CHAPTER VI

ON THE STRUCTURE OF PRE J - RINGS

The materials of this chapter are drawn from references [2], where theorem 6.6 was proven for the special case when $k = 1$, i.e. a pre - p - ring.

The purpose of this chapter is to prove the following main theorem concerning the structure of pre - p^k - rings and pre - J - rings.

Definition 6.1. A commutative ring R whose characteristic is a prime p is called a pre - p^k - ring (k an integer ≥ 1) if $xy^{p^k} = x^{p^k}y$ for x and y in R.

Definition 6.2. A commutative ring R is called a pre - J - ring if there exists a positive integer $n > 1$ such that $xy^n = x^n y$ for $x, y \in R$. Call n the order of the pre - J - ring R

The proof of theorem 6.6 is based an the following lemmas.

Lemma 6.3. Let R be a pre - p^k - ring. If h and l are integers such that $l \geq p^k + 2$ and $h - l = t(p^k - 1)$ for some $t \geq 1$. Then

$$x^h = x^l \qquad \text{for every } x \text{ in } R .$$

<u>proof</u> Let $l = p^k + 2 + u \qquad u \geq 0$

$$h = t(p^k - 1) + 1 \qquad \text{for some } t \geq 1$$

Since R is a pre - p^k- ring,

$$xy^{p^k} = x^{p^k}y \qquad \text{for every } x, y \text{ in } R$$

Substituting x^2 for y in the above equality, we obtain

$$x^{2p^k + 1} = x^{p^k + 2} .$$

Multiplying both sides of the last equality by x^u, we obtain

$$x^{2p^k + 1 + u} = x^{p^k + 2 + u} .$$

Multiplying both sides by $x^{b(p^k - 1)}$ $\quad b \geq 1$

For each $b = 1, 2, \ldots t$ we obtain a sequence of equalities which imply

$$x^{p^k + 2 + u} = x^{2p^k + 1 + u} = x^{3p^k + u} = \ldots \; x^{p^k + 2 + u + t(p^k - 1)}$$

Therefore $\quad x^h = x^l$.

<u>Lemma 6.4</u>. Let R be a pre - p^k- ring, N the nil radical of R and $B = \{ r \in R \mid r^{p^k} = r \}$ (i) Then every element $r \in R$ has a unique representation of the form $r = b + q$, $b \in B$, $q \in N$.
(ii) $bq = 0$

<u>proof</u>. (i) For every r in R, let

$$b = r^{p^{2k}+p^k-1} \quad \text{and}$$

$$q = r - r^{p^{2k}+p^k-1}$$

Then $b + q = r^{p^{2k}+p^k-1} + r - r^{p^{2k}+p^k-1} = r$.

Since $p^{2k}+p^k-1 > p^{2k}+p^k-2 = (p^k+2)(p^k-1)$ and

$$p^{3k}+p^{2k}-p^k-(p^{2k}+p^k-1) = (p^{2k}+p^k-1)(p^k-1) ,$$

it follows from lemma 6.3 that

$$b = r^{p^{2k}+p^k-1} = r^{p^{3k}+p^{2k}-p^k}$$

$$= r^{(p^{2k}+p^k-1)p^k} = b^{p^k} .$$

Thus, for every r in R , b is in B. Moreover since $p^{2k} \geq p^k+2$ and $p^{4k}+p^{3k}-p^{2k}-p^{2k} = p^{2k}(p^k+2)(p^k-1)$.

It is a consequence of lemma 6.3 that

$$r^{p^{2k}} = r^{p^{4k}+p^{3k}-p^{2k}}$$

$$= r^{(p^{2k}+p^k-1)\,p^{2k}}$$

and it follows that, $q^{p^{2k}} = 0$.

Thus, for every r in R, q is in N .

Clearly B and N are subrings of R. Also, from theorem 3.4, $N \cap B = \{0\}$. Finally, we will prove that the representation is unique.

$$\text{If} \quad r = b_1 + q_1 = b_2 + q_2 \qquad b_1, b_2 \in B, \; q_1 q_2 \in N$$

$$\text{Then} \quad b_1 + q_1 - b_2 = b_2 + q_2 - b_2$$

$$b_1 - b_2 = q_2 - q_1$$

Since $N \cap B = \{0\}$. Hence $b_1 - b_2 = 0 \Rightarrow b_1 = b_2$

and $q_2 - q_1 = 0 \Longrightarrow q_2 = q_1$

The proof is completed.

(ii) In the view of definition 6.1 and the definition of a p^k- ring we have that

$$bq = b^{p^k} q = bq^{p^k} = b^{p^k} q^{p^k} = bq^{p^{2k}} = 0$$

Lemma 6.5. Let R be a pre -p^k- ring. Then B and N are ideals of R.

proof. The fact that B and N are subrings of R, as pointed out above, is obvious. Now let r be in R and b be in B. By lemma 6.4

$$r = b' + q' \qquad b' \in B, \; q' \in N$$

$$\begin{aligned} \text{and} \qquad rb &= (b' + q')b \\ &= b'b + q'b \\ &= b'b \end{aligned}$$

Since B is a subring, bb' is in B, and consequently rb is in B , for every r in R and every b in B. Thus, B is an ideal of R. Similarly N is ideal of R.

Theorem 6.6 . Every pre - p^k- ring is a direct sum of a p^k- ring and a nil radical(i.e. $R = B \oplus N$)

proof . By lemma 6.5 , B and N are ideals of R, B is p^k- ring and N is the nil radical of R. The fact that $R = B \oplus N$ then follows from lemma 6.4 .

Now, observe that if a pre - J - ring has characteristic n where n is the order of R, then we can prove that, every pre - J - ring is a direct sum of a J - ring and its nil radical in the same way as we just did for pre - p^k- rings. If we assume that a pre - J - ring R has the property that $\{x \in R \mid xy = 0 \ \ \forall y \in R\} = \{0\}$ then we can still prove theorem 6.6. The proof is almost the same except lemma 6.4 since we don't know that the characteristic of R is n. Therefore in order to complete the proof in this case two remarks and a lemma are needed.

Remark (1) Let R be pre - J - ring with the property that $\{x \in R \mid xy = 0 \ \ \forall y \in R\} = \{0\}$ then we have

$$\left[-\binom{n^2}{1} y^{n^2} z^{-1} + \binom{n^2}{2} y^{n^2-2} z^2 \dots \pm \binom{n^2}{n^2-1} y z^{n^2-1} \right] = 0$$

(altermating sum) $\forall y, \ z \in R$

<u>proof.</u> Since R is pre - J - ring, we have

$$xy^n = x^n y \qquad \text{for } x, y \text{ in } R$$

Therefore $xy^{n^2} = x(y^n)^n = x^n y^n = (x^n)^n y = x^{n^2} y$

Hence $x(y-z)^{n^2} = x^{n^2}(y-z) = x^{n^2} y - x^{n^2} z$ ________ (A)

<u>Case 1</u> If n is odd, from (A) we have

$$x\left[y^{n^2} - \binom{n^2}{1} y^{n^2-1} z + \binom{n^2}{2} y^{n^2-2} z^2 - \binom{n^2}{3} y^{n^2-3} z^3 + \cdots\cdots\cdots + \binom{n^2}{n^2-1} yz^{n^2-1} - z^{n^2}\right] = x^{n^2} y - x^{n^2} z$$

Thus $$+\, x\left[-\binom{n^2}{1} y^{n^2-1} z + \binom{n^2}{2} y^{n^2-2} z^2 \cdots + \binom{n^2}{n^2-1} yz^{n^2-1}\right] = 0$$

Since $\{ x \in R \mid xy = 0 \ \forall y \in R \} = \{0\}$. We have

$$\left[-\binom{n^2}{1} y^{n^2-1} z + \binom{n^2}{2} y^{n^2-2} z^2 + \cdots + \binom{n^2}{n^2-1} yz^{n^2-1}\right] = 0 .$$

<u>Case 2</u> If n is even

Since $x(-z)^n = x^n(-z)$,

Hence $xz^{n^2} = -x^{n^2} z$

From (A) we have

$$x\left[y^{n^2} - \binom{n^2}{1} y^{n^2-1} z + \binom{n^2}{2} y^{n^2-2} z^2 \cdots - \binom{n^2}{n^2-1} yz^{n^2-1} + z^{n^2}\right] = x^{n^2} y - x^{n^2} z$$

$$xy^{n^2} + x\left[-\binom{n^2}{1} y^{n^2-1} z + \binom{n^2}{2} y^{n^2-2} z^2 - \cdots - \binom{n^2}{n^2-1} yz^{n^2-1}\right] + xz^{n^2} = x^{n^2} y + x^{n^2} z.$$

Thus, similarly to case 1, we have

$$\left[-\binom{n^2}{1} y^{n^2-1} z + \binom{n^2}{2} y^{n^2-2} z^2 \cdots - \binom{n^2}{n^2-1} y z^{n^2-1}\right] = 0$$

Remark (2) If R is pre - J - ring whose order n is even, Then $r^{n^4+n^3-n^2} = -r^{n^4+n^3-n^2} \quad \forall\, r \in R$

proof. Since n is even, $n^4+n^3-n^2 =$ even

and $n^4+n^3-1 =$ odd .

Consider $r^{n^4+n^3-n^2} = r^{n^4-n^2} \cdot r^{n^3} = \left(r^{n^3-n}\right)^n r^{n^3}$

$= r^{n^3-n}\ r^{n^4} = r^{n^4+n^3-n}$

$= \left(r^{n^3-1}\right)^n \left(r^{n^3}\right) = r^{n^3-1} \cdot r^{n^4}$.

$= r^{n^4+n^3-1}$

Thus $-r^{n^4+n^3-n^2} = -r^{n^4+n^3-1}$.

Since $(-r)^{n^4+n^3-n^2} = (-r)^{n^4-n^2} (-r)^{n^3}$.

Using the same process as above we have

$(-r)^{n^4+n^3-n^2} = (-r)^{n^4+n^3-1}$.

Or $r^{n^4+n^3-n^2} = -r^{n^4+n^3-1}$.

Thus $r^{n^4+n^3-n^2} = -r^{n^4+n^3-n^2}$.

Lemma 6.7 If R be a pre - J - ring , then

$$(r-r^{n^2+n-1})^{n^2} = 0 \qquad \text{for every } r \text{ in } R .$$

proof Case 1 n = odd

$$(r-r^{n^2+n-1})^{n^2} = r^{n^2} - \binom{n^2}{1} r^{n^2-1} r^{n^2+n-1} + \binom{n^2}{2} r^{n^2-2} \left(r^{n^2+n-1}\right)^2 \cdots$$

$$+ \binom{n^2}{n^2-1} r \left(r^{n^2+n-1}\right)^{n^2-1} - r^{n^4+n^3-n^2}$$

From lemma 6.3 and remark (1) we have

$$(r-r^{n^2+n-1})^{n^2} = 0 .$$

Case 2 n = even

$$(r-r^{n^2+n-1})^{n^2} = r^{n^2} - \binom{n^2}{1} r^{n^2-1} r^{n^2+n-1} + \binom{n^2}{2} r^{n^2-2} \left(r^{n^2+n-1}\right)^2 - \cdots$$

$$- \binom{n^2}{n^2-1} r (r^{n^2+n-1})^{n^2-1} + r^{n^4+n^3-n^2} .$$

From lemma 6.3 , remark (1) and remark (2) we have

$$(r-r^{n^2+n-1})^{n^2} = 0 .$$

Therefore $(r-r^{n^2+n-1})^{n^2} = 0$ for every r in R .