has property (Γ N), hence there exists a Γ - net $\{y_\alpha\}(\alpha < \Gamma)$ in H converging to x. We shall show that $\{g(y_\alpha)\}(\alpha < \Gamma)$ is cauchy.

Let $\varepsilon > 0$ be given. Since $G = \bigcup_{n=0}^{\infty} K_n$ where each K_n is a compact neighborhood of the identity e of G, hence $x \in K_n$ for some n. Since $\{y_\alpha\}(\alpha < \Gamma)$ converges to x, hence there exists an ordinal $\beta_0 < \Gamma$ such that $y_\beta \in K_n$ for all $\beta \geqslant \beta_0$.

From lemma 4.3, there exists a neighborhood N of e such that

$$|g(x) - g(y)| < \varepsilon$$

for all x, y, in K_n and $x - y \in N$.

By theorem 2.5.3 the convergent net $\{y_\alpha\}(\alpha < \Gamma)$ is cauchy . Hence there exists β' such that $y_{\beta_1} - y_{\beta_2} \in N$ for all $\beta_1, \beta_2 \geqslant \beta'$. It follows that

$$|g(y_\alpha) - g(y_\beta)| < \varepsilon$$

for all $\alpha, \beta \geqslant \max(\beta_0, \beta')$.

Therefore $\{g(y_\alpha)\}_{(\alpha<\Gamma)}$ is cauchy. Since $\mathbb{R}$ is complete, hence $\{g(y_\alpha)\}_{(\alpha<\Gamma)}$ is convergent. Therefore lim $(g(y_\alpha), \alpha<\Gamma)$ exists.

To show that lim $(g(y_\alpha), \alpha<\Gamma)$ is independent of the choice of y_α, let $\{y_\alpha\}_{(\alpha<\Gamma)}$ and $\{y'_\alpha\}_{(\alpha<\Gamma)}$ be Γ- nets which converge to x. We must show that.

$$\lim\ (g(y_\alpha), \alpha<\Gamma) = \lim\ (g(y'_\alpha), \alpha<\Gamma)$$

Define

$$z_{\omega\xi + n2} = y_{\omega\xi + n},$$

$$z_{\omega\xi + n2 + 1} = y'_{\omega\xi + n},$$

where ξ and n are ordinals and $n<\omega$.

By lemma 2.5.4, $\{z_\alpha\}_{(\alpha<\Gamma)}$ is cauchy.

Therefore $\{g(z_\alpha)\}_{(\alpha<\Gamma)}$ is cauchy.

Since $\mathbb{R}$ is complete, hence $\{g(z_\alpha)\}_{(\alpha<\Gamma)}$ is convergent. Therefore lim $(g(z_\alpha), \alpha<\Gamma)$ exists.

Since $\{g(y_\alpha)\}_{(\alpha<\Gamma)}$ and $\{g(y'_\alpha)\}_{(\alpha<\Gamma)}$ are subnets of $\{g(z_\alpha)\}_{(\alpha<\Gamma)}$, hence, by theorem 2.5.1 we have

$$\lim\ (g(y_\alpha), \alpha<\Gamma) = \lim\ (g(y'_\alpha), \alpha<\Gamma).$$

We define g(x) as the limit of $g(y_\alpha)$. Hence by theorem 2.7.1 we have g is continuous relative to $\overline{H}$.

Set $F(x, y) = f(x, y) - g(x) - g(y) + g(x + y)$

for all x, y, in $\bar{H}$. F is continuous on $\bar{H} \times \bar{H}$, and by the assumption F is zero on $H \times H$, hence F is identically zero. This proves that

$$f(x, y) = g(x) + g(y) - g(x + y)$$

for all x, y, in $\bar{H}$.

Lemma 4.5 let G be an abelian n - divisible torsion free σ^* - compact Hausdorff topological group which has the property (ΓN) for some limit ordinal Γ . Let $f : G \times G \longrightarrow \mathbb{R}$ be symmetric and continuous and satisfy

(A) $$f(x, y) + f(x + y, z) = f(y, z) + f(x, y + z)$$

for all x, y, z, in G. Let S be a subgroup of G such that there exists a function $g : S \longrightarrow \mathbb{R}$ satisfying

(B) $$f(x, y) = g(x) + g(y) - g(x + y)$$

for all x, y, in S. Let t be in G such that $nt \in S$. Then there exists a continuous extension $\bar{g}$ of g such that f and $\bar{g}$ satisfy (B) on $\overline{S[t]}$, the closure of $S[t]$.

Proof. Let $\hat{g}$ be defined as in case II of lemma 3.5. Then $\hat{g}$ is an extension of g such that f and $\hat{g}$ satisfy (B) on $S[t]$.

By lemma 4.4, g can be extended to $\bar{g}$ which is continuous on $\overline{S[t]}$ and $\bar{g}$ and f satisfy (B) on $\overline{S[t]}$.

Lemma 4.6 Let G be an n - divisible torsion free topological group. Then there exists an ordinal γ such that G is (γ , n) divisible.

<u>Proof</u>. In the case that $G = \{e\}$, the ordinal $\gamma = 1$. $S_0 = \langle \emptyset \rangle = \{e\}$. Hence G is $(1, \gamma)$ divisible.

Assume that $G \neq \{e\}$. First we shall show that there exists an ordinal γ and a family of subgroups $\{S_\alpha\}(\alpha < \gamma)$ such that if $\alpha < \beta < \gamma$, then $S_\alpha \subset S_\beta$ and $\overline{\bigcup_{\alpha<\beta} S_\alpha} = G$. Let c be a choice function for G. Define $S_0 = \{e\}$ and $y_0 = c(G - S_0)$.

Suppose that S_α and y_α have been defined for all $\alpha < \beta$ and $G - \overline{\bigcup_{\alpha<\beta} S_\alpha} \neq \emptyset$. We shall define S_β and y_β as follows

<u>Case</u> 1 $\quad \beta = \delta + 1$ for some ordinal δ.

Define $\quad S_\beta = \overline{\langle S_\delta \cup \{y_\delta\} \rangle}$.

Since $y_\delta \in S_\delta$, hence there exists y^* in G such that $ny^* = y_\delta$.

Define $\quad y_\delta = y^*$.

<u>Case</u> 2 $\quad \beta$ is a limit ordinal.

Define $\quad S_\beta = \overline{\bigcup_{\alpha<\beta} S_\alpha}$, and $y_\beta = c(G - S_\beta)$.

We claim that there exists an ordinal number γ such that $G - \overline{\bigcup_{\alpha<\gamma} S_\alpha} = \emptyset$. Suppose the contrary, i.e. for all γ,

$$\overline{\bigcup_{\alpha<\gamma} S_\alpha} \subset G .$$

Take $\gamma' = \overline{\overline{\mathcal{P}G}}$, where $\mathcal{P}G$ is the power set of G. Hence

$$\overline{\bigcup_{\alpha<\gamma'} S_\alpha} \subset G.$$

(4.6.1) $$\overline{\overline{\bigcup_{\alpha<\gamma'} S_\alpha}} \leq \overline{\overline{G}}.$$

If γ' is finite then by the definition of $\{S_\alpha\}(\alpha<\gamma')$ we have $\{e, y_0, \ldots, y_{\gamma'-2}\} \subseteq S_{\gamma'-1} = \bigcup_{\alpha<\gamma'} S_\alpha$

Hence $$\gamma' \leq \overline{\overline{\bigcup_{\alpha<\gamma'} S_\alpha}}$$

If γ' is infinite cardinal, then by lemma **A-35**, γ' is a limit ordinal. By the definition of $\{S_\alpha\}(\alpha<\gamma')$ we have

$$\{y_\alpha\} \subseteq S_{\alpha+1}$$ and

$$\bigcup_{\alpha<\gamma'}\{y_\alpha\} \subseteq \bigcup_{\alpha<\gamma'} S_{\alpha+1}$$

Since γ' is a limit ordinal, hence

$$\bigcup_{\alpha<\gamma'} S_{\alpha+1} = \bigcup_{\alpha<\gamma'} S_\alpha .$$

Hence $$\bigcup_{\alpha<\gamma'}\{y_\alpha\} \subseteq \bigcup_{\alpha<\gamma'} S_\alpha \subseteq \overline{\bigcup_{\alpha<\gamma'} S_\alpha} .$$

Therefore $$\overline{\overline{\bigcup_{\alpha<\gamma'}\{y_\alpha\}}} \leq \overline{\overline{\bigcup_{\alpha<\gamma'} S_\alpha}}$$

By the choice of y_α and since G is torsion free, it can be shown that y_α $(\alpha < \gamma')$ are distenct. Hence $\gamma' = \overline{\overline{\bigcup_{\alpha<\gamma'} \{y_\alpha\}}}$

Therefore $\gamma' \leq \overline{\overline{\bigcup_{\alpha<\gamma'} S_\alpha}}$.

This together with (4.6.1) imply $\gamma' \leq \overline{\overline{G}}$.

Hence $\overline{\overline{\mathcal{P}G}} \leq \overline{\overline{G}}$ which is a contradiction.

Then there exists an ordinal γ and a family $\{S_\alpha\}(\alpha < \gamma)$ of subgroups of G such that $\overline{\bigcup_{\alpha<\gamma} S_\alpha} = G$.

Let
$$x_\beta = \begin{cases} x^* & \text{if } \beta = \delta + 1 \text{ and } nx^* = x_\delta \\ c(G-S_\beta) & \text{if } \beta \text{ is a limit ordinal.} \end{cases}$$

Observe that $x_\beta = y_\beta$. By the above construction and since G is torsion free, we see that $\{x_\alpha\}(\alpha < \gamma)$ is a γ - sequence and $\{x_\alpha\}(\alpha<\gamma)$ generates the γ - sequence of subgroups $\{S_\alpha\}(\alpha<\gamma)$. Therefore G is (γ - n) divisible.

<u>Theorem</u> 4.7 Let G be an abelian n - divisible torsion free σ^* - compact Hausdorff topological group which has the property(Γ N) for some limit ordinal Γ . Let $f : G \times G \longrightarrow \mathbb{R}$ be symmetric and continuous and satisfy

(A) $$f(x,y) + f(x+y,z) = f(y,z) + f(x,y+z)$$

for all x,y,z, in G. Then there exists a continuous function $g : G \longrightarrow \mathbb{R}$ such that

(B) $$f(x,y) = g(x)+g(y) - g(x+y)$$

for all x,y, in G.

<u>Proof</u> From lemma 4.6, there exists an ordinal γ such that G is (γ ,n) divisible. For each $\alpha < \gamma$, we shall define g_α on S_α

(1) If $\alpha' < \alpha$, then $g_{\alpha'} \subseteq g_{\alpha}$,

(2) f and each g_{α} satisfy (B) on S_{α} .

This will be done by transfinite induction.

Define g_0 on $S_0 = \{e\}$ by putting

$$g_0(e) = e' .$$

Clearly f and g_0 satisfy (B) on S_0 .

Let $\beta < \gamma$ be any ordinal number such that g_{α} have been defined so that f and g_{α} satisfy (B) on S_{α} for all $\alpha < \beta$.

Case 1. $\beta = \delta + 1$ for some ordinal δ .

Since g_{δ} has been defined on S_{δ} , hence by lemma 4.5, there exists are extension $\bar{g}_{\delta}$ of g_{δ} on $\overline{S_{\delta}[x_{\delta}]}$ such that f and $\bar{g}_{\delta}$ satisfy (B) on $\overline{S_{\delta}[x_{\delta}]}$.

Put $$g_{\beta} = \bar{g}_{\delta} .$$

Then g_{β} is defined on $S_{\beta} = \overline{S_{\delta}[x_{\delta}]}$ and f and g_{β} satisfy (B) on S_{β} . It can be shown that (1) holds.

Case 2. β is a limit ordinal.

In this case, we put

$$g_{\beta} = \bigcup_{\alpha<\beta} g_{\alpha} .$$

Clearly (1) holds. From (1), it follows that g_β is well-defined on $S_\beta = \overline{\bigcup_{\alpha<\beta} S_\alpha}$ and f, g_β satisfy (B) on S_β .

Hence, for each $\beta < \gamma$, if g_α has been defined on S_α , and f and g_α satisfy (B) on S_α for all $\alpha < \beta$, then g_β can be defined on S_β so that f and g_β satisfy (B) on S_β . Therefore g_α can be defined on S_α so that f and g_α satisfy (B) on S_α for all $\alpha < \gamma$.

Define
$$g = \bigcup_{\alpha<\gamma} g_\alpha .$$

It follows from (1) that, g is well-defined on $S = \bigcup_{\alpha<\gamma} S_\alpha$. (2) implies that f and g satisfy (B) on S where $\overline{S} = G$. By lemma 4.3 and lemma 4.4, g can be extended to be continuous on G such that f and g satisfy (B) on G .

Corollary 4.8 Let V be a σ^*- compact Hausdorff topological vector space which has the groperty (Γ N) for some limit ordinal Γ. let $\mathcal{B}$ be a basis of V. Let $f : V \times V \longrightarrow \mathbb{R}$ be symmetric and continuous and satisfy

(A) $$f(x, y) + f(x + y, z) = f(y, z) + g(x, y+z)$$

for all x, y, z, in V. Then there exists a continuous function $g : V \longrightarrow \mathbb{R}$ such that

(B) $$f(x, y) = g(x) + g(y) - g(x + y)$$

for all x, y, z, in V.

Proof. Let V be a topological vector space. It follows that (V, +) is a torsion free 2 - divisible abelian topological group. Hence, the conclusion of the corollary holds.

Theorem 4.9 Let G be a group. Let $f : G \times G \longrightarrow \mathbb{R}^{(k)}$ and $f_i : G \times G \longrightarrow \mathbb{R}$, $i = 1, \ldots, k$ be such that

$$f(x, y) = (f_1(x, y), \ldots, f_k(x, y)),$$

for all x, y, in G. Then f satisfies

$$(A) \qquad f(x, y) + f(x + y, z) = f(y, z) + f(x, y + z)$$

for all x, y, z, in G , if and only if, for each $i = 1, 2, \ldots, k$, f_i satisfies

$$(A') \qquad f_i(x, y) + f_i(x + y, z) = f_i(y, z) + f_i(x, y+z)$$

for all x, y, z, in G.

Proof. Assume that f satisfies (A).

For each $i = 1, 2, \ldots, k$, let $p_i : \mathbb{R}^{(k)} \longrightarrow \mathbb{R}$ be defined by $p_i(t_1, \ldots, t_k) = t_i$ where $t_1, \ldots, t_k \in \mathbb{R}$. It is clear that each p_i is linear and $f_i = p_i \circ f$.

Hence we have

$$f_i(x,y) + f_i(x+y,z) = (p_i \circ f)(x,y) + (p_i \circ f)(x+y, z),$$

$$= p_i(f(x,y)) + p_i(f(x + y, z)),$$

$$= p_i(f(y,z) + f(x, y+z) - f(x+y, z)) + p_i(f(x+y, z)),$$

$$= p_i(f(y, z)) + p_i(f(x,y+z)) - p_i(f(x+y, z)) + p_i(f(x+y,z)),$$

$$= f_i(y, z) + f_i(x, y+z) - f_i(x+y, z) + f_i(x+y, z),$$

$$= f_i(y, z) + f_i(x, y+z) .$$

Conversely, assume that f_i, $1 \leqslant i \leqslant k$ satisfy (Λ') . Then

$$\begin{aligned} f(x, y) + f(x+y, z) &= (f_1(x,y),\ldots, f_k(x, y)) + (f_1(x+y, z),\ldots, \\ &\quad f_k(x+y, z)), \\ &= (f_1(x, y) + f_1(x+y, z), \ldots, f_k(x, y) \\ &\quad + f_k(x+y, z)), \\ &= (f_1(y, z) + f_1(x, y+z),\ldots, f_k(y, z) \\ &\quad + f_k(x, y+z)), \\ &= (f_1(y, z), \ldots, f_k(y, z)) + (f_1(x, y+z),\ldots, \\ &\quad f_k(x, y+z)), \\ &= f(y, z) + f(x, y+z). \end{aligned}$$

This completes the proof of theorem 4.9.

Theorem 4.10 Let $(G, +)$ be as in theorem 4.7. Let $f : G \times G \to \mathbb{R}^{(k)}$ be symmetric and continuous and satisfy

(A) $$f(x, y) + f(x+y, z) = f(y, z) + f(x, y+z)$$

for all x, y, z, in G. Then there exists a continuous function $g : G \longrightarrow \mathbb{R}^{(k)}$, such that

(B) $$f(x, y) = g(x) + g(y) - g(x + y)$$

for all x, y, in G.

<u>Proof.</u> Let f_i, $i = 1, \ldots, k$ be such that

$$f(x, y) = (f_1(x, y), \ldots, f_k(x,y))$$

for all x,y, in G. By theorem 4.9, we have

(A$'$) $$f_i(x,y) + f_i(x+y, z) = f_i(y, z) + f_i(x, y+z)$$

for all x,y, z, in G. It can be shown that f_i is symmetric and continuous.

Theorem 4.7 asserts that for each i , there exists a continuous function $g : G \longrightarrow \mathbb{R}$ such that

(B$'$) $$f_i(x, y) = g_i(x) + g_i(y) - g_i(x + y)$$

for all x, y, in G.

Let $g : G \longrightarrow \mathbb{R}^{(k)}$ be given by

$$g(x) = (g_1(x), \ldots, g_k(x)),$$

for x in G. Therefore g is continuous and we have

$$f(x,y) = (f_1(x,y), \dots f_k(x,y)),$$

$$= (g_1(x) + g_1(y) - g_1(x + y), \dots, g_k(x)$$

$$+ g_k(y) - g_k(x + y)),$$

$$= (g_1(x), \dots, g_k(x)) + (g_1(y), \dots, g_k(y))$$

$$- (g_1(x + y), \dots, g_k(x + y)),$$

$$= g(x) + g(y) - g(x + y) .$$

This completes the proof of theorem 4.10

Theorem 4.11 Let G be a group. Let $f : G \times G \longrightarrow (\mathbb{R}, +)$ and let $h : (\mathbb{R}, +) \longrightarrow (\mathbb{R}^+, \cdot)$ be an isomorphism. Let $\varphi = h \circ f$. Then f satisfies

(4.11.1) $$f(x,y) + f(x + y, z) = f(y, z) + f(x,y + z)$$

for all x, y, z, in G, if and only if φ satisfies

(4.11.2) $$\varphi(x, y) \cdot \varphi(x + y, z) = \varphi(y, z) \cdot \varphi(x, y + z)$$

for all x,y, z, in G.

Proof. Assume that f satisfies (4.11.1)

Since h is a homomorphism, hence we have

$$\varphi(x, y).\ \varphi(x+y, z) = (h_o f)(x,y).\ (h_o f)(x+y, z),$$

$$= h(f(x,y)).\ h(f(x+y, z)),$$

$$= h(f(x,y) + f(x+y, z)),$$

$$= h(f(y, z) + f(x, y+z)),$$

$$= h(f(y, z)).\ h(f(x, y+z)),$$

$$= \varphi(y, z).\ \varphi(x, y+z).$$

Conversely, assume that φ satisfies (4.11.2), i.e.

$$\varphi(x, y).\ \varphi(x+y, z) = \varphi(y, z).\ \varphi(x, y+z).$$

Hence

$$(h_o f)(x, y).(h_o f)(x+y, z) = (h_o f)(y,z).(h_o f)(x, y+z).$$

$$h(f(x, y)).\ h(f(x+y, z)) = h(f(y, z)).\ h(f(x, y+z)).$$

Since h is an isomorphism, hence

$$h(f(x, y) + f(x+y, z)) = h(f(y, z) + f(x, y+z)),$$

$$f(x,y) + f(x+y, z) = f(y, z) + f(x, y+z).$$

This completes the proof of theorem 4.11 .

Theorem 4.12 Let (G, +) be as in theorem 4.7. Let $\varphi : G \times G \rightarrow (\mathbb{R}^+, .)$ be symmetric and continuous and satisfy

(4.12.1) $$\varphi(x, y).\ \varphi(x+y, z) = \varphi(y, z).\ \varphi(x, y+z)$$

for all x, y, z, in G. Then there exists a continuous function

$\mathscr{C} : G \rightarrow (\mathbb{R}^+, \cdot)$ such that

$$\varphi(x, y) = \mathscr{C}(x) \cdot \mathscr{C}(y) / \mathscr{C}(x + y) .$$

<u>Proof.</u> Let $h : (\mathbb{R}, +) \longrightarrow (\mathbb{R}^+, \cdot)$ be defined by

$$h(x) = e^x \qquad \text{for all } x \text{ in } \mathbb{R} .$$

Then h is an isomorphism and is continuous.

Set $f(x, y) = \ln(\varphi(x, y))$.

Hence f is continuous and symmetric and $\varphi = h \circ f$.

By theorem 4.11, f satisfies

$$f(x,y) + f(x + y, z) = f(y, z) + f(x, y + z)$$

for all x, y, z, in G.

By theorem 4.7, there exists a continuous function $g : G \longrightarrow \mathbb{R}$ such that

$$f(x, y) = g(x) + g(y) - g(x + y).$$

Set $\mathscr{C}(x) = e^{g(x)}$.

Then $\mathscr{C}$ is continuous.

But $\varphi(x, y) = e^{f(x, y)}$, hence

$$\varphi(x, y) = e^{g(x)+g(y) - g(x+y)} ,$$

$$= e^{g(x)} \cdot e^{g(y)} \cdot e^{-g(x+y)} ,$$

$$= \mathscr{C}(x) \cdot \mathscr{C}(y) / \mathscr{C}(x + y) .$$

This completes the proof of theorem 4.12.

<u>Lemma</u> 4.13 Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function satisfying

(4.13.1) $$h(x + y) = h(x) + h(y) ,$$

for all x, y, in $\mathbb{R}$. Then $h(r) = rh(1)$ for all rational number r.

<u>Proof.</u> Since h is a homomorphism, hence $h(0) = 0$.

Therefore $$h(0.a) = h(0) = 0 = 0.h(a) , \text{ for all } a \in \mathbb{R} .$$

Assume that m is a non - negative integer such that

$$h(ma) = mh(a) .$$

Then, $$\begin{aligned} h((m + 1) a) &= h(ma + a), \\ &= h(ma) + h(a), \\ &= mh(a) + h(a), \\ &= (m + 1) h(a) . \end{aligned}$$

Hence $h(na) = nh(a)$ for all non - negative integer n.

For any negative integer m', $- m'$ is a positive integer.

Hence we have

$$\begin{aligned} 0 = h(0) &= h(m'a + (-m')a) , \\ &= h(m'a) + h((-m') a), \\ &= h(m'a) + (-m') h(a), \end{aligned}$$

Thus $h(m'a) = m'h(a)$.

Therefore $h(na) = nh(a)$ for all integer n.

Let r be any rational number say $r = \frac{p}{q}$ where p, q are in teger and $q > 0$.

$$h(1) = h(q.\frac{1}{q}) = q\,h(\frac{1}{q})$$

Hence $h(\frac{1}{q}) = \frac{1}{q}\,h(1)$.

Now, we have

$$h(r) = h(\frac{p}{q}); = p.h(\frac{1}{q}), = p.\frac{1}{q}\,h(1), = r\,h(1).$$

Lemma 4.14 Let V be a topological vector space with $\mathcal{B}$ as a basis. Let $h : V \longrightarrow \mathbb{R}$ be a continuous function satisfying.

(4.14.1) $h(x + y) = h(x) + h(y)$,

(4.14.2) $h(v) = 0$ for all $v \in \mathcal{B}$.

Then $h(x) = 0$ for all x in V.

Proof. For each $v \in \mathcal{B}$, let $\mathfrak{T}_v : \mathbb{R} \to V$ be defined by

$$\mathfrak{T}_v(x) = xv,$$

for $x \in \mathbb{R}$.

It is clear that each $\mathfrak{T}_v$ is linear and continuous.

Set $$h_v = h \circ \mathfrak{T}_v .$$

Hence $h_v : \mathbb{R} \to \mathbb{R}$ is continuous.

Observe that

$$\begin{aligned} h_v(x+y) &= h \circ \mathfrak{T}_v (x+y), \\ &= h(\mathfrak{T}_v(x+y)), \\ &= h(\mathfrak{T}_v(x) + \mathfrak{T}_v(y)), \\ &= h \circ \mathfrak{T}_v(x) + h \circ \mathfrak{T}_v(y), \\ &= h_v(x) + h_v(y) . \end{aligned}$$

By lemma 4.13, we have

(4.14.3) $$h_v(r) = rh_v(1)$$

for all rational number r.

Let $p_v : V \to \mathbb{R}$ be defined by $p_v(x) = x_v$, where x_v is the unique real number given by the representation

$$x = \sum_{v \in \mathcal{F}} x_v v ,$$

where $\mathcal{F}$ is a finite subset of $\mathcal{B}$.

From $h_v = h \circ \mathbb{T}_v$, we have $h_v \circ p_v = (h \circ \mathbb{T}_v) \circ p_v$.

$$h_v \circ p_v(rv) = h \circ \mathbb{T}_v \circ p_v(rv),$$

$$= h(\mathbb{T}_v(r)),$$

$$= h(rv).$$

This together with (4.14.3) inply

$$h(rv) = h_v \circ p_v (rv),$$

$$= h_v(p_v(rv)),$$

$$= h_v(r),$$

$$= rh_v(1),$$

$$= rh \circ \mathbb{T}_v(1),$$

$$= r\, h(v).$$

Thus

(4.14.4) $$h(rv) = r\, h(v),$$

for all rational number r and all $v \in \mathcal{B}$.

Let $x \in V$. Then $x = \sum_{v \in \mathcal{F}} x_v v$ where $x_v \in \mathbb{R}$ and $\mathcal{F}$ is a finite subset of $\mathcal{B}$. Since the set of rational numbers is dense in $\mathbb{R}$, for each v, we can find a sequence $\{r_{vn}\}$ converges to x_v as $n \to \infty$. Hence $\{\sum_{v \in \mathcal{F}} r_{vn} v\}$ converges to $\sum_{v \in \mathcal{F}} x_v v$. Since h is continuous, hence

$$\lim_{n \to \infty} h\left(\sum_{v \in \mathcal{F}} r_{vn} v\right) = h\left(\sum_{v \in \mathcal{F}} x_v v\right).$$

But

$$\begin{aligned} \lim_{n \to \infty} h\left(\sum_{v \in \mathcal{F}} r_{vn} v\right) &= \lim_{n \to \infty} \left(\sum_{v \in \mathcal{F}} r_{vn} h(v)\right), \\ &= \sum_{v \in \mathcal{F}} \lim_{n \to \infty} r_{vn} h(v), \\ &= \sum_{v \in \mathcal{F}} x_v h(v). \end{aligned}$$

Therefore

$$\begin{aligned} h(x) &= \sum_{v \in \mathcal{F}} x_v h(v), \\ &= 0. \end{aligned}$$

Since x is arbitrary, hence we have

$$h(x) = 0 \quad \text{for all } x \text{ in } V.$$

Theorem 4.15 Let V be a $\mathcal{C}^*$ - compact Hausdorff topological vector space which has the property (ΓN) for some limit ordinal Γ. Let $\mathcal{B}$ be a basis of V. Let $f : V \times V \longrightarrow \mathbb{R}^{(k)}$ be symmetric and continuous and satisfy

(A) $$f(x, y) + f(x + y, z) = f(y, z) + f(x, y + z)$$

for all x, y, z in V. Then there exists a unique continuous function $g : V \longrightarrow \mathbb{R}^{(k)}$ such that $g(v) = \underline{0}$ for all $v \in \mathcal{B}$ and

(B) $$f(x, y) = g(x) + g(y) - g(x + y)$$

for all x, y, z, in V.

Proof. By virtue of theorem 4.9, it suffices to prove this theorem in the case k = 1. Hence we shall assume that $f : V \times V \longrightarrow \mathbb{R}$,

We shall show that V is $(\gamma, 2)$ divisible for some ordinal γ.

If $V = \{\underline{0}\}$, then set $S_0 = \langle \emptyset \rangle = \{\underline{0}\}$. Therefore V is (1, 2) divisible, Assume that $V \neq \{\underline{0}\}$.

Let $\{v_\alpha\}_{(\alpha < \eta)}$ be a well - ordering of the basis $\mathcal{B}$, where η is an ordinal number. We shall show that V is $(\omega\eta, 2)$ divisible.

Define

$$S_0 = \{\underline{0}\} \quad \text{and} \quad x_0 = v_0$$

Let $\beta < \omega\eta$ be any ordinal number such that S_α, x_α have been defined for all $\alpha < \beta$. We shall define S_β and x_β as follows :

Case 1 $\beta = \delta + 1$ for some ordinal δ.

Define $$S_\beta = \overline{\langle S_\delta \cup \{x_\delta\} \rangle} .$$

Since $x_\delta \in S_\delta$, hence there exists x^* in V such that $2x^* = x_\delta$. Define $x_\beta = x^*$.

<u>Case</u> 2. β is a limit ordinal.

By theorem 27(i), $\beta = \omega\xi$ for some ordinal ξ.

Define
$$S_\beta = \overline{\bigcup_{\alpha<\beta} S_\alpha}$$

$$x_\beta = v_\xi .$$

We claim that $\overline{\bigcup_{\beta<\omega\eta} S_\beta} = V$.

Clearly $\overline{\bigcup_{\beta<\omega\eta} S_\beta} \subseteq V$. We will show that $V \subseteq \overline{\bigcup_{\beta<\omega\eta} S_\beta}$.

Let x be in V. Since $\{v_\alpha / \alpha<\eta\}$ forms a basis of V, hence x can be written in the form $\sum_{i=1}^{k} a_{\alpha_i} v_{\alpha_i}$ where $v_{\alpha_1}, \dots, v_{\alpha_k} \in \mathcal{B}$, and $a_{\alpha_i} \in \mathbb{R}$. We may assume that $\alpha_1 < \alpha_2 < \dots < \alpha_k$. Since $v_{\alpha_k} = x_{\omega\alpha_k}$, hence x must be in $\bar{S}_{\omega(\alpha_k+1)}$. But $\bar{S}_{\omega(\alpha_k+1)} \subseteq \overline{\bigcup_{\alpha<\omega\eta} S_\alpha}$, therfore $x \in \overline{\bigcup_{\alpha<\omega\eta} S_\alpha}$.

For all $\beta < \omega\eta$, define g_β as in the proof of theorem 4.7 with the special choice of x_β as given above. Moreover, we define $g_0(0) = 0$ and in the case that $\beta = \omega\xi$ for some ordinal ξ, define $g_{\beta+1}(v_\xi) = 0$.

Let
$$g = \bigcup_{\alpha<\omega\eta} g_\alpha .$$

Hence, it follows from corollary 4.8 that g and f satisfy (B) on V and we also have $g(v) = 0$ for all $v \in \mathcal{B}$.

To prove uniqueness of g, assume that f, g_1 and f, g_2 satisfy (B). Hence

$$g_1(x) + g_1(y) - g_1(x + y) = g_2(x) + g_2(y) - g_2(x + y)$$

This implies

$$g_1(x) - g_2(x) + g_1(y) - g_2(y) = g_1(x + y) - g_2(x + y)$$

Set $h = g_1 - g_2$.

Therefore $h(x) + h(y) = h(x + y)$.

Since g_1 and g_2 are continuous, hence h is continuous.

Observe that

$$h(v) = g_1(v) - g_2(v) = 0 \qquad \text{for all } v \in \mathcal{B}.$$

Hence, by lemma 4.14, we have $h(x) = 0$ for all x in V.

This implies $g_1(x) = g_2(x)$ for all x in V. Therefore g is unique.

<u>Corollary</u> 4.16 Let $\mathbb{R}^{(k)}$ be a real Euclidean space of finite dimension k, with $\{e_1, \ldots, e_k\}$ as a basis. Suppose that $f : \mathbb{R}^{(k)} \times \mathbb{R}^{(k)} \longrightarrow \mathbb{R}$ is symmetric and continuous and satisfies

(A) $$f(x, y) + f(x + y, z) = f(y, z) + f(x, y + z)$$

for all x, y, z, in $\mathbb{R}^{(k)}$, then there exists a unique continuous $g : \mathbb{R}^{(k)} \longrightarrow \mathbb{R}$ such that $g(e_1) = \ldots = g(e_k)$ and

(B) $$f(x, y) = g(x) + g(y) - g(x + y)$$

for all $x, y, z,$ in $\mathbb{R}^{(k)}$.

Proof. It can be shown that $(\mathbb{R}^{(k)}, +)$ is σ^*- compact Hausdorff topological vector space. To show that $\mathbb{R}^{(k)}$ has the property $(\omega^k N)$, let $b = (b_1, \ldots, b_k)$ be any accumulation point of any set $A \subseteq \mathbb{R}^{(k)}$. Hence there exists a sequence $\{a^{(n)}\}$ in A which converges to b. For each n, we have

$$a^{(n)} = (a_1^{(n)}, \ldots, a_k^{(n)})$$

where $\lim a_i^{(n)} = b_i$, $i = 1, 2, \ldots, k$.

We define a ω^k - net as follows :

For $\beta = \omega^{k-1}p_1 + \omega^{k-2}p_2 + \cdots + \omega p_{k-1} + p_k$, define

$$x_\beta = (a_1^{(p_1+1)}, a_2^{(p_1+p_2+1)}, \ldots\ldots, a_k^{(p_1+\cdots p_k+1)}).$$

APPENDIX

Axiom of Choice and Transfinite Numbers.

This appendix is devoted to a brief account on the axiom of choice, ordinal numbers, cardinal numbers,

Before speaking of axiom of choice, a definition is needed.

Definition A-1 A choice function on a set of non - empty sets $\mathcal{A}$ is a function $\theta : \mathcal{A} \longrightarrow \bigcup \mathcal{A}$ such that for all $A \in \mathcal{A}$, $\theta(A) \in A$.

Axiom of choice.

Every set of non - empty sets has a choice function.

We now introduce some definitions which will be useful later.

Definition A-2 let X and Y be sets. Then X and Y are said to be equipotent (denoted by $X \approx Y$) if and only if there exists a one - to - one correspondence between X and Y.

Definition A-3 A set is infinite if it is equipotent to a proper subset of itself. Otherwise a set is finite.

Definition A-4 A well - ordering on a set X is a relation r on X satisfying

i) Reflexive law : $(a, a) \in r$ for all $a \in x$

ii) Antisymmetric law : $(a, b) \in r$ and $(b, a) \in r$ imply $a = b$ for all $a, b \in X$.

iii) Transitive law : $(a, b) \in r$ and $(b,c) \in r$ imply $(a, c) \in r$ for all $a, b, c \in X$.

iv) Every non - empty subset S of X contains an element m such that $(m, x) \in r$ for every $x \in X$.

(X, r) is said to be a <u>well - ordered set.</u>

If there is no confusion we sometimes use X to denote both the well - ordered set and the underlying set on which the well - ordering defined.

It is customary to denote a well - ordering r by $\leq$ and write $x \leq y$ to denote the fact that $(x, y) \in r$. We further agree that $y \geq x$ has the same meaning as $x \leq y$, and that $x \nleq y$ mean that $(x, y) \notin r$.

We agree that $x < y$ is an abbreviation for "$x \leq y$ and $x \neq y$"

<u>Well - ordering theorem.</u>

If X is any set then there exists a relation r such that r is a well - ordering of X.

<u>Remark</u> A - 5 Any two elements x and y in a well - ordered set $(X, \leq)$ is either $x \leq y$ or $y \leq x$.

<u>Definition</u> A - 6 Let $(A, \leq)$ and $(B, \leq)$ be well - ordered sets. A function $f : A \to B$ is called <u>order - isomorphism</u> if it is a one - to - one correspondence from A to B and satisfies the following condition :

For every two elements $x \in A$ and $y \in A$,

$x \leq y$ (in A) if and only if $f(x) \leq f(y)$ (in B).

Definition A - 7 If $(A, \leq)$ and $(B, \leq)$ are well - ordered sets and there exists an order - isomorphism from A to B, we say that A is order - isomorphic with B.

Definition A - 8 Let A be a well - ordered set and suppose $a \in A$. The initial segment of A determined by a is the set I_a, defined as follows :

$$I_a = \{ x \in A / x < a \} .$$

We shall define ordinal numbers as special types of well - ordered set ,

Definition A - 9 Let A be a set on which a well - ordering $\leq$ can be defined such that for all $x \in A$, $x = I_x$. Then A is called an ordinal number.

Definition A - 10 . Let α and β be ordinal numbers. We say that $\alpha \leq \beta$ if and only if $\alpha \subseteq \beta$.

Remark A - 11 It can be shown that if α and β are ordinal numbers, then either $\beta = \alpha$ or $\beta < \alpha$, or $\alpha < \beta$, (using the property that any well - ordered set can not be order - isomorphic with one of its segments.)

Definition A - 12. Let α and β be ordinal numbers such that $\alpha < \beta$. We will call β an immediate successor of α if there is no ordinal number η such that $\alpha < \eta < \beta$.

We may define α to be an immediate predecessor of β if and only if β is an immediate successor of α .

Definition A - 13   Let β be a non-zero ordinal number; if β has no immediate predecessor - that is, if β is not equal to $\alpha \cup \{\alpha\}$ for any ordinal α - then β is called a limit ordinal. Otherwise β is called a nonlimit ordinal.

Definition A - 14   An ordinal number μ is said to be transfinite ordinal if μ is infinite. Otherwise μ is called a finite ordinal.

Remarks A - 15   i) $\emptyset$ is an ordinal. The only relation on $\emptyset$ is $\emptyset$ itself, Clearly it is a well - ordering on $\emptyset$ and there is no $a \in \emptyset$ such that $a \neq I_a$.

ii) If α is an ordinal and $\beta \in \alpha$ then β is also an ordinal.

iii) If α is an ordinal then $\alpha \cup \{\alpha\}$ is an ordinal. From this remark we know that :

$\emptyset$ is an ordinal.

$\emptyset \cup \{\phi\}$ is an ordinal.

$\{\emptyset\} \cup \{\{\phi\}\} = \{\emptyset, \{\phi\}\}$ is an ordinal.

It is customary to denote $\emptyset$ by 0, $\{\emptyset\}$ by 1, $\{\emptyset, \{\emptyset\}\}$ by 2 and so on. We shall define ω to be a set of all finite ordinals. It can be shown that ω is a limit ordinal.

Definition A - 16 If α is an ordinal then $\alpha^+ = \alpha \cup \{\alpha\}$.

Remark A - 17 We now have,

$$\omega \cup \{\omega\} = \omega^+ = \{0, 1, 2, \ldots, \omega\},$$

$$\omega^+ \cup \{\omega^+\} = \omega^{++} = \{0, 1, \ldots, \omega, \omega^+\},$$

and so on

It can be shown that for any well - ordered set $(X, \leq)$, there exists a unique ordinal number that is order-isomorphic to $(X, \leq)$.

Definition A - 18 Let $(X, \leq)$ be a well - ordered set. The ordinal of $(X, \leq)$ denoted by ΘA, is the unique ordinal number that is order- isomorphic to $(X, \leq)$.

Remark A - 19 Let $(A, \leq)$ and $(B, \leq)$ be disjont well - ordered sets. Let $C = A \cup B$ and $\leq$ be defined on C as follows :

for $x, y \in C$, $x \leq y$ if and only if

i) $x \in A$ and $y \in A$ and $x \leq y$ in A or,

ii) $x \in B$ and $y \in B$ and $x \leq y$ in B or,

iii) $x \in A$ and $y \in B$.

Then $(C, \leq)$ is a well - ordered set.

Definition A - 20 Let α and β be ordinal numbers, and let A and B be disjoint well - ordered sets such that $\alpha = \Theta A$ and $\beta = \Theta B$ We will define the sum $\alpha + \beta$ to be the ordinal number of the well - ordered set $(A \cup B, \leq)$ as defined in remark A - 19. i.e, $\alpha + \beta = \Theta(A \cup B)$.

Remark A - 21. Using the above definition of sum of ordinal numbers, it can be seen that $\alpha + 1 = \alpha \cup \{\alpha\}$.

Remark A - 22. Let $(A, \leq)$ and $(B, \leq)$ be well - ordered sets.

Let $C = A \times B$ and define $\leq$ on C as follows :

for $(a, b), (a_1, b_1) \in C$, $(a,b) \leq (a_1, b_1)$ if and only if

i) $a \leq a_1$ and $b = b_1$ or

ii) $b < b_1$.

Then $(C, \leq)$ is a well - ordered set.

Definition A - 23. Let α and β be ordinal numbers. Let A and B be well - ordered sets such that $\alpha = \Theta A$ and $\beta = \Theta B$. We will define the product $\alpha\beta$ to be the ordinal number of the well - ordered set $(A \times B, \leq)$ as defined in remark A - 22 .

The elementary properties of ordinal addition, multiplication and comparison, given in the following theorems can be seen in [6] .

Theorem A - 24. Let α, β and γ be ordinal numbers. Then

i) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$,

ii) $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$,

iii) if $\beta > 0$ then $\alpha < \alpha + \beta$.

Theorem A - 25. Let α and β be ordinals such that $\alpha < \beta$. Then there exists a unique ordinal $\gamma > 0$ such that $\alpha + \gamma = \beta$.

Theorem A - 26. For any ordinals α, β, γ , the folowing rules hold :

i) $\alpha < \beta \implies \gamma + \alpha < \gamma + \beta$,

ii) $\alpha < \beta, \ \gamma > 0 \implies \gamma\alpha < \gamma\beta$,

iii) $\alpha\gamma < \beta\gamma \implies \alpha < \beta$;

iv) $\gamma > 0, \ \gamma\alpha = \gamma\beta \implies \alpha = \beta$.

<u>Theorem</u> A-27 i) α is a limit ordinal if and only if there exists a unique ordinal $\xi > 0$ such that $\alpha = \omega\xi$.

ii) If α is a nonlimit ordinal, there exists a unique ordinal ξ and a unique finite ordinal $n \neq 0$ such that

$$\alpha = \omega\xi + n .$$

<u>Corollary</u> A-28 Let α, α' be limit ordinals and $n, n' \in \omega$.

If $\alpha + n = \alpha' + n'$ then $n = n'$ and $\alpha = \alpha'$.

<u>Proof.</u> If $n' = 0$, then $\alpha + n = \alpha'$ is a limit ordinal, hence $n = 0 = n'$, and it follows that $\alpha = \alpha'$. Next, suppose that $n \neq 0, n' \neq 0$. Since α and α' are limit ordinals, by theorem A - 27(i) there exist the unique ξ and ξ' such that $\alpha = \omega\xi$ and $\alpha' = \omega\xi'$. From our hypothesis we have

$$\alpha + n = \alpha' + n' ,$$

$$\omega\xi + n = \omega\xi' + n' .$$

Again by theorem A - 27(ii) we have $\xi = \xi'$ and $n = n'$.
From $\xi = \xi'$ it follows that $\omega\xi = \omega\xi'$.

<u>Theorem</u> A - 29 Let $n \in \omega$ and α be a transfinite ordinal.
Then $n + \alpha = \alpha$.

In proving this theorem, use the property that any two ordinal numbers are equal if and only if they are order-isomorphic(the details will be omitted).

<u>Theorem A-30</u> Let ξ, ξ', n and n' be any ordinals such that $n, n' \in \omega$

i) If $\omega\xi + n2 \geqslant \omega\xi' + n'2$ then $\omega\xi + n \geqslant \omega\xi' + n'$.

ii) If $\omega\xi + n2 \geqslant \omega\xi' + n'2 + 1$ then $\omega\xi + n \geqslant \omega\xi' + n'$.

<u>Proof.</u> i) Let ξ, ξ', n, n' be ordinals such that $n, n' \in \omega$. **First we assume that none of ξ, ξ', n, n' is zero. Assume that $\omega\xi + n2 = \omega\xi' + n'2$.**

It follows from corollary A-28 that.

$$\omega\xi = \omega\xi' \quad \text{and} \quad n2 = n'2 .$$

Since $n2$ and $n'2$ are finite, hence $n = n'$.

Therefore, $\omega\xi + n = \omega\xi' + n'$.

Assume that $\omega\xi + n2 > \omega\xi' + n'2$.

From theorem A-25, there exists a unique ordinal $\gamma > 0$ such that

$$(\omega\xi' + n'2) + \gamma = \omega\xi + n2 .$$

<u>Case</u> 1 $\gamma < \omega$.

From theorem A-24(i), we have

$$(\omega\xi' + n'2) + \gamma = \omega\xi' + (n'2 + \gamma).$$

Since $\omega\xi$ and $\omega\xi'$ are limit ordinals and $n'2 + \gamma$ and $n2$ are finite, by corollary A-28, we have

$$\omega\xi = \omega\xi' \quad \text{and} \quad n'2 + \gamma = n2.$$

By theorem A-24(iii), we have

$$n'2 < n'2 + \gamma .$$

Therefore, $n'2 < n2$.

By theorem A-26(iii), we have $n' < n$.

By theorem A-26(i), we have

$$\omega\xi' + n' < \omega\xi + n.$$

<u>Case 2</u> $\gamma = \omega\xi'' + n''$ for some ordinal $\xi'' \neq 0$ and $n'' < \omega$.

$$\omega\xi + n2 = (\omega\xi' + n'2) + \omega\xi'' + n'' .$$

By theorem A-24(i), we have

$$(\omega\xi' + n'2) + \omega\xi'' + n'' = \omega\xi' + (n'2 + (\omega\xi'' + n'')) ,$$

$$= \omega\xi' + (n'2 + \omega\xi'') + n'' .$$

By theorem A-29, we have

$$n'2 + \omega\xi'' = \omega\xi''$$

By theorem A-24 (ii), we have

$$\omega\xi' + \omega\xi'' = \omega(\xi' + \xi'') .$$

Hence $\omega\xi + n2 = \omega(\xi' + \xi'') + n''$.

By corollary A-28, we have

$$\omega\xi = \omega(\xi' + \xi'') \quad \text{and} \quad n2 = n'' .$$

By theorem A-26(iv), we have

$$\xi = \xi' + \xi'' .$$

Therefore by theorem A-24(iii), $\xi' < \xi$.

Hence, by theorem A-26(ii), we have $\omega\xi' < \omega\xi$.

Suppose that there is a finite ordinal m' such that

$$\omega\xi' + m' \geq \omega\xi .$$

Hence $\omega\xi' + m' = \omega\xi$ or $\omega\xi' + m' > \omega\xi$.

<u>Case</u> 2.1 Suppose $\omega\xi' + m' = \omega\xi$.

By the corollary A-28, $\omega\xi' = \omega\xi$ and $m' = 0$.

By theorem A-26(iv), $\xi' = \xi$ which is a contradiction.

<u>Case</u> 2.2 Suppose $\omega\xi' + m' > \omega\xi$.

By theorem A-25, there exists a unique ordinal $\eta > 0$ such that

$$\omega\xi' + m' = \omega\xi + \eta .$$

If η is finite then by corollary A-28 we have $\omega\xi' = \omega\xi$ which is a contradiction. If η is transfinite, then by theorem A-27(i), A-24(ii), corollary A-28 and theorem A-24(iii) respectively we have $\xi < \xi'$ this is also a contradiction.

Hence $\omega\xi' + m' < \omega\xi$ for all $m' < \omega$.

Therefore

$$\omega\xi' + m' < \omega\xi + m$$

for all $m', m \in \omega$. **In particular, we have**

In particular, $\omega\xi' + n' < \omega\xi + n$.

Proofs of the cases in which any of ξ, ξ', n, n' are zero can be done in a similar fashion.

We have proved (i). Since $\omega \xi'_{\gamma} + n' 2 + 1 > \omega \xi'_{\gamma} + n' 2$, hence (ii) follows immediately from (i).

Cardinals.

Definition A-31. Let X be a set. The cardinal of X, denoted by $\overline{\overline{X}}$, is the smallest ordinal S for which $S \approx X$.

Definition A-32 Let α and β be cardinals. We say that $\alpha \leq \beta$ if $\alpha \subseteq \beta$, and $\alpha < \beta$ if $\alpha \leq \beta$ but $\alpha \neq \beta$.

The following are facts about cardinals. We state these facts for later refferences. Their proofs can be found in [8] .

Theorem A-33 Let A and B be sets such that α and β be the cardinals of A and B respectively. If A is equipotent to a subset of B, but A and B are not equipotent then $\alpha < \beta$.

Theorem A-34 $\overline{\overline{X}} < \overline{\overline{\mathcal{P}(X)}}$ for every set X.

Lemma A-35 Each infinite cardinal number is a limit number.

Proof. Let α be an infinite cardinal number. Since α is a cardinal, α is also an ordinal.

Suppose that α is not a limit ordinal.

Hence there exists an ordinal β such that

$$\beta + 1 = \alpha .$$

We will show that β is equipotent with $\beta + 1$.

Since $\beta + 1$ is infinite, β is also infinite.

Define $f : \beta + 1 \longrightarrow \beta$ by

$$f(\beta) = 0$$

$$f(n) = n + 1 \quad \text{for } n \in \omega$$

$$f(x) = x \quad \text{where } x \in \beta - \omega .$$

Then $\beta \approx \beta + 1$, i.e, $\beta \approx \alpha$.

But $\beta < \beta + 1$ and $\beta \approx \alpha$, then α is not a cardinal, contrary to to our hypothesis, where α is a limit ordinal.