CHAPTER IV

ON THE STRUCTURE OF J - RINGS

The materials of this chapter are drawn from references [1], [4] and [7].

In this chapter, we prove that every J - ring is a direct sum of rings with prime characteristic. We also investigate further properties of these summands in order to morefully study the structure of J - rings.

__Definition 4.1.__ A ring R is called a p^k- ring if there exist a prime p and integer $k \geqslant 1$ such that $px = 0$ and $x^{p^k} = x$ for every x in R.

__Theorem 4.2.__ If $m_1, m_2, \dots m_r$ are positive integer $\geqslant 2$ then there exists a prime p such that $m_i - 1 \mid p-1 \quad \forall\, i = 1, \dots r$.

__proof__ Let $a = (m_1 - 1)(m_2 - 1) \dots (m_r - 1)$ and $b = 1$, therefore by Dirichlet's theorem, there exists a prime of the form $p = (m_1-1)(m_2-1)\dots(m_r-1)\, n+1$ for some integer n. Thus $m_i - 1 \mid p-1 \quad \forall\, i = 1, \dots, r$.

__Theorem 4.3.__ If n is an arbitrary intger > 1 and if q is a prime then there exist integers. $k \geqslant 1$ and $t \geqslant 0$ such that,

$$q^t(q^k - 1) \equiv 0 \mod(n-1).$$

<u>proof</u>. If $n = 2$, the theorem is true .

Suppose $n > 2$, write $n - 1 = mq^t$ . . . .(1)

where $q \nmid m$. We can do this, since $n - 1 > 1$ and every integer greater than 1 can be written as a product of primes.

Suppose $n - 1 = p_1^{\alpha_1} p_2^{\alpha_2} \cdots\cdots p_{m-1}^{\alpha_{m-1}} \cdot q^t$, by letting

$p_1^{\alpha_1} p_2^{\alpha_2} \cdots\cdot p_{m-1}^{\alpha_{m-1}} = m$ we get the result.

By the Fermat - Euler Theorem , $q^k \equiv 1 \mod(m)$

where $k = \phi(m)$, i.e. $m \mid q^k - 1$

From (1) , $m = \dfrac{n-1}{q^t}$.

Thus $\dfrac{n-1}{q^t} \mid q^k - 1$ which implies that $n-1 \mid q^t(q^k-1)$

Therefore $q^t(q^k - 1) \equiv 0 \mod(n-1)$

<u>Theorem 4.4</u>. If x is an element of a ring R such that $x^n = x$ then for all integers $m \geq 0$, $x^{m(n-1)+1} = x$ and if h and k are positive integers such that $h \equiv k \mod(n-1)$ then $x^h = x^k$.

<u>proof</u>. We shall use induction on m. If $m = 1$, we get.

$$x^{n-1+1} = x^n = x$$

Assume the statement is true for $m = k$. For $m = k+1$, we get

$$x^{(k+1)(n-1)+1} = x^{k(n-1)+1} \cdot x^{n-1}$$

$$= x . x^{n-1} = x^n = x .$$

Thus for all $m \geqslant 0$, $x^{m(n-1)+1} = x$.

We assume that $h \geqslant k$,

if $k = 1$, $h \equiv 1 \mod(n-1)$

i.e. $h = m(n-1)+1$ with $m \geqslant 0$

so that $x^h = x = x^k$.

If $k > 1$, then $h - 1 \equiv k - 1 \mod(n-1)$

so $h - 1 = m(n-1) + k-1$ with $m \geqslant 0$.

Hence we have

$$x^h = x^{h-1} . x = x^{m(n-1)+k-1} . x$$

$$= x^{k-1} . x^{m(n-1)+1} = x^{k-1} . x$$

$$= x^k .$$

Theorem 4.5. A ring R is a J - ring if and only if there exists a prime p such that $x^p = x$ for every x in R .

proof. The sufficiency is obvious.

Necessity, suppose that $x^n = x$ for every x in R. Since $n \geqslant 2$, by theorem 4.2 there exists a prime p such that.

$n-1 \mid p-1$ i.e. $p \equiv 1 \mod(n-1)$

By applying theorem 4.4, we have

$$x^p = x \qquad \text{for every } x \text{ in } R .$$

Lemma 4.6. If a ring R has positive characteristic $n = n_1 n_2$ where n_1 and n_2 are greater than 1 and $(n_1, n_2) = 1$ then $R \cong R_1 \oplus R_2$ where R_i is a ring of characteristic n_i.

proof. Since $(n_1, n_2) = 1$, there exist integers k, l such that

$$1 = n_1 k + n_2 l$$

and hence for every x in R .

$$x = n_1 kx + n_2 lx \qquad \text{------------ (2)}$$

Let $R_2 = \{ n_1 kx \mid x \in R \}$ and $R_1 = \{ n_2 lx \mid x \in R \}$.

It follows that R_2 is a ring and its characteristic does not exceed

n_2 , since $n_2(n_1kx) = nkx = 0$. In like manner, R_1 is a ring whose characteristic does not exceed n_1 .

From (2), we see that .

$$n_1kx = (n_1k)^2x + nklx$$

and since $nx = 0$, this implies that

$$(n_1k)^2x = n_1kx \qquad \text{for every } x \text{ in } R.$$

Similarly, we can show that.

$$(n_2l)^2x = n_2lx \ .$$

Define $\emptyset : R \rightarrow R_2 \oplus R_1$ by

$$\emptyset(x) = (n_1kx,\ n_2lx) \ .$$

We shall now show that $\emptyset$ is an isomorphism of R with the direct sum R_2 and R_1 . If x and y are arbitrary elements of R, then

$$\begin{aligned} \emptyset(x+y) &= \left[n_1k(x+y)\ ,\ n_2l(x+y)\right] \\ &= (n_1kx + n_1ky,\ n_2lx + n_2ly) \\ &= (n_1kx,\ n_2lx) + (n_1ky,\ n_2ly) \\ &= \emptyset(x) + \emptyset(y) \end{aligned}$$

and

$$\emptyset(xy) = \left[n_1k(xy)\ ,\ n_2l(xy)\right]$$

$$\emptyset(xy) = \left[(n_1k)^2(xy)\ ,\ (n_2l)^2(xy)\right]$$

$$= (n_1kx,\ n_2lx)\ .\ (n_1ky,\ n_2ly)$$

$$= \emptyset(x).\ \emptyset(y) \qquad .$$

Furthermore, if $\emptyset(x) = (0,0)$ we see that $n_1kx = n_2lx = 0$ and then (2) show that $x = 0$.

Thus $\emptyset$ is an onto homomorphism having zero kernel and is therefore actually an isomorphism.

To complete the proof of the theorem, we only need to show that the characteristic of R_i is n_i $(i = 1,2)$. If R_i has characteristic $m_i (i = 1,2)$, we see that $m_i \leqslant n_i$ $(i=1,2)$.

If (x_2, x_1) is any element of $R_2 \oplus R_1$, it follows that

$$m_1m_2\ (x_2, x_1) = (m_1m_2x_2,\ m_1m_2x_1\ ) = (0,0)\ .$$

Therefore, the characteristic of $R_2 \oplus R_1$ can not be greater than m_1m_2 .

Since n is the characteristic of R, it is also the characteristic of the isomorphic ring $R_2 \oplus R_1$ and thus $n \leqq m_1m_2$. But $m_i \leqslant n_i$ $(i = 1,2)$ implies that $m_1m_2 \leqslant n_1n_2 = n$, and hence we must have $n = n_1n_2 = m_1m_2$. Again making use of the fact that $m_i \leqq n_i$ $(i = 1,2)$ we see that $m_i = n_i (i = 1,2)$, and the proof is therefore complete .

<u>Lemma 4.7</u>. If R has characteristic $n = p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m}$ where the p_i are distinct primes and each $k_i \geqslant 1$ $(i = 1, \ldots, m)$ then $R \cong R_1 \oplus R_2 \oplus \ldots \oplus R_m$ where R_i has characteristic $p_i^{k_i}$ $(i = 1, \ldots, m)$.

<u>proof</u>. Using induction on m, for $m = 2$, the theorem is true by lemma 4.6. Assume the theorem is true for $m = n-1$. For $m = n$.

$$\text{Let } n_1 = p_1^{k_1} p_2^{k_2} \ldots p_{n-1}^{k_{n-1}} \quad \text{and} \quad n_2 = p_n^{k_n},$$

then $(n_1, n_2) = 1$. It follows from lemma 4.6 that

$$R \cong T_1 \oplus R_n$$

where T_1 has characteristic $p_1^{k_1} p_2^{k_2} \ldots p_{n-1}^{k_{n-1}}$

and R_n has characteristic $p_n^{k_n}$.

Since the theorem is true for $m = n-1$, this implies that

$$T_1 \cong R_1 \oplus R_2 \oplus \ldots \oplus R_{n-1}$$

where R_i has characteristic $p_i^{k_i}$ $(i = 1, 2, \ldots, n-1)$.

Hence $R \cong R_1 \oplus R_2 \oplus \ldots \oplus R_n$ where R_i has characteristic $p_i^{k_i}$ $(i = 1, \ldots, m)$.

Theorem 4.8. If R is a J - ring, there exist rings $R_1, R_2, \dots, R_m$ such that $R = R_1 \oplus \dots \oplus R_m$, R_i $(i = 1, \dots, m)$ has prime characteristic.

proof. First we need only to show that the characteristic of R is a product of distinct primes.

Since $(ax)^n = ax$ we get $(a^n - a)\, x = 0$

for every x in R and every integer a, there exists a least positive integer m such that $mx = 0$. Clearly $m \mid a^n - a$ for every integer a.

We claim now that m is a product of distinct primes. To see this, it will be sufficient to show that if $m = p_1 m_1$ where p_1 is a prime factor of m, then $(p_1, m_1) = 1$.

Suppose that $m_1 = p_1 m_2$ for some integer m_2. Then, for every x in R

$$m_1 x = (m_1 x)^n = m_1^n x = p_1^n\, m_2^n x$$

$$= p_1^{n-2}\, m_2^{n-1}\, mx = 0$$

which contradicts the choice of m. Thus $m = p_1 p_2 \dots\dots p_m$ where p_i are distinct primes. By using lemma 4.7 the proof is complete .

Theorem 4.9. A ring R is a J - ring if and only if R is a direct sum of finitely many p^k- rings.

proof. To prove the sufficiency, by theorem 4.8, we get.

$$R \cong R_1 \oplus \ldots\ldots \oplus R_m \quad ;$$

where $R_i = \{ x \mid x \in R \text{ such that } p_i x = 0 \}$ (i=1, ...,m),

Clearly, R_i is a subring of R with characteristic p_i.

Now we want to prove that R_i is a $p_i^{k(i)}$ - ring. In fact, by theorem 4.3 there exist integers $t(i) \geqslant 0$ and $k(i) \geqslant 1$ such that

$$n - 1 \mid p_i^{t(i)} (p_i^{k(i)} - 1)$$

$$\text{so} \quad p^{t(i)+k(i)} \equiv p_i^{t(i)} \quad \text{mod}(n-1) .$$

By theorem 4.4, we have

$$x^{p_i^{t(i)+k(i)}} = x^{p_i^{t(i)}} \qquad (i=1,\ldots,m) .$$

Hence, for every x in R_i, we have

$$(x - x^{p_i^{k(i)}})^{p_i^{t(i)}} = x^{p_i^{t(i)}} - \binom{p_i^{t(i)}}{1} x^{p_i^{t(i)-1}-1} x^{p_i^{k(i)}} + \ldots\ldots$$

$$\ldots\ldots\ldots - x^{p_i^{t(i)+k(i)}} .$$

$$= x^{p_i^{t(i)}} - x^{p_i^{t(i)+k(i)}}$$

$$= 0 .$$

By theorem 3.4, R has no nilpolent element other than 0. Therefore, we obtain that.

$$x - x^{p_i^{k(i)}} = 0$$

$$\text{i.e.} \quad x = x^{p_i^{k(i)}} .$$

Thus R_i is a $p_i^{k(i)}$- ring .

To prove necesscity, assume that R is a direct sum of finitely many $p_i^{k(i)}$- rings say $R \cong R_1 \oplus \cdots\cdots \oplus R_m$. Since $p_i^{k(i)} \geq 2$ $(i = 1,\ldots,m)$ by theorem 4.2, there exists a prime p such that

$$p_i^{k(i)} - 1 \mid p - 1 \qquad i = 1, \ldots, m .$$

$$\text{or} \qquad p \equiv 1 \quad \text{mod} \left(p_i^{k(i)} - 1\right) .$$

Therefore, by theorem 4.4 we have

$$y^p = y \qquad \text{for every } y \text{ in } R_i .$$

For any x in R,

$$x = (x_1, x_2, \ldots, x_m) \qquad x_i \in R_i$$

$$x^p = \underbrace{(x_1, x_2, \ldots, x_m)(x_1, \ldots, x_m) \ldots\ldots(x_1, \ldots, x_m)}_{p\text{ - times}}$$

$$= (x_1^p, x_2^p, \ldots\ldots, x_m^p)$$

$$= (x_1, x_2, \ldots, x_m)$$

$$= x$$

By theorem 4.5, R is a J - ring.