CHAPTER II

SEMIGROUPS

This chapter will study congruences and partial congruences on semigroups and groups.

2.1 Semigroups

This section will consider the following categories :

1) The category $\mathcal{S}_g$ of semigroups and semigroup homomorphisms.

2) The category $\mathcal{S}_{g,i}$ of semigroups and semigroup isomorphisms.

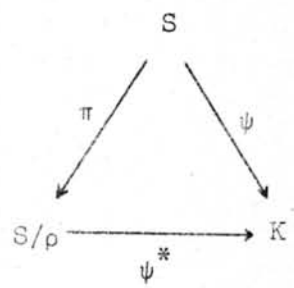
First we shall define naturally equivalent contravariant functors from $\mathcal{S}_g$ to $\mathcal{L}$ by using congruences and quotient semigroups which are defined below.

Remark: We can prove that if ρ is an operation preserving equivalence relation on a semigroup $(S,\cdot)$ then the set S/ρ of equivalence classes of S can be made into a semigroup in natural way and the natural projection map $\pi : S \to S/\rho$ is an onto semigroup homomorphism. Hence the definition of a congrucence on an object $(S,\cdot)$ in $\mathcal{S}_g$(or $\mathcal{S}_{g,i}$) is the same as the definition of an operation preserving equivalence relation on the semigroup $(S,\cdot)$.

Definition 2.1.1 A quotient semigroup of a semigroup S is a pair (K, ψ) where K is a semigroup and $\psi : S \to K$ is an onto semigroup homomorphism.

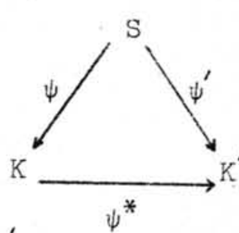
<u>Example</u> $(S/\rho, \pi)$ is a quotient semigroup of a semigroup S where ρ is a congruence on S.

<u>Theorem 2.1.2</u> Let (K,ψ) be a quotient semigroup of a semigroup S and $\rho = \{(a,b) \in S \times S \mid \psi(a) = \psi(b)\}$. Then ρ is a congruence on S and there exists an isomorphism $\psi^*: S/\rho \to K$ such that the following diagram is commutative



<u>Proof.</u> Clearly ρ is a congruence on S since ψ is an onto semigroup homomorphism. Define $\psi^*: S/\rho \to K$ as follows: given $\alpha \in S/\rho$ choose $a \in \alpha$ and let $\psi^*(\alpha) = \psi(a)$. Then ψ^* is an isomorphism such that $\psi^* \circ \pi = \psi$. #

<u>Definition 2.1.3</u> Let (K,ψ) and (K',ψ') be quotient semigroups of a semigroup S. Say that (K,ψ) is <u>strongly equivalent</u> to (K',ψ') iff there exists an isomorphism $\psi^*: K \to K'$ such that the following diagram is commutative



Write this as $(K,\psi) \simeq (K',\psi')$.

<u>Remarks</u> : 1. $\simeq$ is an equivalence relation on the set of quotient semigroups of a semigroup.

2. For each quotient semigroup (K,ψ) of a semigroup S, $(K,\psi) \approx (S/\rho,\pi)$ where $\rho = \{(a,b) \in S \times S \mid \psi(a) = \psi(b)\}$.

<u>Proposition 2.1.4</u> Let $\psi: S \to S'$ be a semigroup homomorphism. If ρ' is a congruence on S' then $(\psi \times \psi)^{-1}(\rho')$ is a congruence on S.

Fix a semigroup S, let C(S) = the set of congruences on S,

Q(S) = the set of equivalence classes of quotient semigroups of S under $\approx$.

Now we shall define natural relations on these sets making them into posets.

1) Let $\subseteq$ on C(S) be set inclusion. Then clearly $(C(S),\subseteq)$ is a poset.

2) Let $\subseteq$ on Q(S) be defined as follows: given $\alpha,\beta \in Q(S)$ choose $(K_1,\psi_1) \in \alpha$, $(K_2,\psi_2) \in \beta$ then say that $\alpha \subseteq \beta$ iff there exists an onto semigroup homomorphism $\psi: K_1 \to K_2$ such that $\psi \circ \psi_1 = \psi_2$. First we shall show that $\subseteq$ is well-defined. Let $(K_1,\psi_1) \approx (K_2,\psi_2)$ and $(K_1',\psi_1') \approx (K_2',\psi_2')$. Suppose $\exists$ an onto homomorphism $\psi: K_1 \to K_1'$ such that $\psi \circ \psi_1 = \psi_1'$. We must show that $\exists$ an onto homomorphism $\psi^*: K_2 \to K_2'$ such that $\psi^* \circ \psi_2 = \psi_2'$. Because $(K_1,\psi_1) \approx (K_2,\psi_2)$ and $(K_1',\psi_1') \approx (K_2',\psi_2')$, $\exists$ an isomorphism $\eta: K_2 \to K_1$ such that $\eta \circ \psi_2 = \psi_1$ and $\exists$ an isomorphism $\eta': K_1' \to K_2'$ such that $\eta' \circ \psi_1' = \psi_2'$. Define $\psi^*: K_2 \to K_2'$ by $\psi^* = \eta' \circ \psi \circ \eta$. Then ψ^* is an onto homomorphism such that $\psi^* \circ \psi_2 = \psi_2'$. Hence $\subseteq$ is well-defined. Next we shall show that $(Q(S),\subseteq)$ is a poset. Clearly $\subseteq$ is reflexive. Let $\alpha \subseteq \beta$ and $\beta \subseteq \alpha$. Choose $(K,\psi) \in \alpha$ and $(K',\psi') \in \beta$. Then $\exists$ an onto homomorphism $\psi^*: K \to K'$ such that $\psi^* \circ \psi = \psi'$ and $\exists$ an onto homomorphism $\psi^{**}: K' \to K$ such that $\psi^{**} \circ \psi' = \psi$. We shall

show that $\overset{*}{\psi}o\overset{*}{\psi} = id_K$. Let $k \in K$ so $\exists$ $a \in S$ such that $\psi(a) = k$ then $\overset{*}{\psi}o\overset{*}{\psi}(k) = \overset{*}{\psi}o\overset{*}{\psi}o\psi(a) = \overset{*}{\psi}o\psi'(a) = \psi(a) = k$. Hence $\overset{*}{\psi}o\overset{*}{\psi} = id_K$. Therefore $\overset{*}{\psi}$ is 1-1. Thus ψ^* is an isomorphism such that $\overset{*}{\psi}o\psi = \psi'$. Hence $\alpha = \beta$, ie. $\subsetneq$ is antisymmetric. Clearly $\subsetneq$ is transitive. Therefore $(Q(S),\subsetneq)$ is a poset.

<u>Theorem 2.1.5</u> For each semigroup S, the posets C(S) and Q(S) are isomorphic.

<u>Proof</u>. Let S be a semigroup. Define $\psi:Q(S) \to C(S)$ as follows: given $\alpha \in Q(S)$ choose $(K,\eta) \in \alpha$ and let $\psi(\alpha) = \rho_\alpha$ where $\rho_\alpha = \{(a,b) \in S \times S | \eta(a) = \eta(b)\}$. First we shall show that ψ is well-defined. Let $(K_1,\eta_1) \simeq (K_2,\eta_2)$ so $\exists$ an isomorphism $\overset{*}{\psi}:K_1 \to K_2$ such that $\overset{*}{\psi}o\eta_1 = \eta_2$. Then clearly $\rho_1 = \rho_2$. Hence ψ is well-defined.

Next we shall show that ψ is 1-1. Let $\alpha,\beta \in Q(S)$ be such that $\psi(\alpha) = \psi(\beta)$. Choose $(K_1,\eta_1) \in \alpha$, $(K_2,\eta_2) \in \beta$ Then $\rho_\alpha = \rho_\beta$. Define $\overset{*}{\psi}:K_1 \to K_2$ as follows: given $k \in K_1$ then $\exists$ $a \in S$ such that $\eta_1(a) = k$, let $\overset{*}{\psi}(k) = \eta_2(a)$. Because $\rho_\alpha \subsetneq \rho_\beta$, ψ^* is well-defined. Since $\rho_\beta \subsetneq \rho_\alpha$ ψ^* is 1-1. Clearly ψ^* is onto. Because η_1,η_2 are homomorphisms, $\overset{*}{\psi}$ is a homomorphism . Hence ψ^* is an isomorphism such that $\overset{*}{\psi}o\eta_1 = \eta_2$, ie. $\alpha = \beta$. Thus ψ is 1-1.

Next we shall show that ψ is onto. Let $\rho \in C(S)$. Then $(S/\rho, \pi)$ is a quotient semigroup of S, and $\psi([S/\rho,\pi]) = \{(a,b) \in S \times S | \pi(a) = \pi(b)\}$ $= \{(a,b) \in S \times S | a\rho b\} = \rho$. Hence ψ is onto.

Next we shall show that ψ is isotone. Let $\alpha,\beta \in Q(S)$ be such that $\alpha \subsetneq \beta$. Choose $(K,\eta) \in \alpha$, $(K',\eta') \in \beta$. Then $\exists$ an onto homomorphism $\overset{*}{\psi}:K \to K'$

such that $\psi^{*} o \eta = \eta'$ .Clearly $\rho_\alpha \subsetneq \rho_\beta$. Hence $\psi(\alpha) \subsetneq \psi(\beta)$. Thus ψ is isotone.

Lastly we shall show that ψ^{-1} is isotone. Let $\rho_1, \rho_2 \in C(S)$ be such that $\rho_1 \subsetneq \rho_2$. Define $\psi': S/\rho_1 \to S/\rho_2$ as follows: given $\gamma \in S/\rho_1$ choose $a \in \gamma$ and then let $\psi'(\gamma) = [a]_2$. Because $\rho_1 \subsetneq \rho_2$, ψ' is well-defined. Clearly ψ' is an onto homomorphism such that $\psi' o \pi_1 = \pi_2$. Thus $\psi^{-1}(\rho_1) \subsetneq \psi^{-1}(\rho_2)$. Therefore ψ^{-1} is isotone. Hence ψ is an isomomorphism, ie. $Q(S)$ is isomorphic to $C(S)$. #

We shall show that for each semigroup S $(C(S), \subsetneq)$, $(Q(S), \subsetneq)$ are lattices. Let S be a semigroup. For each $\rho_1, \rho_2 \in C(S)$ denote $\rho_1 \cap \rho_2$ by $\rho_1 \wedge \rho_2$ and the congruence on S generated by $\rho_1 \cup \rho_2$ by $\rho_1 \vee \rho_2$. Let $\rho_1, \rho_2 \in C(S)$. Then $\rho_1 \wedge \rho_2 =$ g.l.b.$\{\rho_1, \rho_2\}$ and $\rho_1 \vee \rho_2 =$ l.u.b$\{\rho_1, \rho_2\}$. Hence $(C(S), \subsetneq)$ is a lattice. Let $\psi: Q(S) \to C(S)$ be the isomorphism in Theorem 2.1.5. Let $\alpha, \beta \in Q(S)$ then $\psi(\alpha), \psi(\beta) \in C(S)$. So $\psi(\alpha) \wedge \psi(\beta) =$ g.l.b.$\{\psi(\alpha), \psi(\beta)\}$ and $\psi(\alpha) \vee \psi(\beta) =$ l.u.b.$\{\psi(\alpha), \psi(\beta)\}$. Therefore $[(S/\psi(\alpha) \wedge \psi(\beta), \pi)] = \psi^{-1}(\psi(\alpha) \wedge \psi(\beta)) =$ g.l.b.$\{\alpha, \beta\}$ and $[(S/\psi(\alpha) \vee \psi(\beta), \pi')] = \psi^{-1}(\psi(\alpha) \vee \psi(\beta)) =$ l.u.b.$\{\alpha, \beta\}$. Hence $(Q(S), \subsetneq)$ is a lattice.

Now we shall define contravariant functors from $\mathcal{S}_g$ to $\mathcal{L}$.

1) Let S, S' be in Ob $\mathcal{S}_g$ and $\psi: S \to S'$ a semigroup homomorphism. Then $C(S), C(S')$ are in Ob $\mathcal{L}$. Define $C(\psi): C(S') \to C(S)$ by $C(\psi)(\rho) = (\psi \times \psi)^{-1}(\rho)$ $\forall \rho \in C(S')$. Clearly $C(\psi)$ is an isotone map. Since $C(id_S) = id_{C(S)}$ $\forall S$ in Ob $\mathcal{S}_g$ and $C(\psi o \eta) = C(\eta) o C(\psi)$ $\forall$ semigroup homomorphisms ψ, η whenever $\psi o \eta$ is defined, C is a contravariant functor from $\mathcal{S}_g$ to $\mathcal{L}$.

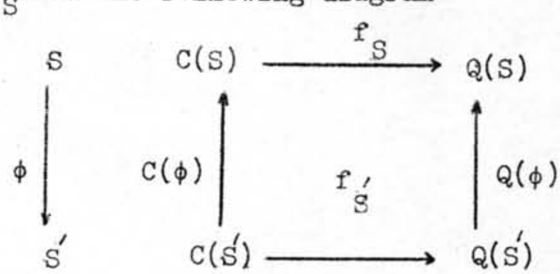
2) Let S, S' be in Ob $\mathcal{S}_g$ and $\psi : S \to S'$ a semigroup homomorphism. Then $Q(S), Q(S')$ are in Ob $\mathcal{L}$. Define $Q(\psi) : Q(S') \to Q(S)$ as follows: given $\alpha \in Q(S')$ choose $(K, \eta) \in \alpha$ and then let $Q(\psi)(\alpha) = [(S/(\psi\times\psi)^{-1}(\rho), \pi)]$ where $\rho = \{(x,y) \in S' \times S' \mid \eta(x) = \eta(y)\}$. First we shall show that $Q(\psi)$ is well-defined. Let $(K_1, \eta_1) \approx (K_2, \eta_2)$ then $\rho_1 = \rho_2$ so $(\psi \times \psi)^{-1}(\rho_1) = (\psi \times \psi)^{-1}(\rho_2)$. Therefore $(S/(\psi\times\psi)^{-1}(\rho_1), \pi_1) = (S/(\psi\times\psi)^{-1}(\rho_2), \pi_2)$.

Hence $Q(\psi)$ is well-defined. Next we shall show that $Q(\psi)$ is isotone. Let $\alpha, \beta \in Q(S)$ be such that $\alpha \subseteq \beta$. Choose $(K_1, \eta_1) \in \alpha$, $(K_2, \eta_2) \in \beta$ then $\rho_1 \subseteq \rho_2$. So $(\psi \times \psi)^{-1}(\rho_1) \subseteq (\psi \times \psi)^{-1}(\rho_2)$. Therefore

$(S/(\psi\times\psi)^{-1}(\rho_1), \pi_1) \subseteq (S/(\psi\times\psi)^{-1}(\rho_2), \pi)$ ie. $Q(\psi)(\rho_1) \subseteq Q(\psi)(\rho_2)$.

Hence $Q(\psi)$ is isotone. Lastly we shall show that Q is a contravariant functor from $\mathcal{S}_g$ to $\mathcal{L}$. Clearly $Q(id_S) = id_{Q(S)}$ $\forall$ S in Ob $\mathcal{S}_g$. Let $\psi : S \to S'$ and $\psi' : S' \to S''$ be semigroup homomorphisms. Then $\psi' o \psi : S \to S''$ is a homomorphism. Let $\alpha \in Q(S'')$ choose $(K, \eta) \in \alpha$ then $(Q(\psi) o Q(\psi'))(\alpha) = Q(\psi)[(S'/(\psi'\times\psi')^{-1}(\rho), \pi')] = [(S/(\psi\times\psi)^{-1} o (\psi'\times\psi')^{-1}(\rho), \pi)] = [(S/(\psi' o\psi \times \psi' o \psi)^{-1}(\rho), \pi)] = Q(\psi' o \psi)(\alpha)$. Therefore $Q(\psi) o Q(\psi') = Q(\psi' o \psi)$. Hence Q is a contravariant functor from $\mathcal{S}_g$ to $\mathcal{L}$.

Next we shall show that C is naturally equivalent to Q. For each S in Ob $\mathcal{S}_g$, define $f_S : C(S) \to Q(S)$ be the map in Theorem 2.1.5. Then f_S is an isomorphism. We shall show that f is a natural equivalence from C to Q. Let S, S' be in Ob $\mathcal{S}_g$ and $\phi : S \to S'$ a semigroup homomorphism. So we have f_S, $f_{S'}$ and the following diagram



We must show that $Q(\phi)of_{S'} = f_S oC(\phi)$. Let $\rho \in C(S')$ then $(Q(\phi)of_{S'})(\rho) = (Q(\phi))\left[(S'/\rho,\pi')\right] = \left[(S/(\phi\times\phi)^{-1}(\rho),\pi)\right] = f_S(\phi\times\phi)^{-1}(\rho)) = f_S oC(\phi)(\rho)$. So $Q(\phi)of_{S'} = f_S oC(\rho)$. Hence f is a natural equivalence from C to Q.

<u>Remark</u>: We see that C is the congruence functor of $\mathcal{S}_g$.

Now we shall define naturally equivalent covariant functors from $\mathcal{S}_{g,i}$ to $\mathcal{Q}$ using equivalence classes of congruences and equivalence classes of quotient semigroups which are defined below.

<u>Definition 2.1.6</u> Let ρ_1 and ρ_2 be congruences on a semigroup S. Say that ρ_1 is <u>equivalent</u> to ρ_2 $(\rho_1 \sim \rho_2)$ iff there exists an automorphism $f:S \to S$ such that $(f\times f)(\rho_1) = \rho_2$.

<u>Remark</u>: $\sim$ is an equivalence relation on the set of congruences on a semigroup.

<u>Definition 2.1.7</u> Let (K,ϕ) and (K',ϕ') be quotient semigroups of a semigroup S. Say that (K,ϕ) is <u>weakly equivalent</u> to (K',ϕ') iff there exist isomorphisms $f:S \to S$ and $f':K \to K'$ such that the following diagram is commutative :

$$\begin{array}{ccc} S & \xrightarrow{\;f\;} & S \\ \phi\downarrow & & \downarrow\phi' \\ K & \xrightarrow{\;f'\;} & K' \end{array}$$

Write this as $(K,\phi) \sim (K',\phi')$

<u>Remarks</u>: 1) $\sim$ is an equivalence relation on the set of quotient semigroups of a semigroup.

2) $(K,\phi) \simeq (K',\phi')$ implies that $(K,\phi) \sim (K',\phi')$. (Just let $f = id_S$)

Fix a semigroup S let $C^*(S)$ = the set of equivalence classes of congruences on S under $\sim$,

$Q^*(S)$ = the set of equivalence classes of quotient semigroups of S under $\sim$.

We shall define binary relations on these sets making them into quasi-ordered sets.

1) Let the binary relation $\leqslant$ on $C^*(S)$ be defined as follows: given $\alpha, \beta \in C^*(S)$ say that $\alpha \leqslant \beta$ iff there exist $\rho_1 \in \alpha$, $\rho_2 \in \beta$ such that $\rho_1 \subsetneq \rho_2$. Clearly $\leqslant$ is well-defined and $(C^*(S), \leqslant)$ is a quasi-ordered set

2) Let the binary relation $\leqslant$ on $Q^*(S)$ be defined as follows: given $\alpha, \beta \in Q^*(S)$ say that $\alpha \leqslant \beta$ iff there exist $(K, \eta) \in \alpha$, $(K', \eta') \in \beta$, an onto homomorphism $\psi : K \to K'$ and an automorphism $\psi' : S \to S$ such that $\psi \circ \eta = \eta' \circ \psi'$. Clearly $\leqslant$ is well-defined and $(Q^*(S), \leqslant)$ is a quasi-ordered set.

<u>Theorem 2.1.8</u> For each semigroup S the quasi-ordered sets $C^*(S)$, $Q^*(S)$ are isomorphic.

<u>Proof</u>. Let S be a semigroup. Define $\psi^* : C^*(S) \to Q^*(S)$ as follows: given $\alpha \in C^*(S)$ choose $\rho \in \alpha$ and then let $\psi^*(\alpha) = [(S/\rho, \pi)]$ First we shall show that ψ^* is well-defined. Let $\rho_1 \sim \rho_2$ then $\exists$ an automorphism $\psi : S \to S$ such that $(\psi \times \psi)(\rho_1) = \rho_2$. Define $\psi' : S/\rho_1 \to S/\rho_2$ as follows: given $\beta \in S/\rho_1$ choose $s \in \beta$ and then let $\psi'(\beta) = [\psi(s)]_2$. Since $(\psi \times \psi)(\rho_1) \subseteq \rho_2$, ψ' is well-defined. Since $\rho_2 \subseteq (\psi \times \psi)(\rho_1)$, ψ' is 1-1. Clearly ψ' is an onto homomorphism such that $\psi' \circ \pi_1 = \pi_2 \circ \psi$. Hence ψ^* is well-defined.

Next we shall show that ψ^* is 1-1. Let ρ_1, ρ_2 be congruences on S such that $(S/\rho_1, \pi_2) \sim (S/\rho_2, \pi_2)$ so $\exists$ an isomorphism $\psi : S/\rho_1 \to S/\rho_2$

and an automorphism $\psi':S \to S$ such that $\psi o \pi_1 = \pi_2 o \psi'$. We want to show that $(\psi' \times \psi')(\rho_1) = \rho_2$. Let $(a,b) \in \rho_1$ then $\pi_1(a) = \pi_1(b)$ so $\pi_2 o \psi'(a) = \pi_2 o \psi'(b)$ ie. $(\psi'(a),\psi'(b)) \in \rho_2$. Therefore $(\psi' \times \psi')(\rho_1) \subseteq \rho_2$. Let $(a,b) \in \rho_2$ then $\exists x,y \in S$ such that $\psi'(x) = a$, $\psi'(y) = b$ then $\psi o \pi_1(x) = \psi o \pi_1(y)$ so $\pi_1(x) = \pi_1(y)$ therefore $(x,y) \in \rho_1$ so $(a,b) \in (\psi' \times \psi')(\rho_1)$. Hence $\rho_2 \subseteq (\psi' \times \psi')(\rho_1)$. Thus $(\psi' \times \psi')(\rho_1) = \rho_2$. So $\rho_1 \sim \rho_2$. Therefore ψ^* is 1-1.

Next we shall show that ψ^* is onto. Let $\alpha \in Q^*(S)$ choose $(K,\eta) \in \alpha$ then define $\rho_\alpha = \{(a,b) \in S \times S | \eta(a) = \eta(b)\}$. So $[\rho_\alpha] \in C^*(S)$ and $\psi([\rho_\alpha]) = [(S/\rho_\alpha,\pi)] = [(K,\eta)] = \alpha$. Therefore ψ^* is onto.

Next we shall show that ψ^* is isotone. Let $\alpha,\beta \in C^*(S)$ be such that $\alpha \leq \beta$. Then $\exists\, \rho_1 \in \alpha, \rho_2 \in \beta$ such that $\rho_1 \subseteq \rho_2$. Define $\psi:S/\rho_1 \to S/\rho_2$ as follows: given $\gamma \in S/\rho_1$ choose $a \in \gamma$ and then let $\psi(\gamma) = [a]_2$. Since $\rho_1 \subseteq \rho_2$, ψ is well-defined. Clearly ψ is an onto homomorphism such that $\psi o \pi_1 = \pi_2 o\, id_S$. Hence $[(S/\rho_1,\pi_1)] \leq [(S/\rho_2,\pi_2)]$ ie. ψ^* is isotone.

Lastly we shall show that ψ^{*-1} is isotone. Let $\alpha,\beta \in Q^*(S)$ be such that $\alpha \leq \beta$ then $\exists\, (K_1,\eta_1) \in \alpha, (K_2,\eta_2) \in \beta$, an onto homomorphism $\psi':K_1 \to K_2$ and an automorphism $\psi:S \to S$ such that $\psi' o \eta_1 = \eta_2 o \psi$. Clearly $(\psi \times \psi)(\rho_1) \subseteq \rho_2$. Hence ψ^{*-1} is isotone. Thus $C^*(S)$ is isomorphic to $Q^*(S)$. #

Now we shall define covariant functors from $\mathcal{S}_{g,i}$ to $\mathcal{Q}$.

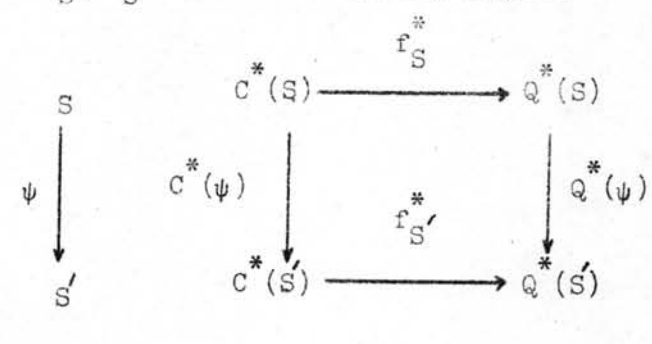
1) Let S,S' be in Ob $\mathcal{S}_{g,i}$ and $\psi:S \to S'$ a semigroup isomorphism.

Then $C^*(S)$, $C^*(S')$ are in Ob $\mathcal{Q}$. Define $C^*(\psi):C^*(S) \to C^*(S')$ as follows: given $\alpha \in C^*(S)$ choose $\rho \in \alpha$ and then let $(C^*(\psi))(\alpha) = [(\psi \times \psi)(\rho)]$. First we shall show that $C^*(\psi)$ is well-defined. Let $\rho_1 \sim \rho_2$ so $\exists$ an isomorphism $\psi^*:S \to S$ such that $(\psi^* \times \psi^*)(\rho_1) = \rho_2$. We want to show that $(\psi \times \psi)(\rho_1) \sim (\psi \times \psi)(\rho_2)$. Define $\psi':S' \to S'$ by $\psi' = \psi o \psi^* o \psi^{-1}$. Then ψ' is an isomorphism such that $(\psi' \times \psi')(\psi \times \psi)(\rho_1) = (\psi \times \psi)(\rho_2)$. Hence $C^*(\psi)$ is well-defined. Next we shall show that $C^*(\psi)$ is isotone. Let $\alpha,\beta \in C^*(S)$ be such that $\alpha \leqslant \beta$ then $\exists\ \rho_1 \in \alpha$, $\rho_2 \in \beta$ such that $\rho_1 \subseteq \rho_2$. Clearly $(\psi \times \psi)(\rho_1) \subseteq (\psi \times \psi)(\rho_2)$. Hence $C^*(\psi)(\alpha) \leqslant C^*(\psi)(\beta)$. Therefore $C^*(\psi)$ is isotone. Lastly we shall show that C^* is a covariant functor from $\mathcal{S}_{g,i}$ to $\mathcal{Q}$. Clearly $C^*(id_S) = id_{C^*(S)}$ $\forall$ S in Ob $\mathcal{S}_{g,i}$. Let $\psi:S \to S'$ and $\psi':S' \to S''$ be semigroup isomorphisms. Then $\psi' o \psi:S \to S''$ is a semigroup isomorphism. Let $\alpha \in C^*(S)$ choose $\rho \in \alpha$ then $C^*(\psi' o \psi)(\alpha) = [(\psi' o \psi \times \psi' o \psi)(\rho)] = [(\psi' \times \psi') o (\psi \times \psi)(\rho)] = C^*(\psi')[(\psi \times \psi)(\rho)] = (C^*(\psi') o C^*(\psi))(\alpha)$ Hence $C^*(\psi' o \psi) = C^*(\psi') o C^*(\psi)$. Therefore C^* is a covariant functor from $\mathcal{S}_{g,i}$ to $\mathcal{Q}$.

2) Let S,S' be in Ob $\mathcal{S}_{g,i}$ and $\psi:S \to S'$ a semigroup isomorphism. Then $Q^*(S)$, $Q^*(S')$ are in Ob $\mathcal{Q}$. Define $Q^*(\psi):Q^*(S) \to Q^*(S')$ as follows: given $\alpha \in Q^*(S)$ choose $(K,\eta) \in \alpha$ and then let $Q^*(\psi)(\alpha) = [(S/_{(\psi \times \psi)(\rho_\alpha)},\pi)]$ where $\rho_\alpha = \{(a,b) \in S \times S | \eta(a) = \eta(b)\}$. First we shall show that $Q^*(\psi)$ is well defined. Let $(K_1,\eta_1) \sim (K_2,\eta_2)$. Then by the proof of Theorem 2.1.8 , $\rho_1 \sim \rho_2$ hence $(\psi \times \psi)(\rho_1) \sim (\psi \times \psi)(\rho_2)$ therefore $(S/_{(\psi\times\psi)(\rho_1)},\pi_1) \sim (S/_{(\psi\times\psi)(\rho_2)},\pi_2)$ ie. Q^* is well-defined. Next we shall show that $Q^*(\psi)$ is isotone. Let $\alpha,\beta \in Q^*(S)$ be such that $\alpha \leqslant \beta$ then by the proof of Theorem 2.1.8 , $[\rho_\alpha] \leqslant [\rho_\beta]$ hence $[(\psi \times \psi)(\rho_\alpha)] \leqslant [(\psi \times \psi)(\rho_\beta)]$

therefore $[(S/_{(\psi\times\psi)(\rho_\alpha)},\pi_\alpha)] \leq [S/_{(\psi\times\psi)(\rho_\beta)},\pi_\beta)]$. Hence $Q^*(\psi)(\alpha) \leq Q^*(\psi)(\beta)$ ie. $Q^*(\psi)$ is isotone. Lastly we shall show that Q^* is a covariant functor from $\mathcal{S}_{g,i}$ to $\mathcal{Q}$. Clearly $Q^*(id_S) = id_{Q^*(S)}$ $\forall S$ in Ob $\mathcal{S}_{g,i}$. Let $\psi:S \to S'$ and $\psi':S' \to S''$ be semigroup isomorphisms. Then $\psi' o\psi:S \to S''$ is a semigroup isomorphism. Let $\alpha \in Q^*(S)$ choose $(K,\eta) \in \alpha$ then $(Q^*(\psi')oQ^*(\psi))(\alpha) = Q^*(\psi')[(S'/_{(\psi\times\psi)(\rho_\alpha)},\pi')] =$ $[(S''/(\psi'\times\psi')o(\psi\times\psi)(\rho_\alpha),\pi'')] = [(S''/(\psi'o\psi\times\psi'o\psi)(\rho_\alpha),\pi'')] = (Q^*(\psi'o\psi))(\alpha)$. Hence $Q^*(\psi')oQ^*(\psi) = Q^*(\psi'o\psi)$. Therefore Q^* is a covariant functor from $\mathcal{S}_{g,i}$ to $\mathcal{Q}$.

Next we shall show that C^* is naturally equivalent to Q^*. For each S in Ob $\mathcal{S}_{g,i}$, define $f_S^*:C^*(S) \to Q^*(S)$to be the map in Theorem 2.1.8. Then f_S^* is an isomorphism. We shall show that f^* is a natural equivalence from C^* to Q^*. Let S,S' in Ob $\mathcal{S}_{g,i}$ and $\psi:S \to S'$ a semigroup isomorphism. So we have f_S^*, $f_{S'}^*$ and the following diagram



We must show that $Q^*(\psi)of_S^* = f_{S'}^* o\ C^*(\psi)$. Let $\alpha \in C^*(S)$ choose $\rho \in \alpha$ then $(Q^*(\psi)of_S^*)(\alpha) = Q^*(\psi)[(S/\rho,\pi)] = [(S'/_{(\psi\times\psi)(\rho)},\pi)] = f_{S'}^*[(\psi\times\psi)(\rho_\alpha)] =$ $(f_S^* \ o\ C^*(\psi))(\alpha)$. So $Q^*(\psi)of_S^* = f_{S'}^* o\ C^*(\psi)$. Hence f^* is a natural equivalence from C^* to Q^*.

Hence there exist naturally equivalence covariant functors C^*, Q^* from $\mathcal{S}_{g,i}$ to $\mathcal{Q}$.

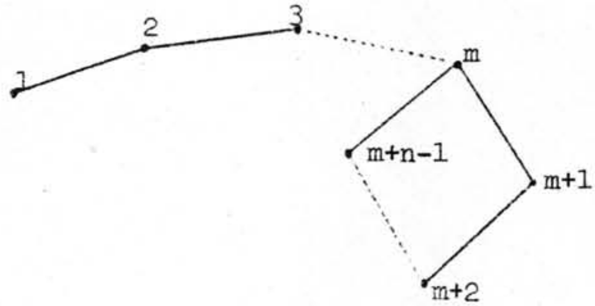
Next we shall consider properties of semigroups and give some theorems.

We have that $(\mathbb{N},+)$ is a semigroup. For each pair (m,n) of elements in $\mathbb{N}$, we shall define a new semigroup denoted by $\mathbb{N}_{(m,n)}$. Let $m,n \in \mathbb{N}$. Put $s = m + n$. Let $\mathbb{N}_{(m,n)} = \{1,2,\ldots,s-1\}$. We shall define a binary operation $*$ on $\mathbb{N}_{(m,n)}$ making $(\mathbb{N}_{(m,n)},*)$ is a semigroup. For each $i,j \in \mathbb{N}_{(m,n)}$ let

$$A_{(i,j)} = \{k \in \mathbb{N} \mid k > \frac{i+j-s}{n}\} \quad \text{and}$$

$$l_{(i,j)} = \begin{cases} 0 & \text{if } i+j < s, \\ \min A_{(i,j)} & \text{otherwise,} \end{cases}$$

then $i + j - l_{(i,j)}n \in \mathbb{N}_{(m,n)}$. Define $*$ on $\mathbb{N}_{(m,n)}$ by $i * j = i + j - l_{(i,j)}n$. By [2] $(\mathbb{N}_{(m,n)},*)$ is a semigroup and the cardinality $= m + n - 1$.



Theorem 2.1.9 $\mathbb{N}_{(m,n)} \cong \mathbb{N}_{(p,q)}$ iff $m = p$ and $n = q$.

Proof. Assume that $\mathbb{N}_{(m,n)} \cong \mathbb{N}_{(p,q)}$. Let $\psi: \mathbb{N}_{(m,n)} \to \mathbb{N}_{(p,q)}$ be an isomorphism. We shall show that $m = p$. Suppose that $m \neq p$. Assume that $m > p$. Claim that $\forall a \in \mathbb{N}_{(m,n)}$ [$a < m$ implies that $\psi(a) < p$]. It suffices to show that $\forall a \in \mathbb{N}_{(m,n)}$ [$\psi(a) \geq p$ implies that $a \geq m$]. Let $a \in \mathbb{N}_{(m,n)}$ be such that $\psi(a) \geq p$ so $\psi(a) * q = \psi(a)$. Because ψ is an isomorphism, $a * \psi^{-1}(q) = \psi^{-1}(\psi(a) * q) = \psi^{-1}(\psi(a)) = a$ so $a \geq m$. Hence we have the claim. By the claim, we have that $m \leq p$ which is a contradiction. Hence $m = p$. Because $m + n - 1 = p + q - 1$, $n = q$.

Conversely, if $m = p$ and $n = q$ then clearly $\mathbb{N}_{(m,n)} \cong \mathbb{N}_{(p,q)}$.

#

Theorem 2.1.10 Let $(S,+)$ be a semigroup with one generator. Then $S \cong \mathbb{N}$ or $\mathbb{N}_{(m,n)}$ for some $m,n \in \mathbb{N}$.

Proof. Let x be a generator of S. Consider $\{n\,x\}_{n \in \mathbb{N}}$

case 1 $m \neq n$ implies that $m\,x \neq n\,x$. Define $\phi: \mathbb{N} \to S$ by $\phi(m) = m\,x$. Clearly ϕ is an isomorphism ie. $S \cong \mathbb{N}$.

case 2 $\exists\, m \neq n$ such that $mx = nx$. Let $\mathcal{A} = \{k \in \mathbb{N} \mid \exists\, d \in \mathbb{N} \ni dx = kx\}$ then $\mathcal{A} \neq \emptyset$. Let $m = \min \mathcal{A}$. Let $\mathcal{B} = \{k \in \mathbb{N} \setminus \{m\} \mid k\,x = m\,x\}$ then $\mathcal{B} \neq \emptyset$. Let $s = \min \mathcal{B}$. Put $n = s - m$. Claim that $S \cong \mathbb{N}_{(m,n)}$. To prove this, for each $a \in S$ let $S_a = \{k \in \mathbb{N} \mid a = k\,x\}$ and $k_a = \min S_a$, then $k_a \in \mathbb{N}_{(m,n)}$. Define $\psi: S \to \mathbb{N}_{(m,n)}$ by $\psi(a) = k_a$. Then clearly ψ is an isomorphism. Hence $S \cong \mathbb{N}_{(m,n)}$. #

Now we shall find all quotient semigroups of $(\mathbb{N},+)$. Let (S,ϕ) be a quotient semigroup of $(\mathbb{N},+)$. Then $\phi: \mathbb{N} \to S$ is an onto homomorphism. We know that $\mathbb{N}$ is the semigroup generated by 1. We shall show that S is the semigroup generated by $\phi(1)$. Let $s \in S$ then $\exists\, n \in \mathbb{N}$ such that $\phi(n) = s$. So $\phi(n) = \phi(\underbrace{1 + 1 + \ldots + 1}_{n \text{ times}}) = \underbrace{\phi(1) + \phi(1) + \ldots + \phi(1)}_{n \text{ times}} = n\,\phi(1)$.

Hene $S = \langle\phi(1)\rangle$. By above Theorem, $S \cong \mathbb{N}$ or $\mathbb{N}_{(m',n')}$ for some $n',m' \in \mathbb{N}$.

Theorem 2.1.11 Let $m_o, n_o \in \mathbb{N}$ be such that $m_o < n_o$. Let $\langle(m_o,n_o)\rangle$ denote the congruence on $(\mathbb{N},+)$ generated by (m_o,n_o). Then

$\langle(m_o,n_o)\rangle = \{(a,a) \mid a \in \mathbb{N}\} \cup$

$\{(a,b) \in \mathbb{N}\times\mathbb{N} \mid \exists\, k \in \mathbb{N} \ni$ either $a+km_o = b+kn_o$ and $b \geq m_o$ or $a+kn_o = b+km_o$ and $a \geq m_o\}$

Proof. Let $\rho = \{(a,a) \mid a \in \mathbb{N}\} \cup$

$\{(a,b) \in \mathbb{N}\times\mathbb{N} \mid \exists\, k \in \mathbb{N} \ni$ either $a+km_o = b+kn_o$ and $b \geq m_o$ or $a+kn_o = b+km_o$ and $a \geq m_o\}$

First we shall show that ρ is a congruence on $(\mathbb{N},+)$. Clearly ρ is reflexive and symmetric. Let $(a,b),(b,c) \in \rho$ If $a = b$ or $b = c$ then then clearly $(a,c) \in \rho$. We may assume that $a \neq b$ and $b \neq c$. Then $\exists\, k \in \mathbb{N}$ such that either $(a + km_o = b + kn_o$ and $b \geq m_o)$ or $(a + kn_o = b + km_o$ and $a \geq m_o)$ and $\exists\, k' \in \mathbb{N}$ such that either $(b + k'm_o = c + k'n_o$ and $c \geq m_o)$ or $(b + k'n_o + c + k'm_o$ and $b \geq m_o)$

case 1 $a + km_o = b + kn_o$, $b \geq m_o$ and $b + k'm_o = c + k'n_o$, $c \geq m_o$. Then Then $a - b = k(n_o - m_o)$ and $b - c = k'(n_o - m_o)$ so $a - c = (k + k')(n_o - m_o)$ therefore $a + (k + k')\,m_o = c + (k + k')n_o$ and $c \geq m_o$ ie. $(a,c) \in \rho$

case 2 $a + km_o = b + kn_o$, $b \geqslant m_o$ and $b + k'n_o = c + k'm_o$, $b \geqslant m_o$. Then $a - b = k(n_o - m_o)$ and $b - c = k'(m_o - n_o)$. If $k > k'$ then $k - k' \in \mathbb{N}$. Since $a - c = (k - k')(n_o - m_o)$, $a + (k - k')m_o = c + (k - k')n_o$. Because $b - c = k'(m_o - n_o) < 0$, $c > b \geqslant m_o$. Therefore $(a,c) \in \rho$. If $k < k'$ them $k' - k \in \mathbb{N}$. Since $a - c = (k' - k)(m_o - n_o)$, $a + (k' - k)n_o = c + (k' - k)m_o$. Because $a - b = k(n_o - m_o) > 0$, $a > b \geqslant m_o$. Therefore $(a, c) \in \rho$. If $k = k'$ then $a + km_o = c + km_o$ ie. $a = c$. Therefore $(a, c) \in \rho$

case 3 $a + kn_o = b + km_o$, $a \geqslant m_o$ and $b + k'm_o = c + k'n_o$, $c \geqslant m_o$. Then $a - b = k(m_o - n_o)$ and $b - c = k'(n_o - m_o)$. If $k = k'$ then $a = c$ ie. $(a,c) \in \rho$ If $k > k'$ then $k - k' \in \mathbb{N}$. Since $a - c = (k - k')(m_o - n_o)$, $a + (k - k')n_o = c + (k - k')m_o$. So $(a, c) \in \rho$. If $k' > k$ then $k' - k \in \mathbb{N}$. Since $a - c = (k' - k)(n_o - m_o)$, $a + (k' - k)m_o = b + (k' - k)n_o$. So $(a,c) \in \rho$.

case 4 $a + kn_o = b + km_o$, $a \geqslant m_o$ and $b + k'n_o = c + k'm_o$, $b \geqslant m_o$. Then $a - b = k(m_o - n_o)$ and $b - c = k'(m_o - n_o)$, so $a - c = (k + k')(m_o - n_o)$. Therefore $a + (k + k')n_o = c + (k + k')m_o$ ie. $(a, c) \in \rho$.

Hence ρ is transitive. Let $(a, b) \in \rho$ and $c \in \mathbb{N}$. Then $\exists k \in \mathbb{N}$ such that either ($a + km_o = b + kn_o$ and $b \geqslant m_o$) or ($a + kn_o = b + km_o$ and $a \geqslant m_o$). Assume that $a + km_o = b + kn_o$ and $b \geqslant m_o$ Then $c + a + km_o = c + b + kn_o$ and $b + c > b \geqslant m_o$ so $(c + a, c + b) \in \rho$. Hence ρ is a congruence on $(\mathbb{N},+)$.

Next we shall show that ρ is the smallest congruence on $(\mathbb{N},+)$ containing (m_o,n_o). Let ρ' be a congruence on $(\mathbb{N},+)$ containing (m_o,n_o). Let $(a,b) \in \rho$. Assume that $a > b$. Claim that $(m_o + k(n_o - m_o), m_o) \in \rho'$ for all $k \in \mathbb{N}$. We shall prove the claim by induction, if $k = 1$

then $(m_o + k(n_o - m_o), m_o) = (n_o, m_o) \in \rho'$. Suppose $(m_o + k(n_o - m_o), m_o) \in \rho'$ so $(m_o + (k+1)(n_o - m_o), n_o) = (m_o + k(n_o - m_o) + (n_o - m_o),\ m_o + (n_o - m_o)) \in \rho'$. Since $(n_o, m_o) \in \rho'$ and ρ' is transitive, $(m_o + (k+1)(n_o - m_o), m_o) \in \rho'$. Hence $(m_o + k(n_o - m_o), m_o) \in \rho'\ \forall k \in \mathbb{N}$. So we have the claim. Because $(a,b) \in \rho$, $\exists\, k \in \mathbb{N}$ such that $(a + km_o = b + kn_o$ and $b \geq m_o)$ or $(a + kn_o = b + km_o$ and $a \geq m_o)$. Since $a > b$, $a + km_o = b + kn_o$ so $b \geq m_o$. If $b = m_o$ then by the claim, $(b + k(n_o - m_o), b) \in \rho'$ so $(a,b) \in \rho'$. Assume that $b > m_o$. Then by the claim $(m_o + k(n_o - m_o), m_o) \in \rho$. Because ρ is a congruence on $\mathbb{N}$ and $b - m_o \in \mathbb{N}$, $(b - m_o + m_o + k(n_o - m_o),\ b - m_o + m_o) \in \rho'$ so $(a,b) = (b + k(n_o - m_o), b) \in \rho'$. Hence $\rho \subsetneq \rho'$. Thus $\rho = \langle (m_o, n_o) \rangle$. #

<u>Theorem 2.1.12</u> Let ρ be a congruence on $(\mathbb{N}, +)$. Then ρ is generated by one element.

<u>Proof</u>. Let $\pi : \mathbb{N} \to \mathbb{N}/\rho$ be the natural projection map. Hence $(\mathbb{N}/\rho, \pi)$ is a quotient semigroup of $\mathbb{N}$. Then $\mathbb{N}/\rho \cong \mathbb{N}$ or $\mathbb{N}/\rho \cong \mathbb{N}_{(m,n)}$ for some $m,n \in \mathbb{N}$. If $\mathbb{N}/\rho \cong \mathbb{N}$ then $\rho = \Delta = \langle (1,1) \rangle$ so we are done. We may assume that $\mathbb{N}/\rho \cong \mathbb{N}_{(m,n)}$ for some $m,n \in \mathbb{N}$. Let $\phi : \mathbb{N} \to \mathbb{N}_{(m,n)}$ be defined as follows:

$$\phi(p) = \begin{cases} p & \text{if } p \leq m, \\ m + k & \text{if } p > m \text{ and } p = m + in + k \text{ for some } i \in \mathbb{N}_o, \\ & k \in \{0,1,\ldots,n-1\}. \end{cases}$$

Then clearly ϕ is an onto homomorphism. Let $\rho^* = \{(a,b) \in \mathbb{N} \times \mathbb{N} \mid \phi(a) = \phi(b)\}$. Then $(\mathbb{N}_{(m,n)}, \phi) \simeq (\mathbb{N}/_{\rho^*}, \pi^*)$ where $\pi^* : \mathbb{N} \to \mathbb{N}/_{\rho^*}$ is the natural projection map. Hence $(\mathbb{N}/_{\rho}, \pi) \simeq (\mathbb{N}/_{\rho^*}, \pi^*)$. By Theorem 2.1.5., $\rho = \rho^*$. Claim that $\rho = \langle(m,m+n)\rangle$. To prove this, clearly $(m,m+n) \in \rho$ so $\langle(m,m+n)\rangle \subsetneq \rho$. We shall show that $\rho \subsetneq \langle(m,m+n)\rangle$. Let $(a,b) \in \rho = \rho^*$ so $\phi(a) = \phi(b)$. If $a = b$ then $(a,b) \in \langle(m,m+n)\rangle$ so we are done. We may assume that $a = m + in + k$ and $b = m + jn + k$ for some $i,j \in \mathbb{N}_0$, $i \neq j$ and $k \in \{0,1,\ldots, n-1\}$. Therefore if $a > b$ then $a = b + (i-j)n$ so $a = b + ln$ where $l = i - j \in \mathbb{N}$. So $a + lm = b + lm + ln = b + l(m+n)$. Therefore $(a,b) \in \langle(m,m+n)\rangle$. Similarly if $b > a$ then $(a,b) \in \langle(m,m+n)\rangle$ Hence $\rho \subsetneq \langle(m,m+n)\rangle$. Thus $\rho = \langle(m,m+n)\rangle$. #

<u>Definition 2.1.13</u> Let S be a commutative semigroup and $a \in S$. Then a is said to be <u>cancellative</u> iff for each $x,y \in S$ $x.a = y.a$ implies that $x = y$.

<u>Theorem 2.1.14</u> Let S be a commutative semigroup containing at least one cancellative element. Then there exists an extension semigroup S' of S such that every cancellative element in S has a inverse in S'.

<u>Proof</u>. Let S be a commutative semigroup. Let $U = \{a \in S \mid a$ is cancellative$\}$. Then $U \neq \phi$. Clearly U is a subsemigroup of S. Define a binary operation. on $S \times U$ by $(x,u)\cdot(x',u') = (x.x', u.u')$ Then $(S \times U, \cdot)$ is a commutative semigroup. Define $\sim = \{((s,u),(s',u')) \in (S\times U)\times(S\times U) \mid su' = s'u\}$. Claim that $\sim$ is a congruence on $S \times U$. To prove this, clearly $\sim$ is reflexive and symmetric. Let $(s,u) \sim (s',u')$ and $(s',u') \sim (s'',u'')$ then $su' = s'u$ and $s'u'' = s''u'$. Because $\cdot$ is commutative, $su'u'' = s'uu'' = s'u''u = s''u'u$ and hence $(su'')u' = (s''u)u'$. Since $u' \in U$, $su'' = s''u$ ie. $(s,u) \sim (s'',u'')$. Hence $\sim$ is transitive. Let $(s,u) \sim (s',u')$ and $(s'',u'') \in S \times U$ Then $su' = s'u$ so $su's''u'' = s'us''u''$. Because $\cdot$ is commutative, $ss''uu'' = s's''u'u''$ hence $(ss'',uu'') \sim (s's'',u'u'')$.

Therefore $(s,u)(s'',u'') \sim (s',u')(s'',u'')$ Thus $\sim$ is a congruence on $S \times U$. Hence $(S \times U/\sim, \cdot)$ is a commutative semigroup.

Next we shall show that S is isomorphic to a subsemigroup of $S \times U/\sim$ and every cancellative element in S has an inverse. Let $u,u' \in U$ then $(su,u) \sim (su',u')$ $\forall s \in S$. Fix $u \in U$. Define $\phi : S \to S \times U/\sim$ by $\phi(s) = [(su,u)]$. Then ϕ is well-defined. Next we shall show that ϕ is 1-1. Let $s_1,s_2 \in S$ be such that $\phi(s_1) = \phi(s_2)$. Then $(s_1u,u) \sim (s_2u,u)$ so $s_1uu = s_2uu$. Because $uu \in U$, $s_1 = s_2$. Hence ϕ is 1-1. Now we shall show that ϕ is a homomorphism. Let $a,b \in S$ then $\phi(a).\phi(b) = [(au,u)]\cdot[(bu,u)] = [(au.bu,uu)] = [(abuu,uu)] = [(abu,u)] = \phi(a,b)$. Hence ϕ is a homomorphism. Therefore $S \cong \phi(S)$ and $\phi(S)$ is a subsemigroup of $S \times U/\sim$. Thus $S \times U/\sim$ is an extension semigroup of S and $\forall u \in U$ $[(u,u)]$ is the identity of $S \times U/\sim$. Let $u \in U \subsetneq S$ then $\phi(u) = [(uu,u)]$. Because $[(u,uu)] \in S \times U/\sim$ and $[(u,uu)] \cdot [(uu,u)] = [(u,u)] = [(uu,u)] \cdot [(u,uu)]$, $(\phi(u))^{-1} = [(u,uu)]$. Thus every cancellative element in S has an inverse. #

<u>Remark</u>: The above construction can be applied to any subsemigroup of U.

2.2 <u>Semigroup-spaces</u>.

In this section we shall work with left congruences on a semigroup S and left S-spaces. But everything that we prove for left congruences and left S-spaces can be similarly proved for right congruences on a semigroup S and right S-space. As in Section 2.1, we shall consider the categories $\mathcal{S}_g$ and $\mathcal{S}_{g,i}$. First we shall define naturally equivalent contravariant functors from $\mathcal{S}_g$ to $\mathcal{L}$ by using left congruences and quotient left semigroup-spaces are defined below.

Definition 2.2.1 A left congruence on a semigroup $(S,\cdot)$ is an equivalence relation ρ on S such that $x \,\rho\, y$ implies that $(a.x) \,\rho\, (a.y)$ for all $x,y,a \in S$.

Definition 2.2.2 Let S be a semigroup and X a nonempty set. A left action of S on X is a map $\cdot : S \times X \to X$ such that $(s.r).x = s.(r.x)$ for all $s,r \in S$, $x \in X$. Then $(X,\cdot)$ is said to be a left S-space.

Remark: For each semigroup $(S,\cdot)$, $(S,\cdot)$ is a left S-space.

Definition 2.2.3 Let $(X,\cdot)$ and $(Y,*)$ be left S-spaces and $\phi: X \to Y$ a map. Then ϕ is said to be left S-equivariant iff $\phi(s.x) = s * \phi(x)$ for all $s \in S$, $x \in X$.

Remarks: 1) If ϕ is a bijectively S-equivalent map then ϕ^{-1} is also left S-equivariant. We shall call such a map a left S-space isomorphism.

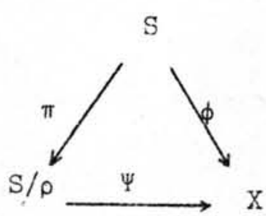
2) If ρ is a left congruence on a semigroup $(S,\cdot)$ then the set S/ρ of equivalence classes of S can be made into a left S-space in natural way and the natural projection map $\pi: S \to S/\rho$ is an onto left S-equivariant map.

Definition 2.2.4 Let S be a semigroup. A quotient left S-space is a pair

(X,ϕ) where X is a left S-space and $\phi:S \to X$ is an onto left S-equivariant map.

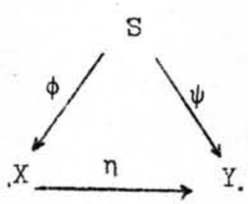
<u>Example</u> $(S/\rho,\pi)$ is a quotient left S-space where ρ is a left congruence on S.

<u>Theorem 2.2.5</u> Let S be a semigroup and (X,ϕ) a quotient left S-space. Let $\rho = \{(a,b) \in S \times S \mid \phi(a) = \phi(b)\}$. Then ρ is a left congruence on S and there exists a left S-space isomorphism ψ from S/ρ to X such that the following diagram commutes.



<u>Proof</u>. It is similar to the proof of Theorem 2.1.2

<u>Definition 2.2.6</u> Let (X,ϕ) and (Y,ψ) be quotient left S-spaces. Say that (X,ϕ) is <u>strongly equivalent</u> to (Y,ψ) iff there exists a left S-space isomorphism $\eta:X \to Y$ such that the following diagram commutes.



Write this as $(X,\phi) \simeq (Y,\psi)$

<u>Remarks</u>: 1) $\simeq$ is an equivalence relation on the set of quotient left S-spaces.

2) For each quotient left S-space (X,ϕ), $(X,\phi) \simeq (S/\rho,\pi)$ where $\rho = \{(a,b) \in S \times S | \phi(a) = \phi(b)\}$.

<u>Proposition 2.2.7</u> Let $\phi: S \to S$ be a semigroup homomorphism. If ρ' is a left congruence on S' then $(\phi \times \phi)^{-1}(\rho')$ is a left congruence on S.

Fix a semigroup S let LC(S) = the set of left congruences on S,
LQ(S) = the set of equivalence classes of quotient left S - spaces under $\simeq$.

We define natural relations $\subseteq$ on LC(S) and LQ(S) as $\subseteq$ on C(S) and Q(S) in Section 2.1 respectively. Then the proof that (LC(S), $\subseteq$) and (LQ(S), $\subseteq$) are posets is similar to the proof that (C(S), $\subseteq$) and (Q(S), $\subseteq$) are posets respectively.

<u>Theorem</u> 2.2.8 For each semigroup S, the posets LC(S) and LQ(S) are isomorphic.

<u>Proof</u>. It is similar to the proof of Theorem 2.1.5 and the isomorphism has the same form as in Theorem 2.1.5.

<u>Remark</u> Fix a semigroup S, let $\rho_1, \rho_2 \in LC(S)$. Then $\rho_1 \cap \rho_2 =$ g.l.b.$\{\rho_1,\rho_2\}$ and the left congruence on S generated by $\rho_1 \cup \rho_2 =$ l.u.b.$\{\rho_1,\rho_2\}$. Hence LC(S) is a lattice. Therefore LQ(S) is a lattice also.

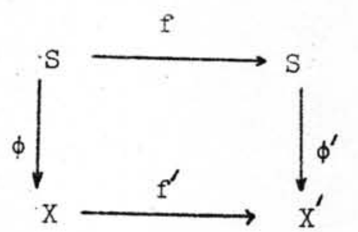
We define contravariant functors LC and LQ from $\mathcal{S}_g$ to $\mathcal{L}$ as we defined the contravariant functors C and Q from $\mathcal{S}_g$ to $\mathcal{L}$ in Section 2.1 respectively. Then the proof that LC is naturally equivalent to LQ is similar to the proof that C is naturally equivalent to Q in Section 2.1.

Next we shall define naturally equivalent covariant functors from $\mathcal{S}_{g,i}$ to $\mathcal{Q}$.

<u>Definition 2.2.9</u> Let ρ_1 and ρ_2 be left congruences on a semigroup S. Say that ρ_1 is <u>equivalent</u> to ρ_2 ($\rho_1 \sim \rho_2$) iff there exists a semigroup automorphism $\phi: S \to S$ such that $(\phi \times \phi)(\rho_1) = \rho_2$.

<u>Remark</u>: $\sim$ is an equivalence relation on the set of left congruences on a semigroup.

<u>Definition 2.2.10</u> Let (X,ϕ), (X',ϕ') be quotient left S-space. Say that (X,ϕ) is <u>weakly equivalent</u> to (X',ϕ') iff there exist a semigroup automorphism $f: S \to S$ and a left S-space isomorphism $f': X \to X'$ such that the following diagram commutes.



Write this as $(X,\phi) \sim (X',\phi')$

<u>Remarks</u>: 1) $\sim$ is an equivalence relation on the set of quotient left S-spaces.

2) $(X,\phi) \simeq (X',\phi')$ implies that $(X,\phi) \sim (X',\phi')$.

Fix a semigroup S, let $LC^*(S)$ = the set of equivalence classes of left congruences on S under $\sim$,

$LQ^*(S)$ = the set of equivalence classes of quotient left S-spaces under $\sim$.

We define binary relation $\leq$ on $LC^*(S)$ and $LQ^*(S)$ as $\leq$ on $C^*(S)$ and $Q^*(S)$ in Section 2.1, respectively. Then the proof that $(LC^*(S),\leq)$ and $(LQ^*(S),\leq)$ are quasi-ordered sets is similar to the proof that $(C^*(S),\leq)$ and $(Q^*(S),\leq)$ are quasi-ordered sets.

<u>Theorem 2.2.11</u> For each semigroup S, the quasi-ordered sets $LC^*(S)$ and $LQ^*(S)$ are isomorphic.

<u>Proof</u>. It is similar to the proof of Theorem 2.1.8, and the isomorphism has the same form as in Theorem 2.1.8.

We define covariant functors LC^* and LQ^* from $\mathcal{S}_{g,i}$ to $\mathcal{Q}$ as we defined the covariant functors C^* and Q^* from $\mathcal{S}_{i,g}$ to $\mathcal{Q}$ in Section 2.1, respectively. Then the proof that LC^* is naturally equivalent to LQ^* is similar to the proof that C^* is naturally equivalent to Q^*.

2.3 Groups

This section will consider the following subcategories of $\mathcal{S}_g$:

1) The category $\mathcal{G}$ of groups and group-homomorphisms.

2) The category $\mathcal{G}_o$ of groups and onto group homomorphisms.

3) The category $\mathcal{G}_i$ of groups and group isomorphisms.

We shall show that $\mathcal{G}$ has a congruence set so we shall define naturally equivalent contravariant functors from $\mathcal{G}$ to $\mathcal{L}$ by using congruences, normal subgroups and quotient groups which are defined below.

Remarks: 1) If ρ is an operation preserving equivalence relation on a group $(G,\cdot)$ then the set G/ρ of equivalence classes of G can be made into a group in natural way and the natural projection map $\pi:G \to G/\rho$ is an onto group homomorphism. Hence the definition of a congruence on an object $(G,\cdot)$ in $\mathcal{G}$ is the same as the definition of an operation preserving equivalence relation on the group $(G,\cdot)$.

2) Let ρ be a congruence on a group G. Then $[1]_\rho = \{a \varepsilon G | a\rho 1\} \trianglelefteq G$.

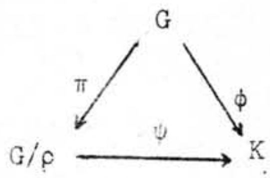
3) Let N be a normal subgroup of a group G ($N \trianglelefteq G$). Then $\{(a,b) \varepsilon G \times G | \; a^{-1}b \varepsilon N\}$ is a congruence on G.

Definition 2.3.1 A quotient group of a group G is a pair (K,ϕ) where K is a group and $\phi:G \to K$ is an onto group homomorphism.

Examples 1) $(G/\rho,\pi)$ is a quotient group of a group G where ρ is a congruence on G.

2) Let N be a normal subgroup of a group G. Let $\rho = \{(a,b) \in G \times G | a^{-1}b \in N\}$ and $G/N = G/\rho$. Then $(G/N,\pi)$ is a quotient group of G.

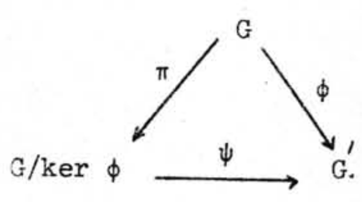
<u>Theorem 2.3.2</u> Let (K,ϕ) be a quotient group of a group G and $\rho = \{(a,b) \in G \times G | \phi(a) = \phi(b)\}$. Then ρ is a congruence on G and there exists an isomorphism $\psi : G/\rho \to K$ such that the following diagram is commutative



<u>Proof</u>. It is similar to the proof of Theorem 2.1.2.

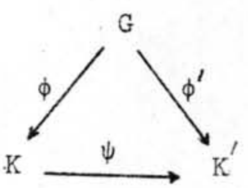
The following result is well-known, it is called <u>the first isomorphism theorem</u> of group theory.

<u>Theorem 2.3.3</u> Let $\phi : G \to G'$ be an onto group homomorphism. Then $\ker \phi \trianglelefteq G$ and there exists a natural isomorphism $\psi : G/_{\ker\phi} \to G'$ such that the following diagram is commutative



<u>Proof</u>. Clearly $\ker \phi \trianglelefteq G$ since ϕ is an onto group homomorphism. Define $\psi : G/_{\ker\phi} \to G'$ as follows: given $\alpha \in G/_{\ker\phi}$ choose $a \in \alpha$ and let $\psi(\alpha) = \phi(a)$. Then ψ is an isomorphism such that $\psi \circ \pi = \phi$. #

Definition 2.3.4 Let (K,ϕ), (K',ϕ') be quotient groups of a group G. Say that (K,ϕ) is strongly equivalent to (K',ϕ') iff there exists an isomorphism $\psi : K \to K'$ such that the following diagram is commutative



Write this as $(K,\phi) \simeq (K',\phi')$.

Remarks: 1. $\simeq$ is an equivalence relation on the set of quotient groups of a group.

2. For each quotient group (K,ϕ) of a group G, $(K,\phi) \simeq (G/\rho,\pi)$ where $\rho = \{(a,b) \in G \times G \mid \phi(a) = \phi(b)\}$.

3. For each quotient group (K,ϕ) of a group G, $(K,\phi) \simeq (G/_{\ker\phi},\pi)$.

Proposition 2.3.5 Let $\phi : G \to G'$ be a group homomorphism. If ρ' is a congruence on G' then $(\phi \times \phi)^{-1}(\rho')$ is a congruence on G. If N' is a normal subgroup of G' then $\phi^{-1}(N')$ is a normal subgroup of G.

Proposition 2.3.6 Let $\phi : G \to G'$ be an onto group homomorphism. If ρ is a congruence on G then $(\phi\times\phi)(\rho)$ is a congruence on G'. If N is a normal subgroup of G then $\phi(N)$ is a normal subgroup of G'.

Proof. Assume that ρ is a congruence on G. Clearly $(\phi \times \phi)(\rho)$ are reflexive and symmetric. Next we shall show that $(\phi \times \phi)(\rho)$ is transitive. Let $(a,b),(b,c) \in (\phi \times \phi)(\rho)$ then $\exists\, (x,y),(y',z) \in \rho$ such that $a = \phi(c)$, $\phi(y) = b = \phi(y')$, $c = \phi(z)$. So $\phi(yy'^{-1}) = 1$ ie. $yy'^{-1} \in \ker\phi$

therefore $y'y^{-1} = k$ for some $k \in \ker\phi$ so $y' = ky$. Since $(x,y) \in \rho$, $(kx,ky) \in \rho$. Because $(ky,z) = (y',z) \in \rho$ and ρ is transitive, $(kx,z) \in \rho$ Then $(a,c) = (\phi(x),\phi(z)) = (\phi(kx),\phi(z)) = (\phi \times \phi)(kx,z) \in (\phi \times \phi)(\rho)$. Hence $(\phi \times \phi)(\rho)$ is transitive. Let $(a,b) \in (\phi \times \phi)(\rho)$ and $c \in G'$ Then clearly $(ac,bc), (ca,cb) \in (\phi \times \phi)(\rho)$. Hence $(\phi \times \phi)(\rho)$ is a congruence on G'. The proof of the second part is standard. #

Fix a group G, let $C(G)$ = the set of congruences on G,

$N(G)$ = the set of normal subgroups of G,

$Q(G)$ = the set of equivalence classes of quotient groups of G under $\simeq$.

We define natural relations $\subsetneq$ on $C(G)$ and $Q(\,)$ as $\subsetneq$ on $C(S)$ and $Q(S)$ in Section 2.1 respectively. Then the proof that $(C(G),\subsetneq)$ and $(Q(G),\subsetneq)$ are posets is similar to the proof that $(C(S),\subsetneq)$ and $(Q(S),\subsetneq)$ are posets, respectively. Let $\subsetneq$ on $N(G)$ be set inclusion. Then clearly $(N(G),\subsetneq)$ is a poset.

Theorem 2.3.7 For each group G, the posets $C(G)$ and $Q(G)$ are isomorphic.

Proof. It is similar to the proof of Theorem 2.1.5 and the isomorphism has the same form as in Theorem 2.1.5.

Theorem 2.3.8 For each group G, the posets $C(G)$ and $N(G)$ are isomorphic.

Proof. Let G be a group. Define $\phi: C(G) \to N(G)$ by $\phi(\rho) = [1]_\rho = \{g \in G \mid g\rho 1\}$ $\forall \rho \in C(G)$. Then ϕ is well-defined. First we shall show that ϕ is 1-1. Let $\rho_1,\rho_2 \in C(G)$ be such that $\phi(\rho_1) = \phi(\rho_2)$. Must show that $\rho_1 = \rho_2$, let $(a,b) \in \rho_1$ so $(ab^{-1},1) \in \rho_1$ then $ab^{-1} \in \phi(\rho_1) = \phi(\rho_2)$ so $(ab^{-1},1) \in \rho_2$ therefore $(a,b) \in \rho_2$. Hence $\rho_1 \subseteq \rho_2$. Similarly we can show that $\rho_2 \subseteq \rho_1$. So $\rho_1 = \rho_2$. Hence ϕ is 1-1. Next we shall show that ϕ is onto. Let $N \in N(G)$. Define $\rho = \{(a,b) \in G \times G | a^{-1}b \in N\}$. Then $\rho \in C(G)$ and $\phi(\rho) = \{a \in G | a\rho\, 1\} = \{a \in G | a \in N\} = N$. Thus ϕ is onto. Next we shall show that ϕ is isotone. Let $\rho_1,\rho_2 \in C(G)$ be such that $\rho_1 \subseteq \rho_2$. Must show that $\phi(\rho_1) \subseteq \phi(\rho_2)$, let $a \in \phi(\rho_1)$ then $(a,1) \in \rho_1 \subseteq \rho_2$ so $a \in \phi(\rho_2)$. Hence $\phi(\rho_1) \subseteq \phi(\rho_2)$. Thus ϕ is isotone. Lastly we shall show that ϕ^{-1} is isotone. Let $N_1,N_2 \in N(G)$ be such that $N_1 \subseteq N_2$. Must show that $\phi^{-1}(N_1) \subseteq \phi^{-1}(N_2)$, let $(a,b) \in \phi^{-1}(N_1) = \{(x,y) \in G \times G | x^{-1}y \in N_1\}$ then $a^{-1}b \in N_1 \subseteq N_2$ so $(a,b) \in \phi^{-1}(N_2)$. Hence $\phi^{-1}(N_1) \subseteq \phi^{-1}(N_2)$. Therefore ϕ^{-1} is isotone. Thus ϕ is an isomorphism ie. $C(G)$ is isomorphic to $N(G)$. #

Corollary 2.3.9 For each group G, the posets $N(G)$ and $Q(G)$ are isomorphic.

Proposition 2.3.10 Let N_1,N_2 be normal subgroups of a group G. Then $N_1.N_2 = \{n_1.n_2 | n_1 \in N_1, n_2 \in N_2\}$ is the normal subgroup of G generated by $N_1 \cup N_2$.

Proof. It is standard.

Proposition 2.3.11 Let ρ_1,ρ_2 be congruences on a group G. Then $\rho_1.\rho_2 = \{(a_1.a_2, b_1.b_2) | (a_1,b_1) \in \rho_1, (a_2,b_2) \in \rho_2\}$ is the congruence on G generated by $\rho_1 \cup \rho_2$.

Proof. First we shall show that $\rho_1.\rho_2$ is a congruence on G. Clearly $\rho_1.\rho_2$ is reflexive and symmetric. Let $(a,b),(b,c) \in \rho_1.\rho_2$ then $a = a_1a_2, b = b_1b_2 = b_1'b_2'$, $c = c_1c_2$ where $(a_1,b_1),(b_1',c_1) \in \rho_1, (a_2,b_2)$, $(b_2',c_2) \in \rho_2$. Then $(b_1,c_1b_1'^{-1}b_1) = (b_1'b_1'^{-1}b_1, c_1b_1'^{-1}b_1) \in \rho_1$ and $(b_2,b_2b_2'^{-1}c_2) = (b_2b_2'^{-1}b_2', b_2b_2'^{-1}c_2) \in \rho_2$. Since ρ_1 and ρ_2 are transitive, $(a_1,c_1b_1'^{-1}b_1) \in \rho_1$ and $(a_2,b_2b_2'^{-1}c_2) \in \rho_2$. Then $(a,c) = (a_1a_2,c_1c_2) = (a_1a_2,c_1b_1'^{-1}b_1b_2b_2'^{-1}c_2) \in \rho_1.\rho_2$. Hence $\rho_1.\rho_2$ is transitive. Let $(a,b) \in \rho_1.\rho_2$ and $c \in G$. Then $a = a_1a_2$, $b = b_1b_2$ where $(a_1,b_1) \in \rho_1$, $(a_2,b_2) \in \rho_2$. So $(ca,cb) = (ca_1,cb_1)(a_2,b_2) \in \rho_1.\rho_2$ and $(ac,bc) = (a_1,b_1)(a_2c,b_2c) \in \rho_1.\rho_2$. Hence $\rho_1.\rho_2$ is a congruence on G. Clearly $\rho_1 \subseteq \rho_1.\rho_2$ and $\rho_2 \subseteq \rho_1.\rho_2$. Let ρ be a congruence on G containing $\rho_1 \cup \rho_2$. Let $(a,b) \in \rho_1.\rho_2$. Then $a = a_1a_2$, $b = b_1b_2$ where $(a_1,b_1) \in \rho_1$, $(a_2,b_2) \in \rho_2$. So $(a_1,b_1),(a_2,b_2) \in \rho$. Then $(a_1a_2,b_1a_2),(b_1a_2,b_1b_2) \in \rho$ Hence $(a,b) = (a_1a_2, b_1b_2) \in \rho$. Thus $\rho_1.\rho_2 \subseteq \rho$. Hence $\rho_1.\rho_2$ is the congruence on G generated by $\rho_1 \cup \rho_2$. #

We shall show that $(C(G),\subseteq)$, $(N(G),\subseteq)$ and $(Q(G),\subseteq)$ are lattices for all groups G. Let G be a group. Let $N_1,N_2 \in N(G)$. Then $N_1 \cap N_2 = \text{g.l.b.}\{N_1,N_2\}$ and $N_1.N_2 = \text{l.u.b.}\{N_1,N_2\}$. Hence $(N(G),\subseteq)$ is a lattice. Let $\rho_1,\rho_2 \in C(G)$. Then $\rho_1 \cap \rho_2 = \text{g.l.b.}\{\rho_1,\rho_2\}$ and $\rho_1.\rho_2 = \text{l.u.b.}\{\rho_1,\rho_2\}$. Hence $(C(G),\subseteq)$ is a lattice. Therefore $(Q(G),\subseteq)$ is a lattice also.

We define contravariant functors C,Q from $\mathcal{G}$ to $\mathcal{L}$ as the contravariant functors C,Q from $\mathcal{S}_g$ to $\mathcal{L}$ in Section 2.1 respectively. Next we shall define a contravariant functor N from $\mathcal{G}$ to $\mathcal{L}$. Let G,

G' be in Ob$\mathscr{G}$ and $\phi : G \to G'$ a group homomorphism. Then $N(G), N(G')$ are in Ob$\mathscr{L}$ Define $N(\phi) : N(G') \to N(G)$ by $N(\phi)(A) = \phi^{-1}(A)$. Then $N(\phi)$ is an isotone map. Since $N(id_G) = id_{N(G)}$ for all G in Ob$\mathscr{G}$ and $N(\phi o \eta) = N(\eta) o N(\phi)$ for all group homomorphisms ϕ, η whenever $\phi o \eta$ is defined, N is a contravariant functor from $\mathscr{G}$ to $\mathscr{L}$.

The proof that C is naturally equivalent to Q is similar to the proof that C is naturally equivalent to Q in Section 2.1. Next we shall show that N is naturally equivalent to C. For each G in Ob$\mathscr{G}$, define $f_G : N(G) \to C(G)$ be the map in Theorem 2.3.8. Then f_G is an isomorphism. We shall show that f is a natural equivalence from N to C. Let G, G', be in Ob$\mathscr{G}$ and $\phi : G \to G'$ a group homomorphism. So we have f_G, $f_{G'}$ and the following diagram.

$$\begin{array}{ccccc} G & \quad & N(G) & \xrightarrow{\;f_G\;} & C(G) \\ \phi \big\downarrow & & N(\phi) \big\uparrow & & \big\uparrow C(\phi) \\ G' & & N(G') & \xrightarrow[\;f_{G'}\;]{} & C(G') \end{array}$$

We must show that $C(\phi) o f_{G'} = f_G o N(\phi)$. Let $A \in N(G')$. Then $(C(\phi) o f_{G'})(A) = (C(\phi))(\rho_A) = (\phi \times \phi)^{-1}(\rho_A)$ where $\rho_A = \{(a,b) \in G' \times G' | a^{-1}b \in A\}$, and $(f_G o N(\phi))(A) = f_G(\phi^{-1}(A)) = \rho_{\phi^{-1}(A)} = \{(a,b) \in G \times G | a^{-1}b \in \phi^{-1}(A)\}$.

Clearly $(\phi \times \phi)^{-1}(\rho_A) = \rho_{\phi^{-1}(A)}$ Then $(C(\phi) o f_{G'})(A) = (f_G o N(\phi))(A)$.

Hence $C(\phi) o f_{G'} = f_G o N(\phi)$. Therefore f is a natural equivalence from N to C. Thus there exist three naturally equivalent contravariant functors C,N,Q from $\mathscr{G}$ to $\mathscr{L}$.

Remark As a result we see that C is the congruence functor of $\mathscr{G}$, $\mathscr{G}$ has a congruence set and normal subgroups of a group are congruence sets with respect to N.

Next we shall define three naturally equivalent covariant functors from $\mathscr{G}_o$ to $\mathscr{L}$. For each group G, let $C'(G) = C(G)$, $N'(G) = N(G)$ and $Q'(G) = Q(G)$.

1) Let G, G' be in Ob $\mathscr{G}_o$ and $\phi : G \to G'$ an onto group homomorphism. Then $C'(G), C'(G')$ are in Ob $\mathscr{L}$. Define $C'(\phi) : C'(G) \to C'(G')$ by $C'(\phi)(\rho) = (\phi \times \phi)(\rho)$ $\forall \rho \in C'(G)$. Then $C'(\phi)$ is an isotone map. Since $C'(id_G) = id_{C'(G)}$ $\forall$ G in Ob $\mathscr{G}_o$ and $C'(\phi o \eta) = C'(\phi) o C'(\eta)$ $\forall$ onto group homomorphisms ϕ, η whenever $\phi o \eta$ is defined, C' is a covariant functor from $\mathscr{G}_o$ to $\mathscr{L}$.

2) Let G, G' be in Ob $\mathscr{G}_o$ and $\phi : G \to G'$ an onto group homomorphism. Then $N'(G), N'(G')$ are in Ob $\mathscr{L}$. Define $N'(\phi) : N'(G) \to N'(G')$ by $N'(\phi)(N) = \phi(N)$ $\forall N \in N'(G)$. Then $N'(\phi)$ is an isotone map. Since $N'(id_G) = id_{N'(G)}$ $\forall$ G in Ob $\mathscr{G}_o$ and $N'(\phi o \eta) = N'(\phi) o N'(\eta)$ $\forall$ onto group homomorphisms $\phi o \eta$ whenever $\phi o \eta$ is defined, N' is a covariant functor from $\mathscr{G}_o$ to $\mathscr{L}$.

3) Let G, G' be in Ob $\mathscr{G}_o$ and $\phi : G \to G'$ an onto group homomorphism. Then $Q'(G), Q'(G')$ are in Ob $\mathscr{L}$. Define $Q'(\phi) : Q'(G) \to Q(G')$ as follows: given $\alpha \in Q'(G)$ choose $(K, \eta) \in \alpha$ and then let $(Q'(\phi))(\alpha) = [(G'/_{(\phi \times \phi)(\rho)}, \pi)]$ where $\rho = \{(a,b) \in G \times G \mid \eta(a) = \eta(b)\}$. First we shall show that $Q(\phi)$ is well-defined. Let $(K_1, \eta_2) \simeq (K_2, \eta_2)$. Then by the proof of Theorem 2.3.7, $\rho_1 = \rho_2$ so $(\phi \times \phi)(\rho_1) = (\phi \times \phi)(\rho_2)$ and hence $(G'/_{(\phi \times \phi)(\rho_1)}, \pi_1) = (G'/_{(\phi \times \phi)(\rho_2)}, \pi_2)$. Hence $Q'(\phi)$ is well-defined. Next we shall show that

$Q'(\phi)$ is isotone. Let $\alpha,\beta \in Q'(G)$ be such that $\alpha \subseteq \beta$. Choose $(K_1,\eta_1) \in \alpha$. $(K_2,\eta_2) \in \beta$. Then by the proof of Theorem 2.3.10, $\rho_1 \subseteq \rho_2$. So $(\phi \times \phi)(\rho_1) \subseteq (\phi \times \phi)(\rho_2)$ and hence $[(G'/_{(\phi\times\phi)(\rho_1)},\pi'_1)] \subseteq [(G'/_{(\phi\times\phi)(\rho_2)},\pi'_2)]$ ie. $Q'(\phi)(\alpha) \subseteq (Q'(\phi)(\beta)$. Hence $Q(\phi)$ is isotone. Lastly we shall show that Q' is a covariant functor from $\mathcal{G}_o$ to $\mathcal{L}$. Clearly $Q'(id_G) = id_{Q'(G)}$ $\forall$ G in Ob$\mathcal{G}_o$. Let $\phi:G \to G'$ and $\phi':G' \to G''$ be onto group homomorphisms. Let $\alpha \in Q'(G)$ choose $(K,\eta) \in \alpha$. Then $(Q'(\phi')oQ'(\phi))(\alpha) = Q'(\phi')\ [(G'/_{(\phi\times\phi)(\rho)},\pi')] = [G''/_{(\phi'\times\phi')o(\phi\times\phi)(\rho)},\pi''] = [G''/_{(\phi' o\phi\times\phi' o\phi)(\rho)},\pi''] = Q'(\phi' o\phi)(\alpha)$ where $\rho = \{(a,b) \in G \times G \mid \eta(a) = \eta(b)\}$. Therefore $Q'(\phi')oQ'(\phi) = Q'(\phi' o\phi)$. Hence Q' is a covariant functor from $\mathcal{G}_o$ to $\mathcal{L}$.

Next we shall show that N',C',Q' are naturally equivalent.

1) For each G in Ob$\mathcal{G}_o$ define $f_G:N'(G) \to C'(G)$ be the map in Theorem 2.3.8. Then f_G is an isomorphism. We shall show that f_G is a natural equivalence from N' to C'. Let G,G' be in Ob$\mathcal{G}_o$ and $\phi:G \to G'$ be an onto group homomorphism so we have $f_G, f_{G'}$ and the following diagram

$$
\begin{array}{ccccc}
G & & N'(G) & \xrightarrow{\ f_G\ } & C'(G) \\
\phi\downarrow & & N'(\phi)\downarrow & & \downarrow C'(\phi) \\
G' & & N'(G') & \xrightarrow{\ f_{G'}\ } & C'(G')
\end{array}
$$

We must show that $C'(\phi) o f_G = f_{G'} o N'(\phi)$. Let $N \in N'(G)$. Then $(C'(\phi) o f_G)(N) = C'(\phi)\{(a,b) \in G \times G \mid a^{-1}b \in N\} = (\phi \times \phi)\{(a,b) \in G \times G \mid a^{-1}b \in N\} = \{(\phi(a),\phi(b)) \mid a^{-1}b \in N\}$ and $(f_{G'} o N'(\phi))(N) = f_{G'}(\phi(N)) = \{(x,y) \in G' \times G' \mid x^{-1}y \in \phi(N)\}$. We want to show that $\{(\phi(a),\phi(b)) \mid a^{-1}b \in N\} = \{(x,y) \in G' \times G' \mid x^{-1}y \in \phi(N)\}$. Clearly $\{(\phi(a),\phi(b)) \mid a^{-1}b \in N\} \subsetneq \{(x,y) \in G \times G \mid x^{-1}y \in \phi(N)\}$. So we must show that $\{(x,y) \in G \times G \mid x^{-1}y \in \phi(N)\} \subsetneq \{(\phi(a),\phi(b)) \mid a^{-1}b \in N\}$. First we shall show that $\phi^{-1}(\phi(N))$ is the subgroup of G generated by N and ker ϕ. Let $a,b \in \phi^{-1}(\phi(N))$. Then $\phi(a),\phi(b) \in \phi(N)$. So $\phi(a^{-1}b) = \phi(a^{-1}).\phi(b) = (\phi(a))^{-1}(\phi(b)) = (\phi(n_1))^{-1}(\phi(n_2) = \phi(n_1^{-1}n_2) \in \phi(N)$ where $\phi(a) = \phi(n_1)$, $\phi(b) = \phi(n_2)$ and $n_1,n_2 \in N$. Hence $a^{-1}b \in \phi^{-1}(\phi(N))$. Thus $\phi^{-1}(\phi(N)) \leqslant G$. Clearly $N \subsetneq \phi^{-1}(\phi(N))$ and ker $\phi \subseteq \phi^{-1}(\phi(N))$. Let M be a subgroup of G containing N and ker ϕ. Must show that $\phi^{-1}(\phi(N)) \subseteq M$, let $a \in \phi^{-1}(\phi(N))$ so $\phi(a) \in \phi(N)$ then $\phi(a) = \phi(n)$ for some $n \in N$. Then $\phi(n^{-1}a) = 1$ so $n^{-1}a \in \ker\phi \subseteq M$. Therefore $a = n.n^{-1}a \in M$. Hence $\phi^{-1}(\phi(N)) \subseteq M$. Thus $\phi^{-1}(\phi(N)$ is the subgroup of G generated by N and ker ϕ . Because $N \trianglelefteq G$ and ker $\phi \trianglelefteq G$, $N.\ker\phi = \phi^{-1}(\phi(N))$. Now we can show that $\{(x,y) \in G' \times G' \mid x^{-1}y \in \phi(N)\} \subseteq \{(\phi(a),\phi(b)) \mid a^{-1}b \in N\}$. Let $(x,y) \in G' \times G'$ be such that $x^{-1}y \in \phi(N)$. Since ϕ is onto, $\exists c,d \in G$ such that $x = \phi(c)$, $y = \phi(d)$. Then $\phi(c^{-1}d) = x^{-1}y \in \phi(N)$ so $c^{-1}d \in (\phi^{-1} o \phi)(N)$. Therefore $\exists n \in N$, $m \in \ker\phi$ such that $c^{-1}d = n.m$ so $c^{-1}(dm^{-1}) = nmm^{-1} = n \in N$. Since $\phi(c) = x$ and $\phi(dm^{-1}) = \phi(d).\phi(m^{-1}) = \phi(d) = y$, $(x,y) = (\phi(c),\phi(dm^{-1})) \in \{(\phi(a),\phi(b)) \mid a^{-1}b \in N\}$. Hence $\{(x,y) \in G' \times G' \mid x^{-1}y \in \phi(N)\} \subsetneq \{(\phi(a),\phi(b) \mid a^{-1}b \in N\}$. Therefore $(C'(\phi) o f_G)(N) = (f_{G'} o N'(\phi))(N)$. Thus

$C'(\phi)\circ f_G = f_{G'}\circ N'(\phi)$. Therefore N' is naturally equivalent to C'.

2) For each G in $Ob\mathcal{G}_o$ define $h_G : C'(G) \to Q'(G)$ be the map in Theorem 2.3.7. Then h_G is an isomorphism. We shall show that h is a natural equivalence from C' to Q'. Let G, G' be in $Ob\mathcal{G}_o$ and $\phi : G \to G'$ be an onto group homomorphism so we have $h_G, h_{G'}$ and the following diagram

$$\begin{array}{ccccc} G & \quad & C'(G) & \xrightarrow{\;h_G\;} & Q'(G) \\ \phi\downarrow & & C'(\phi)\downarrow & & \downarrow Q'(\phi) \\ G' & & C'(G') & \xrightarrow{\;h_{G'}\;} & Q'(G') \end{array}$$

We must show that $Q'(\phi)\circ h_G = h_{G'}\circ C'(\phi)$. Let $\rho \in C'(G)$. Then $Q'(\phi)\circ h_G(\rho) = Q'(\phi)[G/\rho, \pi] = [G'/_{(\phi\times\phi)(\rho)}, \pi'] = h_{G'}((\phi\times\phi)(\rho)) = (h_{G'}\circ C'(\phi))(\rho)$. Hence $Q'(\phi)\circ h_G = h_{G'}\circ C'(\phi)$. Therefore h is a natural equivalence from C' to Q'. Thus C', N', Q' are naturally equivalent.

Now we shall define naturally equivalent covariant functors from $\mathcal{G}_i$ to $\mathcal{Q}$ using equivalence classes of congruences, equivalence classes of normal subgroups and equivalence classes of quotient groups which are defined below.

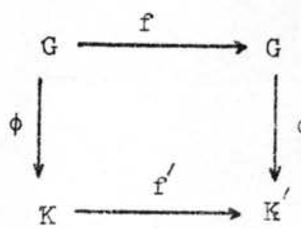
Definition 2.3.12 Let ρ_1 and ρ_2 be congruences on a group G. Say that ρ_1 is equivalent to ρ_2 ($\rho_1 \sim \rho_2$) iff there exists an automorphism $f : G \to G$ such that $(f \times f)(\rho_1) = \rho_2$.

<u>Remark</u>: $\sim$ is an equivalence relation on the set of congruences on a group.

<u>Definition 2.3.13</u> Let N_1 and N_2 be normal subgroups of a group G. Say that N_1 is <u>equivalent</u> to N_2 ($N_1 \sim N_2$) iff there exists an automorphism $f:G \to G$ such that $f(N_1) = N_2$.

<u>Remark</u>: $\sim$ is an equivalence relation on the set of normal subgroups of a group.

<u>Definition 2.3.14</u> Let (K,ϕ), (K',ϕ') be quotient groups of G. Say that (K,ϕ) is <u>weakly equivalent</u> to (K',ϕ') iff there exist isomorphisms $f:G \to G$ and $f':K \to K'$ such that the following diagram is commutative



Write this as $(K,\phi) \sim (K',\phi')$

<u>Remarks</u>: 1) $\sim$ is an equivalence relation on the set of quotient groups of a group.

2) $(K,\phi) \simeq (K',\phi')$ implies that $(K,\phi) \sim (K',\phi')$.

Fix a group G let $C^*(G)$ = the set of equivalence classes of congruences on G under $\sim$,

$N^*(G)$ = the set of equivalence classes of normal subgroups of G under $\sim$,

$Q^*(G)$ = the set of equivalence classes of quotient groups of G under $\sim$.

We define binary relations $\leqslant$ on $C^*(G)$ and $Q^*(G)$ as $\leqslant$ on $C^*(S)$ and $Q^*(S)$ in Section 2.1 respectively. Then the proof that $(C^*(G),\leqslant)$ and $(Q^*(G),\leqslant)$ are quasi-ordered sets is similar to the proof that $(C^*(S),\leqslant)$ and $(Q^*(S),\leqslant)$ are quasi-ordered set respectively. Next we shall define a binary relation $\leqslant$ on $N^*(G)$ as follows: given $\alpha,\beta \in N^*(G)$ say that $\alpha \leqslant \beta$ iff there exist $N_1 \in \alpha$, $N_2 \in \beta$ such that $N_1 \subsetneq N_2$. Clearly $\leqslant$ is well-defined and $(N^*(G),\leqslant)$ is a quasi-ordered set.

<u>Theorem 2.3.15</u> For each group G the quasi-ordered sets $C^*(G)$ and $Q^*(G)$ are isomorphic.

<u>Proof</u>. It is similar to the proof of Theorem 2.1.8, and the isomorphism has the same form as in Theorem 2.1.8.

<u>Theorem 2.3.16</u> For each group G the quasi-ordered sets $C^*(G)$ and $N^*(G)$ are isomorphic.

<u>Proof</u>. Let G be a group. Define $\phi:C^*(G) \to N^*(G)$ as follows: given $\alpha \in C^*(G)$ choose $\rho \in \alpha$ and then let $\phi(\alpha) = [[1]_\rho]$. First we shall show that ϕ is well-defined. Let $\rho_1 \sim \rho_2$. Then $\exists$ an automorphism $f:G \to G$ such that $(f \times f)(\rho_1) = \rho_2$ We must show that $[1]_{\rho_1} \sim [1]_{\rho_2}$ To do this we shall show that $f([1]_{\rho_1}) = [1]_{\rho_2}$. Let $x \in [1]_{\rho_2}$ so

$(x,1) \varepsilon \rho_1$ then $(f(x), f(1)) \varepsilon (f \times f)(\rho_1) = \rho_2$ so $f(x) \varepsilon [1]_{\rho_2}$. Hence $f([1]_{\rho_1}) \subseteq [1]_{\rho_2}$. Let $y \varepsilon [1]_{\rho_2}$ so $(y,1) \varepsilon \rho_2$ and $\exists x \varepsilon G$ such that $f(x) = y$, hence $(f \times f)(x,1) = (f(x),f(1)) \varepsilon \rho_2 = (f \times f)(\rho_1)$. Because f is a bijection, $(x,1) \varepsilon \rho_1$ therefore $x \varepsilon [1]_{\rho_1}$ so $y = f(x) \varepsilon f([1]_{\rho_1})$. Hence $[1]_{\rho_2} \subseteq f([1]_{\rho_1})$. Therefore $f([1]_{\rho_1}) = [1]_{\rho_2}$. Thus ϕ is is well-defined.

Next we shall show that ϕ is 1-1. Let ρ_1, ρ_2 be congruences on G such that $[1]_{\rho_1} \sim [1]_{\rho_2}$. So $\exists$ an automorphism $f:G \to G$ such that $f([1]_{\rho_1}) = [1]_{\rho_2}$. Must show that $\rho_1 \sim \rho_2$, to do this we shall show that $(f \times f)(\rho_1) = \rho_2$. Let $(x,y) \varepsilon \rho_1$ so $x^{-1}y \varepsilon [1]_{\rho_1}$ therefore $(f(x))^{-1}(f(y)) = f(x^{-1}y) \varepsilon f([1]_{\rho_1}) = [1]_{\rho_2}$. Hence $(f(x),f(y)) \varepsilon \rho_2$. So $(f \times f)(\rho_1) \subseteq \rho_2$. Let $(a,b) \varepsilon \rho_2$ so $a^{-1}b \varepsilon [1]_{\rho_2} = f([1]_{\rho_1})$ then $f(d) = a^{-1}b$ for some $d \varepsilon [1]_{\rho_1}$. Therefore $(f^{-1}(a))^{-1}(f^{-1}(b)) = f^{-1}(a^{-1}b) = d \varepsilon [1]_{\rho_1}$. Thus $(f^{-1}(a),f^{-1}(b)) \varepsilon \rho_1$ so $(a,b) \varepsilon (f \times f)(f^{-1}(a), f^{-1}(b)) \varepsilon (f \times f)(\rho_1)$. Therefore $(f \times f)(\rho_1) = \rho_2$. Thus ϕ is 1-1.

Next we shall show that ϕ is onto. Let $N \trianglelefteq G$. Define $\rho_N = \{(a,b) \varepsilon G \times G | a^{-1}b \varepsilon N\}$. Then ρ_N is a congruence on G So $[\rho_N] \varepsilon C^*(G)$ and $\phi([\rho_N]) = [\{a \varepsilon G | a\rho 1\}] = [N]$. Hence ϕ is onto.

Next we shall show that ϕ is isotone. Let $\alpha, \beta \in C^*(G)$ be such that $\alpha \leqslant \beta$ Then $\exists \rho_1 \in \alpha$, $\rho_2 \in \beta$ such that $\rho_1 \subsetneq \rho_2$. So $[1]_{\rho_1} \leqslant [1]_{\rho_2}$ ie. $\phi(\alpha) \leqslant \phi(\beta)$. Hence ϕ is isotone

Lastly we shall show that ϕ^{-1} is isotone. Let $\alpha, \beta \in N^*(G)$ be such that $\alpha \leqslant \beta$. Choose $N_1 \in a$, $N_2 \in \beta$ such that $N_1 \subsetneq N_2$. So $\rho_{N_1} \subsetneq \rho_{N_2}$ ie. $\phi^{-1}(\alpha) \leqslant \phi^{-1}(\beta)$. Hence ϕ^{-1} is isotone. Therefore ϕ is an isomorphism. #

<u>Corollary 2.3.17</u> For each groups G the quasi-ordered sets $N^*(G)$ and $Q^*(G)$ are isomorphic.

We define covariant functors C^*, Q^* from $\mathcal{G}_i$ to $\mathcal{Q}$ as the covariant functors C^*, Q^* from $\mathcal{S}_{g,i}$ to $\mathcal{Q}$ in Section 2.1 respectively. Next we shall define a covariant functor N^* from $\mathcal{G}_i$ to $\mathcal{Q}$. Let G, G' be in $\mathrm{Ob}\mathcal{G}_i$ and $\phi: G \to G'$ a group isomorphism. Then $N^*(G), N^*(G')$ are in $\mathrm{Ob}\,\mathcal{Q}$. Define $N^*(\phi): N^*(G) \to N^*(G')$ as follows: given $\alpha \in N^*(G)$ choose $N \in \alpha$ and then let $(N^*(\phi))(\alpha) = [\phi(N)]$. First we shall show that $N^*(\phi)$ is well-defined. Let $N_1 \sim N_2$ then $\exists$ an automorphism $\eta: G \to G$ such that $\eta(N_1) = N_2$. We must show that $\phi(N_1) \sim \phi(N_2)$. To do this, define $\eta^*: G' \to G'$ by $\eta^* = \phi \circ \eta \circ \phi^{-1}$. Then η^* is an automorphism such that $\eta^*(\phi(N_1)) = \phi(N_2)$. Hence $\phi(N_1) \sim \phi(N_2)$. Therefore $N^*(\phi)$ is well-defined Next we shall show that $N^*(\phi)$ is isotone. Let $\alpha, \beta \in N^*(G)$ be such that $\alpha \leqslant \beta$. Then $\exists\, N_1 \in \beta$, $N_2 \in \beta$ such that $N_1 \subsetneq N_2$. So $\phi(N_1) \subsetneq \phi(N_2)$ and hence $[\phi(N_1)] \leq [\phi(N_2)]$ ie. $(N^*(\phi))(\alpha) \leq (N^*(\phi))(\beta)$. Therefore $N^*(\phi)$ is isotone. Lastly we shall show that N^* is a covariant functor from $\mathcal{G}_i$ to $\mathcal{Q}$. Clearly $N^*(id_G) = id_{N^*(G)}$ $\forall G$ in Ob $\mathcal{G}_i$. Let $\phi:G \to G'$, $\phi':G' \to G''$ be group isomorphisms. We must show that $N^*(\phi'o\phi) = N^*(\phi')oN^*(\phi)$. Let $\alpha \in N^*(G)$ choose $N \in \alpha$ then $(N^*(\phi')oN^*(\phi))(\alpha) = (N^*(\phi'))[\phi(N)] = [\phi'o\phi(N)] = (N^*(\phi'o\phi))(\alpha)$. Hence $N^*(\phi'o\phi) = N^*(\phi')oN^*(\phi)$ Therefore N^* is a covariant functor from $\mathcal{G}_i$ to $\mathcal{Q}$.

The proof that C^* is naturally equivalent to Q^* is similar to the proof that C^* is naturally equivalent to Q^* in Section 2.1. Next we shall show that N^* is naturally equivalent to C^*. For each G in Ob $\mathcal{G}_i$ define $f_G:N^*(G) \to C^*(G)$ be the map in Theorem 2.3.16. Then f_G is an isomorphism. We shall show that f is a natural equivalence from N^* to C^*. Let G,G' in Ob $\mathcal{G}_i$ and $\phi:G \to G'$ a group isomorphism. So we have f_G, $f_{G'}$ and the following diagram.

$$\begin{array}{ccccc} G & & N^*(G) & \xrightarrow{f_G} & C^*(G) \\ \phi\downarrow & & N^*(\phi)\downarrow & & \downarrow C(\phi) \\ G' & & N^*(G') & \xrightarrow{f_{G'}} & C^*(G') \end{array}$$

We must show that $C^*(\phi)o f_G = f_{G'}oN^*(\phi)$. Let $\alpha \in N^*(G)$ choose $N \in \alpha$ Then $(C^*(\phi)of_G)(\alpha) = C^*(\alpha)[\rho_N] = [(\phi \times \phi)(\rho_N)]$ and $(f_{G'}oN^*(\phi))(\alpha) = f_{G'}([\phi(N)]) = [\rho_{\phi(N)}]$. Since ϕ is an isomorphism, $(\phi \times \phi)\rho_N = \rho_{\phi(N)}$. Hence $(C^*(\phi)of_G)(\alpha) = (f_{G'}oN^*(\phi))(\alpha)$. Thus f is a natural equivalence from N^* to C^*. Thus there exist three naturally equivalent covariant functors C^*, N^*, Q^* from $\mathcal{G}_i$ to $\mathcal{Q}$.

Next, we shall consider some theorems which use normal subgroups (ie. congruence sets).

Let G_1, G_2 be groups. Let $G = G_1 \times G_2$ and define a binary operation $\cdot$ on G by $(x_1,x_2)\cdot(y_1,y_2) = (x_1 \cdot y_1, x_2 \cdot y_2)$ Then $(G,\cdot)$ is a group. Let $H_1 = \{(x,1) | x \in G_1\}$ and $H_2 = \{(1,y) | y \in G_2\}$ Then

i) H_1 and H_2 are normal subgroups of G,

ii) $H_1 \cap H_2 = \{(1,1)\}$,

iii) H_1 and H_2 generate G.

<u>Theorem 2.3.18</u> Let G be a group having two normal subgroups H_1, H_2 such that $H_1 \cap H_2 = \{1\}$ and H_1, H_2 generate G. Then $G \cong H_1 \times H_2$.

<u>Proof</u>. Claim that $\forall x \in H_1, \forall y \in H_2$ $x.y = y.x$. To prove this, let $x \in H_1$, $y \in H_2$ then $x.y.x^{-1} \in H_2$ and $yx^{-1}y^{-1} \in H_1$ (because $H_1 \trianglelefteq G$, $H_2 \trianglelefteq G$) so $(xyx^{-1})y^{-1} \in H_2$ and $x(yx^{-1}y^{-1}) \in H_1$ ie. $xyx^{-1}y^{-1} \in H_1 \cap H_2 = \{1\}$ so $xy = yx$.

Define $\phi: H_1 \times H_2 \to G$ by $\phi(h_1,h_2) = h_1.h_2$ $\forall (h_1,h_2) \in H_1 \times H_2$. Then ϕ is well-defined. We shall show that ϕ is 1-1. Let (h_1,h_2), $(h_1',h_2') \in H_1 \times H_2$ be such that $h_1h_2 = h_1'h_2'$ then $h_1^{-1}h_1' = h_2h_2'^{-1} \in H_1 \cap H_2 = \{1\}$ so $h_1 = h_1'$ and $h_2 = h_2'$ ie. $(h_1,h_2) = (h_1',h_2')$. Hence ϕ is 1-1. Next, we shall show that ϕ is onto. Because $H_1 \trianglelefteq G$, $H_2 \trianglelefteq G$ and H_1, H_2 generate G, $G = H_1 . H_2$. Let $g \in G$ then $\exists\, h_1 \in H_1$, $h_2 \in H_2$ such that $g = h_1 h_2$. So $(h_1, h_2) \in H_1 \times H_2$ and $\phi\,(h_1, h_2) = h_1 h_2 = g$ Hence ϕ is onto. Lastly, we shall show that ϕ is a homomorphism. Let $(h_1, h_2), (h_1', h_2') \in H_1 \times H_2$ then $\phi(h_1, h_2) \cdot \phi(h_1', h_2') = = (h_1 h_2)(h_1' h_2') = h_1 (h_2 h_1') h_2' = h_1 h_1' h_2 h_2' = \phi(h_1 h_1', h_2 h_2') = \phi[(h_1, h_2) \cdot (h_1', h_2')]$. Hence ϕ is a homomorphism. Therefore $G \simeq H_1 \times H_2$. #

Remark: We see that normal subgroups (congruence sets) are the factors in the direct product of groups.

Theorem 2.3.19 Let N be a normal subgroup of a group G. Then there exists a bijection between the set of subgroups of G containing N and the set of subgroups of G/N, and this bijection take maximal subgroups to maximal subgroups, normal subgroups to normal subgroups and maximal normal subgroups to maximal normal subgroups.

Proof. Let $\mathcal{A}$ = the set of subgroups of G containing N.

$\mathcal{B}$ = the set of subgroups of G/N.

For each $P \in \mathcal{A}$, let $\pi(P) = \{\pi(g) \mid g \in P\}$ where $\pi: G \to G/N$ is a natural homomorphism. Then $\pi(P) \leq G/N$. Define $\phi: \mathcal{A} \to \mathcal{B}$ by $\phi(P) = \pi(P)$ $\forall\, P \in \mathcal{A}$. Clearly ϕ is well-defined. We shall show that ϕ is 1-1. Let $P_1, P_2 \in \mathcal{A}$ be such that $\phi(P_1) = \phi(P_2)$. Let $a \in P_1$ then $\pi(a) \in \phi(P_1) = \phi(P_2)$ so $\exists\, b \in P_2$ such that $\pi(a) = \pi(b)$ therefore $ab^{-1} \in \ker \pi = N \subseteq P_2$

so $a = (ab^{-1})b \in P_2$. Hence $P_1 \subseteq P_2$. Similarly $P_2 \subseteq P_1$. So $P_1 = P_2$. Thus ϕ is 1-1. Next, we shall show that ϕ is onto. Let $Q \in \mathcal{B}$. Define $P = \{x \in G \mid [x] \in Q\}$. Then $P \in \mathcal{A}$ and $\phi(P) = \pi(P) = Q$. So ϕ is onto. Therefore ϕ is a bijection.

i) Let P be a maximal subgroup of G containing N. We must show that $\phi(P)$ is a maximal subgroup of G/N. Let L be a subgroup of G/N such that $\phi(P) \subsetneq L \subseteq G/N$. Then $P \subsetneq \phi^{-1}(L) \subseteq G$. Because P is a maximal subgroup of G, $P = \phi^{-1}(L)$ hence $\phi(P) = L$. Therefore $\phi(P)$ is a maximal subgroup of G/N. Similarly, if Q is a maximal subgroup of G/N then $\phi^{-1}(Q)$ is a maximal subgroup of G containing N.

ii) Let P be a normal subgroup of G containing N. We must show that $\phi(P) \trianglelefteq G/N$. Clearly $\phi(P) \leqslant G/N$. Let $\alpha \in G/N$ and $\beta \in \phi(P)$. Then $\exists\ a \in G$, $b \in P$ such that $\alpha = [a]$ and $\beta = [b]$ so $\alpha^{-1}\beta\alpha = [a]^{-1}[b][a] = [a^{-1}ba] = \pi(a^{-1}ba)$. Because $P \trianglelefteq G$ and $b \in P$, $a^{-1}ba \in P$ so $\alpha^{-1}\beta\alpha \in \pi(P) = \phi(P)$. Hence $\phi(P) \trianglelefteq G/N$. Let $Q \trianglelefteq G/N$. We must show that $\phi^{-1}(Q)$ is a normal subgroup of G containing N. Clearly $N \leqslant \phi^{-1}(Q) \leqslant G$. Let $g \in G$ and $a \in \phi^{-1}(Q)$ then $[a] \in Q$ and $[g] \in G/N$. Because $Q \trianglelefteq G/N$, $[g^{-1}ag] = [g]^{-1}[a][g] \in Q$ so $g^{-1}ag \in \phi^{-1}(Q)$. Hence $\phi^{-1}(Q) \trianglelefteq G$.

iii) By i and ii, we have that P is a maximal normal subgroup of G containing N iff $\phi(P)$ is a maximal normal subgroup of G/N. #

<u>Definition 2.3.20</u> Let G be a group. G is said to be <u>simple</u> iff G has no normal subgroups except {1} and G.

Corollary 2.3.21 Let N be a maximal normal subgroup of a group G. Then G/N is simple.

Corollary 2.3.22 If G is a simple group and $x, y \in G \setminus \{1\}$ then there exist $m \in \mathbb{N}$, $n_1, \ldots, n_m \in \mathbb{Z}$ and $g_1, \ldots, g_m \in G$ such that

$$y = \prod_{i=1}^{m} g_i^{-1} x^{n_i} g_i .$$

Proof. Assume G is a simple group and $x, y \in G \setminus \{1\}$. We have that $\{ \prod_{i}^{\text{finite}} g_i^{-1} x^{n_i} g_i \mid g_i \in G,\ n_i \in \mathbb{Z} \}$ is the normal subgroup of G generated by x. Because G is simple, $G = \{ \prod_{i}^{\text{finite}} g_i^{-1} x^{n_i} g_i \mid g_i \in G, n_i \in \mathbb{Z} \}$. Since $y \in G$, $\exists m \in \mathbb{N}$, $n_1, \ldots, n_m \in \mathbb{Z}$ and $g_1, \ldots, g_m \in G$ such that

$$y = \prod_{i=1}^{m} g_i^{-1} x^{n_i} g_i .$$ #

2.4 Group spaces.

In this section we shall work with left congruences on a group. But everything that we prove for left congruences can be similarly proved for right congruences also. As in Section 2.3, we shall consider the categories $\mathcal{G}$, $\mathcal{G}_o$ and $\mathcal{G}_i$.

First we shall define natural equivalent covariant functors from $\mathcal{G}$, $\mathcal{G}_o$ to $\mathcal{L}$ by using left congruences, subgroups and pointed homogeneous left group-spaces which are defined below.

Definition 2.4.1 A left congruence on a group G is an equivalence relation ρ on G such that $x \rho y$ implies $(a.x)\rho(a.y)$ for all $x,y,a \in G$.

Remarks: 1) If ρ is a left congruence on a group G then $[1]_\rho = \{a \in G | a \rho 1\}$ is a subgroup of G.

2) If S is a subgroup of a group G then $\{(a,b) \in G \times G | a^{-1}.b \in S\}$ is a left congruence on G.

Definition 2.4.2 Let G be a group and X be a nonempty set. A left action of G on X is a map $\cdot : G \times X \to X$ such that $1.x = x$ for all $x \in X$ and $(g.h).x = g.(h.x)$ for all $g,h \in G$, $x \in X$. Then $(X,\cdot)$ is said to be a left G-space.

Definition 2.4.3 Let G be a group and $(X,\cdot)$ be a left G-space. $\cdot$ is said to be transitive iff for each $x,y \in X$ there exists an element g in G such that $y = g.x$. In this case $(X,\cdot)$ is said to be a homogeneous left G-space.

Proposition 2.4.4 If ρ is a left congruence on a group G then the set G/ρ of equivalence classes of G can be made into a homogeneous left G-space.

Proof. Let ρ be a left congruence on a group G and G/ρ = the set of equivalence classes of G. Define a map $\cdot : G \times G/\rho \to G/\rho$ as follows: given $g \in G$, $\alpha \in G/\rho$ choose $a \in \alpha$ and let $g.\alpha = [g.a]$. Clearly $\forall \alpha \in G/\rho$ $1.\alpha = \alpha$ and $\forall g,h \in G$, $\alpha \in G/\rho$ $(g.h).\alpha = g.(h.\alpha)$. Hence $(G/\rho,\cdot)$ is a left G - space. Next, we shall show that $\cdot$ is transitive.

Let $\alpha,\beta \in G/\rho$, choose $a \in \alpha$, $b \in \beta$. Since $a,b \in G$, $ab^{-1} \in G$ and $\alpha = [a] = [(ab^{-1}).b] = (ab^{-1}).\beta$. Hence $(G/\rho,\cdot)$ is a homogeneous left G-space.

#

<u>Example</u> Let H be a subgroup of a group G. Define $\rho = \{(a,b) \in G \times G | a^{-1}b \in H\}$. As in the case when N is a normal subgroup of G, we can show that ρ is a left congruence on G. So $G/H \cong G/\rho$. Then $(G/H,\cdot)$ is a homogeneous left G-space.

<u>Definition 2.4.5</u> Let G be a group, (X,x) a pointed set and $\cdot$ a left action of G on X. Then $(X,\cdot,x)$ is said to be a <u>pointed left G-space.</u>

<u>Remark</u>: For each group G, each left G-space $(X,\cdot)$ and each $x \in X$, denote $\{g \in G | g.x = x\}$ by G_x. Then G_x is a subgroup of G and is called the <u>isotropy subgroup corresponding to x</u>. Hence if $(X,\cdot,x_o)$ is a pointed left G-space then G_{x_o} is a subgroup of G.

<u>Definition 2.4.6</u> Let G be a group, $(X,\cdot)$, $(Y,*)$ left G-spaces and $\phi: X \to Y$ a map. Then ϕ is said to be <u>G-equivariant</u> iff $\phi(g.u) = g * \phi(u)$ for all $g \in G$, $u \in X$.

<u>Remark</u>: If ϕ is a bijective G-equivariant map then ϕ^{-1} is also G-equivariant. We call such a map a <u>G-space isomorphism.</u>

<u>Definition 2.4.7</u> Let G be a group, $(X,\cdot,x)$ and $(Y,*,y)$ pointed left G-spaces. Say that $(X,\cdot,x)$ is <u>equivalent</u> to $(Y,*,y)$ $((X,\cdot,x) \sim (Y,*,y))$ iff there exists a G-space isomorphism $\phi:(X,x) \to (Y,y)$.

Remark: $\sim$ is an equivalence relation on the set of pointed left G-spaces.

Example Let $(X,\cdot,x)$ be a pointed homogeneous left G-space. Let $u \in X$ then there exists a $g \in G$ such that $u = g.x$. So define $\phi: X \to G/G_x$ by $\phi(u) = [g]$. Then ϕ is well-defined isomorphism such that $\phi(x) = [1]$ Hence $(X,\cdot,x) \sim (G/G_x,\cdot,[1])$.

For each group G, let $S(G)$ = the set of subgroups of G,

$L_o(G)$ = the set of left congruences on G,

$P(G)$ = the set of equivalence classes of pointed homogeneous left G-spaces.

Now we shall define natural relations on these sets making them into posets.

1.) Let $\subsetneq$ on $S(G)$ be set inclusion. Then $(S(G),\subsetneq)$ is a poset.

2.) Let $\subsetneq$ on $L_o(G)$ be set inclusion. Then $(L_o(G),\subsetneq)$ is a poset.

3.) Let $\subsetneq$ on $P(G)$ be defined as follow: given $\alpha,\beta \in P(G)$ choose $(X,\cdot,x) \in \alpha$ and $(Y,*,y) \in \beta$ say that $\alpha \subsetneq \beta$ iff there exists an onto G-equivariant map $\phi:(X,x) \to (Y,y)$. First, we shall show that $\subsetneq$ is well-defined. Let $(X,\cdot,x) \sim (X',\cdot',x')$, $(Y,*,y) \sim (Y',*',y')$ and $\exists$ an onto G-equivariant map $\phi:(X,x) \to (Y,y)$. We must show that $\exists$ an onto G-equivariant map $\phi': (X',x') \to (Y',y')$. Because $(X,.,x) \sim (X',.',x')$ and $(Y,*,y) \sim (Y',*',y')$, $\exists$ an isomorphism $\psi:(X',x') \to (X,x)$ and $\exists$ an isomorphism $\psi':(Y,y) \to (Y',y')$. Define $\phi':X' \to Y'$ by $\phi' = \psi' \circ \phi \circ \psi$. Then ϕ' is an onto G-equivariant map. Hence $\subsetneq$ is well-defined. Next we shall show that

$(P(G),\subsetneq)$ is a poset. Clearly, $\subsetneq$ is reflexive. Let $\alpha \subsetneq \beta$ and $\beta \subsetneq \alpha$. Choose $(X,\cdot,x) \in \alpha, (Y,*,y) \in \beta$. Then $\exists$ an onto G-equivariant map $\phi:(X,x) \to (Y,y)$ and $\exists$ an onto G-equivariant map $\phi':(Y,y) \to (X,x)$. We want to show that ϕ is 1-1, it suffices to show that $\phi' \circ \phi = id_X$. Let $u \in X$ then $\exists g \in G$ such that $u = g.x$ so $\phi' o \phi(u) = \phi'(g*\phi(x)) = g.(\phi' o \phi(x)) = g.(\phi'(y)) = g.x = u$ hence $\phi' o \phi = id_X$ ie. ϕ is 1-1. Therefore $(X,\cdot,x) \sim (Y,*y)$. Thus $\alpha = \beta$. Hence $\subsetneq$ is antisymmetric. Let $\alpha \subsetneq \beta$ and $\beta \subsetneq \gamma$. Choose $(X,\cdot,x) \in \alpha$, $(Y,*,y) \in \beta$ and $(Z,\Delta,z) \in \gamma$. Then $\exists$ an onto G-equivariant map $\phi:(X,x) \to (Y,y)$ and $\exists$ an onto G-equivariant map $\phi':(Y,y) \to (Z,z)$. Define $\phi'':X \to Z$ by $\phi'' = \phi' o \phi$. Then ϕ'' is an onto G-equivariant map. Hence $\alpha \subsetneq \gamma$. Then $\subsetneq$ is transitive. Therefore $(P(G),\subsetneq)$ is a poset.

<u>Theorem 2.4.8</u> For each group G, the posets $L_o(G)$ and $S(G)$ are isomorphic.

<u>Proof</u>. It is similar to the proof of Theorem 2.3.8.

<u>Theorem 2.4.9</u> For each group G, the posets $P(G)$ and $S(G)$ are isomorphic.

<u>Proof</u>. Let G be a group. Define $\phi:P(G) \to S(G)$ as follows: given $\alpha \in P(G)$ choose $(X,\cdot,x) \in \alpha$ and let $\phi(\alpha) = G_x$. First, we shall show that ϕ is well-defined. Let $(X,\cdot,x) \sim (Y,*,y)$. So $\exists$ an isomorphism $f:(X,x) \to (Y,y)$ We must show that $G_x = G_y$. Let $g \in G_x$ then $g.x = x$ so $g*f(x) = f(g.x) = f(x)$. therefore $g \in G_{f(x)} = G_y$ so $G_x \subsetneq G_y$. Let $g \in G_y$ then $g*f(x)$ so $g.x = (f^{-1} of)(g.x) = f^{-1} of(x) = x$ hence $g \in G_x$. Then $G_x = G_y$. Thus ϕ is well-defined

Next, we shall show that ϕ is 1-1. Let $(X,\cdot,x)$, $(Y,*,y)$ be pointed homogeneous left G-spaces such that $G_x = G_y$. We must show that

$(X,\cdot,x) \sim (Y,*,y)$. Given $u \in X$ $\exists g \in G$ such that $u = g.x$ therefore $g*y \in Y$. Define $f:(X,x) \to (Y,y)$ by $f(u) = g*y$. First, we shall show that f is well-defined. Let $g,g' \in G$ be such that $g.x = g'.x$ so $x = g^{-1}.g.x = g^{-1}.g'.x$ then $g^{-1}.g' \in G_x = G_y$ hence $(g^{-1}.g') * y = y$ therefore $g * y = g' * y$ ie. $f(g.x) = f(g'.x)$ hence f is well-defined. Next, we shall show that $f(g.u) = g * f(u)$ $\forall g \in G, u \in X$. Let $g \in G$, $u \in X$ then $\exists a \in G$ such that $u = a.x$ so $f(g.u) = f(g.a.x) = (g.a) * y = g * (a * y) = g * f(a.x) = g * f(u)$. Next, we shall show that f is 1-1. Let $u,u' \in X$ be such that $f(u) = f(u')$ so $\exists g,g' \in G$ such that $u = g.x$, $u = g'.x$ and $g * y = g' * y$ so $(g^{-1}g') * y = y$ therefore $g^{-1}g' \in G_y = G_x$, hence $(g^{-1}g')x = x$ so $gx = g'x$ then $u = u'$. Thus f is 1-1. Lastly, we shall show that f is onto. Let $v \in Y$ then $\exists g \in G$ such that $v = g * y$ so $g.x \in X$ and $f(g.x) = g * y = v$. Hence f is onto thus f is an isomorphism. Therefore $(X,\cdot,x) \sim (Y,*,y)$ ie. ϕ is 1-1.

Next, we shall show that ϕ is onto. Let $A \leqslant G$. Define $\rho = \{(a,b) \in G \times G | a^{-1}b \in A\}$ then $(G/A,\cdot,[1])$ is a pointed homogeneous left G-space. So $f([G/A,\cdot,[1]]) = G_{[1]} = A$. Hence ϕ is onto.

Next, we shall show that ϕ is isotone. Let $\alpha,\beta \in P(G)$ be such that $\alpha \subsetneq \beta$. Choose $(X,\cdot,x) \in \alpha$ and $(Y,*,y) \in \beta$ then $\exists$ an onto G-equivariant map $f:(X,x) \to (Y,y)$. We must show that $\phi(\alpha) \subsetneq \phi(\beta)$ ie. $G_x \subseteq G_y$. Let $g \in G_x$ so $g.x = x$ then $g * y = g * f(x) = f(g.x) = f(x) = y$ so $g \in G_y$ hence $G_x \subsetneq G_y$ ie. $\phi(\alpha) \subsetneq \phi(\beta)$. Thus ϕ is isotone.

Lastly, we shall show that ϕ^{-1} is isotone. Let $A,B \in S(G)$ be such that $A \subsetneq B$. We must show that $\phi^{-1}(A) \subsetneq \phi^{-1}(B)$. Define

$f:(G/A,[1]_A) \to (G/B,[1]_B)$ as follows: given $\alpha \in G/A$ choose $a \in \alpha$ and let $f(\alpha) = [a]_B$. Because $A \subseteq B$, f is well defined. Clearly f is onto and $f([1]_A) = [1]_B$. Let $g \in G$, $\alpha \in G/A$ choose $a \in \alpha$ then $f(g.\alpha) = f([g.a]_A) = [g.a]_B = g * [a]_B = g * f(\alpha)$. Hence $(G/A,\cdot,[1]_A) \sim (G/B,*,[1]_B)$ ie. $\phi^{-1}(A) \subseteq \phi^{-1}(B)$. Therefore ϕ^{-1} is isotone. Hence S(G) is isomorphic to P(G). #

Corollary 2.4.10 For each group G, the posets $L_o(G)$ and P(G) are isomorphic.

Remark: Fix a group G, S(G) is a lattice so $L_o(G)$ and P(G) are lattices also.

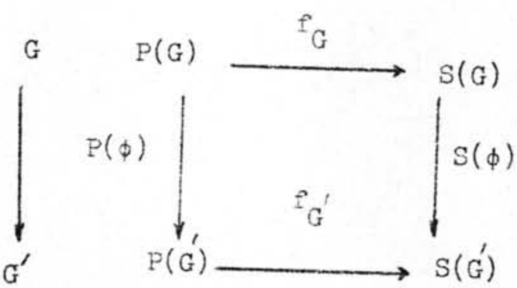
Now we shall define covariant functors from $\mathcal{G}$ to $\mathcal{L}$.

1) Let G, G' be in Ob $\mathcal{G}$ and $\phi: G \to G'$ a group-homomorphism. Then S(G), S(G') are in Ob $\mathcal{L}$. Define $S(\phi): S(G) \to S(G')$ by $S(\phi)(H) = \phi(H)$ for all $H \in S(G)$. The proof that S is covariant functor is similar to the proof that N is a covariant functor in Section 2.3.

2) Let G, G' be in Ob $\mathcal{G}$ and $\phi: G \to G'$ a group homomorphism. Then P(G), P(G') are in Ob $\mathcal{L}$. Define $P(\phi): P(G) \to P(G')$ as follows: given $\alpha \in P(G)$ choose $(X,\cdot,x) \in \alpha$ and let $P(\phi)(\alpha) = [(G'/\phi(G_x),\cdot,[1])]$. First, we shall show that $P(\phi)$ is well-defined. Let $(X,\cdot,x) \sim (Y,*,y)$ then $G_x = G_y$ therefore $\phi(G_x) = \phi(G_y)$. Then $(G/\phi(G_x),\cdot,[1]) = (G/\phi(G_y),\cdot,[1])$. Hence $P(\phi)$ is well-defined. Next, we shall show that $P(\phi)$ is isotone. Let $\alpha, \beta \in P(G)$ be such that $\alpha \subseteq \beta$. Choose $(X,\cdot,x) \in \alpha$ and $(Y,*,y) \in \beta$ then $\exists$ an onto G-equivariant map $\psi:(X,x) \to (Y,y)$. Because ψ is G-equivariant

and $y = \psi(x)$, $G_x \subseteq G_y$. Hence $P(\phi)(\alpha) \subseteq P(\phi)(\beta)$. Therefore $P(\phi)$ is isotone. Next, we shall show that P is a covariant functor from $\mathcal{G}$ to $\mathcal{L}$. Clearly $P(id_G) = id_{P(G)}$ $\forall G$ in Ob $\mathcal{G}$. Let $\phi: G \to G'$ and $\phi': G' \to G''$ be group homomorphisms. Then $\phi' o \phi: G \to G''$. Let $\alpha \in P(G)$ choose $(X,\cdot,x) \in \alpha$ then $(P(\phi') o P(\phi))(\alpha) = P(\phi')\ [(G'/\phi(G_x),\cdot,[1])] = [(G''/\phi'(G'_{[1]}),\cdot,[1'])] = [(G''/\phi'(\phi(G_x)),\cdot,[1'])] = (P(\phi' o \phi))(\alpha)$. Hence $P(\phi') o P(\phi) = P(\phi' o \phi)$. Therefore P is a covariant functor from $\mathcal{G}$ to $\mathcal{L}$.

Now we shall show that P and S are naturally equivalent. For each G in Ob $\mathcal{G}$, define $f_G: P(G) \to S(G)$ to be the map in Theorem 2.4.9. Then f_G is an isomorphism. Claim that f is a natural equivalemce from P to S. To prove this, let G, G' be in Ob $\mathcal{G}$ and $\phi: G \to G'$ be a group homomorphism so we have $f_G, f_{G'}$ and the following diagram



We must show that $S(\phi)\ o\ f_G = f_{G'}\ o P(\phi)$. Let $\alpha \in P(G)$ choose $(X,\cdot,x) \in \alpha$ then $(f_{G'} o\ P(\phi))(\alpha) = f_{G'}\ [(G'/\phi(G_x),\cdot,[1])] = G'_{[1]} = \phi(G_x) = S(\phi)(G_x) = (S(\phi)\ o\ f_G)(\alpha)$. Hence $S(\phi)\ o\ f_G = f_{G'}\ o\ P(\phi)$. Therefore f is a natural equivalence from P to S.

Now we shall define covariant functors from $\mathcal{G}_o$ to $\mathcal{L}$.

1) Define $S_o: \mathcal{G}_o \to \mathcal{L}$ by $S_o = S_{|\mathcal{G}_o}$. Then S_o is a functor.

2) Define $P_o : \mathcal{G}_o \to \mathcal{L}$ by $P_o = S|_{\mathcal{G}_o}$. Then P_o is a functor.

3) Let G, G' be in Ob $\mathcal{G}_o$ and $\phi : G \to G'$ be an onto group homomorphism. Then $L_o(G), L_o(G')$ are in Ob $\mathcal{L}$. Define $L_o(\phi) : L_o(G) \to L_o(G')$ by $L_o(\phi)(\rho) = (\phi \times \phi)(\rho)$ for all $\rho \in L_o(G)$. Then the proof that L_o is a covariant functor is similar to the proof that C' is a covariant functor in Section 2.3.

Now we shall show that S_o, L_o, P_o are naturally equivalent. The proof that S_o and L_o are naturally equivalent is similar to the proof that N' and C' are naturally equivalent in Section 2.3 The proof that P_o and S_o are naturally equivalent is similar to the proof that P and S are naturally equivalent in this section. Hence S_o, L_o, P_o are naturally equivalent.

Next we shall define naturally equivalent covariant functors from $\mathcal{G}$, $\mathcal{G}_o$ to $\mathcal{L}$.

<u>Definition 2.4.11</u> Let G be a group and H_1, H_2 subgroups of G. Say that H_1 is <u>strongly equivalent</u> to H_2 ($H_1 \simeq H_2$) iff there exists a $g \in G$ such that $g^{-1}H_1 g = H_2$.

<u>Remark:</u> $\simeq$ is an equivalence relation on the set of subgroups of G.

<u>Definition 2.4.12</u> Let G be a group and ρ a left congruence on G. Then for each $a \in G$, let $\rho.a = \{(x.a, y.a) | x \rho y\}$.

<u>Remark:</u> $\rho.a$ is a left congruence on G for all $a \in G$ where ρ is a left congruence on G.

<u>Definition 2.4.13</u> Let G be a group and ρ_1, ρ_2 be left congruences on G. Say that ρ_1 is <u>strongly equivalent</u> to ρ_2 $(\rho_1 \simeq \rho_2)$ iff there exists a $g \in G$ such that $\rho_1 \cdot g = \rho_2$.

<u>Remark:</u> $\simeq$ is an equivalence relation on the set of left congruences on G.

<u>Definition 2.4.14</u> Let G be a group and $(X,\cdot)$, $(Y,*)$ be homogeneous left G-spaces. Say that $(X,\cdot)$ is <u>equivalent</u> to $(Y,*)$ $((X,\cdot) \simeq (Y,*))$ iff there exists an isomorphism $\phi : X \to Y$.

<u>Remarks:</u> 1) $\simeq$ is an equivalence relation on the set of homogeneous left G-spaces.

2) For each homogeneous left G-space $(X,\cdot)$, $(X,\cdot) \simeq (G/G_x,\cdot)$ for all $x \in X$.

For each group G, let $S'(G)$ = the set of equivalence classes of subgroups of G under $\simeq$,

$L'_o(G)$ = the set of equivalence classes of left congruences on G under $\simeq$,

$H'(G)$ = the set of equivalence classes of homogeneous left G-spaces under $\simeq$.

Now we shall define binary relations on these sets making them into quasi-ordered sets.

1) Let $\leqslant$ on $S'(G)$ be defined as follows: given $\alpha, \beta \in S'(G)$ say that $\alpha \leqslant \beta$ iff there exist $H_1 \in \alpha$ and $H_2 \in \beta$ such that $H_1 \subsetneq H_2$. Then clearly

$\leqslant$ is well-defined and $(S'(G),\leqslant)$ is a quasi-ordered set.

2) Let $\leqslant$ on $L_o'(G)$ be defined as follows: given $\alpha,\beta \in L_o'(G)$ say that $\alpha \leqslant \beta$ iff there exist $\rho_1 \in \alpha$ and $\rho_2 \in \beta$ such that $\rho_1 \subsetneq \rho_2$. Then clearly $\leqslant$ is well-defined and $(L_o'(G),\leqslant)$ is a quasi-ordered set.

3) Let $\leqslant$ on $H'(G)$ be defined as follows: given $\alpha,\beta \in H'(G)$ say that $\alpha \leqslant \beta$ iff there exist $(X,\cdot) \in \alpha$, $(Y,*) \in \beta$and an onto G-equivariant map $\phi:X \to Y$. Clearly $\leqslant$ is well-defined. Then $(H'(G),\leqslant)$ is a quasi-ordered set.

<u>Theorem 2.4.15</u> For each group G, the quasi-ordered sets $S'(G)$ and $L_o'(G)$ are isomorphic.

<u>Proof</u>. Let G be a group. Define $f:S'(G) \to L_o'(G)$ as follows: given $\alpha \in S(G)$ choose $H \in \alpha$ and let $f(\alpha) = [\rho]$ where $\rho = \{(ab,) \in G \times G \mid a^{-1}b \in H\}$. First, we shall show that f is well-defined. Let $H_1 \leqslant G, H_2 \leqslant G$ be such that $H_1 \simeq H_2$ so $\exists\, g \in G$ such that $g^{-1}H_1g= H_2$. We want to show that $\rho_1 \cdot g = \rho_2$. Let $(a,b) \in \rho_1$ then $a^{-1}b \in H_1$ so $(ag)^{-1}(bg) = g^{-1}(a^{-1}b)g \in g^{-1}H_1g = H_2$. Hence $(ag,bg) \in \rho_2$ ie. $\rho_1 \cdot g \subsetneq \rho_2$. Let $(a,b) \in \rho_2$ then $a^{-1}b \in H_2 = g^{-1}H_1g$ so $(ag^{-1})^{-1}(bg^{-1}) = g(a^{-1}b)g^{-1} \in H_1$ hence $(ag^{-1},bg^{-1}) \in \rho_1$. Because $(a,b)=(ag^{-1}g,bg^{-1}.g) \in \rho_1 \cdot g$, $\rho_2 \subsetneq \rho_1 \cdot g$. Hence $\rho_1 g = \rho_2$. Thus $\rho_1 \simeq \rho_2$ so f is well-defined.

Next, we shall show that f is 1-1. Let $\alpha,\beta \in S'(G)$ be such that $f(\alpha) = f(\beta)$. Choose $H_1 \in \alpha, H_2 \in \beta$ then $\rho_1 \simeq \rho_2$ ie. $\exists\, g \in G$ such that $\rho_1 \cdot g = \rho_2$. We want to show $g^{-1}H_1g = H_2$. Let $a \in H_1$ then $(1,a) \in \rho_1$ so $(g,a.g) \in \rho_1 \cdot g = \rho_2$ then $g^{-1}ag \in H_2$. Hence $g^{-1}H_1g \subsetneq H_2$. Let $b \in H_2$

then $(1,b) \in \rho_2 = \rho_1 . g$ so $(g^{-1}, bg^{-1}) \in \rho_1$ and therefore $gbg^{-1} \in H_1$. Since $b = g^{-1}(gbg^{-1})g$, $b \in g^{-1}H_1g$. Hence $H_2 \subseteq g^{-1}H_1g$. Therefore $H_2 = g^{-1}H_1g$. Thus $H_1 \simeq H_2$ ie. $\alpha = \beta$. Then f is 1-1.

Next we shall show that f is onto. Let $\alpha \in L_o'(G)$ choose $\rho \in \alpha$ then $[1]_\rho = \{a \in G | a \rho 1\} \leqslant G$. So $f([[1]_\rho]) = [\{(a,b) \in G \times G | a^{-1}b \in [1]_\rho\}] = [\{(a,b) \in G \times G | a^{-1}b \rho 1\}] = [\rho] = \alpha$. Hence f is onto.

Next we shall show that f is isotone. Let $\alpha, \beta \in S(G)$ be such that $\alpha \leqslant \beta$. Then $\exists H_1 \in \alpha, H_2 \in \beta$ such that $H_1 \subseteq H_2$. We want to show that $\rho_1 \subseteq \rho_2$. Let $(a,b) \in \rho_1$ then $a^{-1}b \in H_1 \subseteq H_2$ so $(a,b) \in \rho_2$. Hence $\rho_1 \subseteq \rho_2$ ie. $f(\alpha) \leqslant f(\beta)$ Therefore f is isotone.

Lastly we shall show that f^{-1} is isotone. Let α, β $L_o(G)$ be such that $\alpha \leqslant \beta$. Then $\exists \rho_1 \in \alpha, \rho_2 \in \beta$ such that $\rho_1 \subseteq \rho_2$. Then clearly $[1]_{\rho_1} \subseteq [1]_{\rho_2}$. Hence $f^{-1}(\rho_1) \leqslant f^{-1}(\rho_2)$. Therefore f^{-1} is isotone. Hence $S'(G)$ is isomorphic to $L_o'(G)$. #

<u>Theorem 2.4.16</u> For each group G, the quasi-ordered sets $S'(G)$ and $H'(G)$ are isomorphic.

<u>Proof</u>. Let G be a group. Define $f:H'(G) \to S'(G)$ as follows: given $\alpha \in H'(G)$ choose $(X,\cdot) \in \alpha$ and choose $x \in X$ then let $f(\alpha) = [G_x]$. First we shall show that f is well-defined. Let $(X,\cdot) \simeq (Y,*)$ then $\exists$ an isomorphism $\phi : X \to Y$. We want to show that $G_x \simeq G_{\phi(x)}$. Let $a \in G_x$ then $a * \phi(x) = \phi(a.x) = \phi(x)$ so $a \in G_{\phi(x)}$ Hence $G_x \subseteq G_{\phi(x)}$. Let $b \in G_{\phi(x)}$ then $b.x = (\phi^{-1} o \phi)(b.x) = \phi^{-1}(\phi(b.x)) = \phi^{-1}(b * \phi(x)) = \phi^{-1}(\phi(x)) = x$ so

$b \in G_x$. Hence $G_{\phi(x)} \subsetneq G_x$. Therefore $G_x = G_{\phi(x)}$. Because $G_{\phi(x)} \simeq G_y$ $\forall y \in Y$, $G_x \simeq G_y$ $\forall y \in Y$. Hence f is well-defined.

Next we shall show that f is 1-1. Let $\alpha, \beta \in H'(G)$ be such that $f(\alpha) = f(\beta)$. Choose $(X,\cdot) \in \alpha$, $(Y,*) \in \beta$ and $x \in X$, $y \in Y$ then $G_x \simeq G_y$ ie. $\exists g \in G$ such that $g^{-1}G_x g = G_y$. Define $\phi: X \to Y$ as follows: given $u \in X$ $\exists h \in G$ such that $u = h.x$ so let $\phi(u) = (h.g) * y$. We must show that ϕ is an isomorphism. Let $h, h' \in G$ be such that $h.x = h'.x$. then $h^{-1}h' \in G_x$ so $(hg)^{-1}.(h'g) = g^{-1}h^{-1}h'g \in G_y$ ie. $(hg)^{-1}(h'g) * y = y$. Hence $(hg) * y = (h'g) * y$. Thus ϕ is well-defined. Let $u, u' \in X$ be such that $\phi(u) = \phi(u')$. Then $\exists$ $h, h' \in G$ such that $u = h.x$ and $u' = h'.x$ so $(h'g) * y = (hg) * y$. Therefore $(hg)^{-1}(h'g) \in G_y$ so $h^{-1}h' \in G_x$. Hence $hx = h'x$ ie. $u = u'$. Therefore ϕ is 1-1. Let $u \in X$, $a \in G$ then $\exists$ $h \in G$ such that $u = h.x$ so $\phi(a.u) = (ah.x) = (a.hg) * y = a * ((hg) * y) = a * \phi(h.x) = a * \phi(u)$. Hence ϕ is G-equivariant. Let $v \in Y$ then $\exists a \in G$ such that $v = a * y$ then $\phi(ag^{-1}x) = (a.g^{-1}.g) * y = a * y = v$. Hence ϕ is onto. Thus ϕ is an isomorphism ie. $(X,\cdot) \simeq (Y,*)$. Then f is 1-1.

Next we shall show that f is onto. Let $\alpha \in S'(G)$ choose $A \in \alpha$ then $(G/A,\cdot)$ is a homogeneous left G-space. So $f([G/A,\cdot]) = [\{a \in G | a.[1] = [1]\}] = [A]$. Hence f is onto.

Next we shall show that f is isotone. Let $\alpha, \beta \in H'(G)$ be such that $\alpha \leqslant \beta$. Then $\exists (X,\cdot) \in \alpha$, $(Y,*) \in \beta$ and an onto G-equivariant map $\phi: X \to Y$. Choose $x \in X$. Because ϕ is G-equivariant, $G_x \subseteq G_{\phi(x)}$. Hence $f(\alpha) \leqslant f(\beta)$ Thus f is isotone.

Lastly we shall show that f^{-1} is isotone. Let $\alpha,\beta \in S'(G)$ be such that $\alpha \leq \beta$. Then $\exists H_1 \in \alpha$, $H_2 \in \beta$ such that $H_1 \subseteq H_2$. Define $\phi: G/H_1 \to G/H_2$ as follows: given $\gamma \in G/H_1$ choose $a \in \gamma$ and let $\phi(\gamma) = [a]_2$. Because $H_1 \subseteq H_2$, ϕ is well-defined. Clearly ϕ is onto, and $\phi(g.\gamma) = [g.a]_2 = g.[a]_2 = g.\phi(\gamma)$ $\forall g \in G, \gamma \in G/H_1$. Hence ϕ is an onto G-equivariant map. Therefore $f^{-1}(\alpha) \leq f^{-1}(\beta)$. Thus f^{-1} is isotone. Then $H'(G)$ is isomorphic to $S'(G)$. #

Corollary 2.4.17 For each group G, the quasi-ordered sets $L_o'(G)$ and $H'(G)$ are isomorphic.

Now we shall define covariant functors from $\mathcal{G}$ to $\mathcal{Q}$.

1) Let G, G' be in Ob $\mathcal{G}$ and $\phi: G \to G'$ be a group homomorphism. Then $S'(G)$, $S'(G')$ are in Ob $\mathcal{Q}$. Define $S'(\phi): S'(G) \to S'(G')$ as follows: given $\alpha \in S'(G)$ choose $H \in \alpha$ and let $(S'(\phi))(\alpha) = [\phi(H)]$. First we shall show that $S'(\phi)$ is well-defined. Let $H_1 \simeq H_2$ then $\exists g \in G$ such that $g^{-1}H_1g = H_2$. Because ϕ is a homomorphism, $(\phi(g))^{-1}.(\phi(H_1)).(\phi(g)) = \phi(g^{-1}H_1g) = \phi(H_2)$ hence $\phi(H_1) \simeq \phi(H_2)$. Therefore $S(\phi)$ is well-defined. Then the proof that S' is a covariant functor is similar to the proof that N^* is a covariant functor in Section 2.3.

2) Let G, G' be in Ob $\mathcal{G}$ and $\phi: G \to G'$ be a group-homomorphism. Then $H'(G)$, $H'(G')$ are in Ob $\mathcal{Q}$. Define $H'(\phi): H'(G) \to H'(G')$ as follows: given $a \in H'(G)$ choose $(X,\cdot) \in \alpha$ and choose $x \in X$ then let $(H'(\phi))(\alpha) = [(G/\phi(G_x),\cdot)]$. First we shall show that $H'(\phi)$ is well-defined. Let

$(X,\cdot) \simeq (Y,*)$. Choose $x \in X, y \in Y$. Then $G_x \simeq G_y$ so $\phi(G_x) \simeq \phi(G_y)$. Hence $(G/\phi(G_x),\cdot) \simeq (G/\phi(G_y),*)$. Hence $H'(\phi)$ is well-defined. Next we shall show that $H'(\phi)$ is isotone. Let $\alpha,\beta \in H'(G)$ be such that $\alpha \leqslant \beta$ Then $\exists (X,\cdot) \in \alpha$, $(Y,*) \in \beta$ and an onto G-equivariant map $\psi : X \to Y$. Choose $x \in X$ so $\psi(x) \in Y$. Because ψ is G-equivariant, $G_x \subsetneq G_{\psi(x)}$. Then $\phi(G_x) \subsetneq \phi(G_{\psi(x)})$. So $[(G'/\phi(G_x),\cdot')] \leqslant [(G'/\phi(G_{\psi(x)}),*')]$. Hence $(H'(\phi))(\alpha) \leqslant (H'(\phi))(\beta)$. Thus $H'(\phi)$ is isotone. Next we shall show that H is a covariant functor from $\mathcal{G}$ to $\mathcal{Q}$. Clearly $H'(id_G) = id_{H'(G)}$ $\forall$ G in Ob $\mathcal{G}$. Let $\phi : G \to G'$ and $\phi' : G' \to G''$ be group homomorphisms. Then $\phi' o \phi : G \to G''$. Let $\alpha \in H'(G)$, choose $(X,\cdot) \in \alpha$ and $x \in X$ then $(H'(\phi') o H'(\phi))(\alpha) =$ $(H'(\phi'))[(G'/\phi(G_x),\cdot)] = [(G''/\phi'(G_{[1]}),\cdot)] = [(G''/\phi'(\phi(G_x)),\cdot)] = (H'(\phi' o \phi))(\alpha)$. Hence $H'(\phi') o H'(\phi) = H'(\phi' o \phi)$. Therefore H' is a covariant functor from $\mathcal{G}$ to $\mathcal{Q}$.

Now we shall show that H' and S' are naturally equivalent. For each $G \in$ Ob $\mathcal{G}$, define $f_G : H'(G) \to S'(G)$ to be the map in Theorem 2.4.16. Then f_G is an isomorphism. Claim that f is a natural equivalence from H' to S'. To prove this, let $G, G' \in$ Ob $\mathcal{G}$ and $\phi : G \to G'$ be a group homomorphism. So we have f_G, $f_{G'}$ and the following diagram

$$\begin{array}{cccc} G & H'(G) & \xrightarrow{\ f_G\ } & S'(G) \\ \phi \downarrow & H'(\phi) \downarrow & & \downarrow S'(\phi) \\ G' & H'(G') & \xrightarrow{\ f_{G'}\ } & S'(G') \end{array}$$

We must show that $S'(\phi) o f_G = f_{G'} o H'(\phi)$. Let $\alpha \in H'(G)$ choose $(X,\cdot) \in \alpha$ and

$x \in X$ then $(f_{G'} \circ H(\phi))(\alpha) = f_{G'}[(G'/\phi(G_x), \cdot')] = [G'_{[1']}] = [\phi(G_x)] = S'(\phi)[G_x] = (S'(\phi) \circ f_G)(\alpha)$. Hence $S'(\phi) \circ f_G = f_{G'} \circ H'(\phi)$. Therefore f is a natural equivalence from H' to S'.

Now we shall define covariant functors from $\mathcal{G}_o$ to $\mathcal{Q}$.

1) Define $S'_o : \mathcal{G}_o \to \mathcal{Q}$ by $S'_o = S'|_{\mathcal{G}_o}$. Then S'_o is a functor.

2) Define $H'_o : \mathcal{G}_o \to \mathcal{Q}$ by $H'_o = H'|_{\mathcal{G}_o}$. Then H'_o is a functor.

3) Let G, G' be in Ob $\mathcal{G}_o$ and $\phi : G \to G'$ be an onto group-homomorphism. Then $L'_o(G)$, $L'_o(G')$ are in Ob $\mathcal{Q}$. Define $L'_o(\phi) : L'_o(G) \to L'_o(G')$ as follows: given $\alpha \in L'_o(G)$, choose $\rho \in \alpha$ and let $(L'_o(\phi))(\alpha) = [(\phi \times \phi)(\rho)]$. First, we shall show that $L'_o(\phi)$ is well-defined. Let $\rho_1 \simeq \rho_2$ then $\exists\, g \in G$ such that $\rho_1 . g = \rho_2$. Because ϕ is a homomorphism, $(\phi \times \phi)(\rho_1) . (g) = (\phi \times \phi)(\rho_2)$ ie. $(\phi \times \phi)(\rho_1) \simeq (\phi \times \phi)(\rho_2)$. Hence $L'_o(\phi)$ is well-defined. The proof that L'_o is a covariant functor from $\mathcal{G}_o$ to $\mathcal{Q}$ is similar to the proof that C^* is a covariant functor from $\mathcal{G}_i$ to $\mathcal{Q}$.

Now we shall show that S'_o, L'_o, H'_o are naturally equivalent. The proof that S'_o and L'_o are naturally equivalent is similar to the proof that N^* and C^* are naturally equivalent in Section 2.3. The proof that H'_o and S'_o are naturally equivalent is similar to the proof that H and S are naturally equivalent in this section. Hence S'_o, L'_o, H'_o are naturally equivalent.

Next we shall define naturally equivalent covariant functors from $\mathcal{G}_i$ to $\mathcal{Q}$.

Definition 2.4.18 Let G be a group and H_1, H_2 subgroups of G. Say that H_1 is weakly equivalent to H_2 ($H_1 \sim H_2$) iff there exists an automorphism $\phi: G \to G$ such that $\phi(H_1) = H_2$.

Remarks: 1) $\sim$ is an equivalence relation on the set of subgroups of G.

2) $H_1 \simeq H_2$ implies that $H_1 \sim H_2$.

Proof. 2) Let $H_1 \simeq H_2$ then $\exists g \in G$ such that $g^{-1}H_1 g = H_2$. Define $\phi: G \to G$ by $\phi(a) = g^{-1}.a.g$. Then ϕ is an automorphism such that $\phi(H_1) = H_2$. Hence $H_1 \sim H_2$. #

Definition 2.4.19 Let G a group and ρ_1, ρ_2 left congruences on G. Say that ρ_1 is weakly equivalent to $\rho_2 (\rho_1 \sim \rho_2)$ iff there exists an automorphism $\phi: G \to G$ such that $(\phi \times \phi)(\rho_1) = \rho_2$.

Remarks: 1) $\sim$ is an equivalence relation on the set of left congruences on G.

2) $\rho_1 \simeq \rho_2$ implies that $\rho_1 \sim \rho_2$.

Proof. 2) Let $\rho_1 \simeq \rho_2$ then $\exists g \in G$ such that $\rho_1 g = \rho_2$. Define $\phi: G \to G$ by $\phi(a) = g^{-1}.a.g$. Then ϕ is an isomorphism. Let $(a,b) \in \rho_1$ then $(\phi \times \phi)(a,b) = (g^{-1}.a.g, g^{-1}.b.g)$. Since $(g^{-1}.a, g^{-1}.b) \in \rho_1$, $(g^{-1}.a.g, g^{-1}.b.g) \in \rho_1.g = \rho_2$ so $(\phi \times \phi)(\rho_1) \subseteq \rho_2$. Let $(x,y) \in \rho_2$ then $(gx, gy) \in \rho_2$ so $(gxg^{-1}, gyg^{-1}) \in \rho_1$. So $(x,y) = (g^{-1}g\, x\, g^{-1}g\, ,\, g^{-1}g\, y\, g^{-1}g) = (\phi \times \phi)(gxg^{-1}, gyg^{-1}) \in (\phi \times \phi)(\rho_1)$, hence $\rho_2 \subseteq (\phi \times \phi)(\rho_1)$. Thus $(\phi \times \phi)(\rho_1) = \rho_2$. Therefore $\rho_1 \sim \rho_2$. #

<u>Definition 2.4.20</u> Let G be a group and $(X,\cdot),(Y,*)$ homogeneous left G-spaces. Say that $(X,\cdot)$ is <u>weakly equivalent</u> to $(Y,*)$ ($(X,\cdot) \sim (Y,*)$) iff there exist an automorphism $\psi: G \to G$ and a 1-1 onto map $\phi: X \to Y$ such that $\phi(g.u) = \psi(g) * \phi(u)$ for all $g \in G$, $u \in X$.

<u>Remarks</u>: 1) $\sim$ is an equivalence relation on the set of homogeneous left G-spaces.

2) $(X,\cdot) \simeq (Y,*)$ implies $(X,\cdot) \sim (Y,*)$.

For each group G, let $S_i(G)$ = the set of equivalence classes of subgroups of G under $\sim$,

$L_i(G)$ = the set of equivalence classes of left congruences on G under $\sim$,

$H_i(G)$ = the set of equivalence classes of homogeneous left G - spaces under $\sim$.

Now we shall define binary relations on these sets making them into quasi-ordered sets.

1) Let $\leqslant$ on $S_i(G)$ be defined as follows : given $\alpha,\beta \in S_i(G)$ say that $\alpha \leqslant \beta$ iff $\exists H_1 \in \alpha$, $H_2 \in \beta$ such that $H_1 \subsetneq H_2$. Clearly $\leqslant$ is well-defined. The proof that $(S_i(G),\leqslant)$ is a quasi-ordered set is similar to the proof that $(N^*(G),\leqslant)$ is a quasi-ordered set.

2) Let $<$ on $L_i(G)$ be defined as follows: given $\alpha,\beta \in L_i(G)$ say that $\alpha \leqslant \beta$ iff $\exists \rho_1 \in \alpha$, $\rho_2 \in \beta$ such that $\rho_1 \subsetneq \rho_2$. Clearly $\leqslant$ is well-defined. The proof that $(L_i(G),\leqslant)$ is a quasi-ordered set is similar

to the proof that $(C^*(G),\leqslant)$ is a quasi-ordered set.

3) Let $\leqslant$ on $H_i(G)$ be defined as follows: given $\alpha,\beta \in H_i(G)$ say that $\alpha \leqslant \beta$ iff $\exists (X,\cdot) \in \alpha$, $(Y,*) \in \beta$ an onto map $\phi : X \to Y$ and an automorphism $\psi : G \to G$ such that $\phi(g.x) = \psi(g) * \phi(x)$ for all $g \in G$, $x \in X$. Clearly $\leqslant$ is well - defined. Then $(H_i(G),\leqslant)$ is a quasi-ordered set.

__Theorem 2.4.21__ For each group G, the quasi-ordered sets $S_i(G)$ and $L_i(G)$ are isomorphic.

__Proof__. It is similar to Theorem 2.3.16.

__Theorem 2.4.22__. For each group G, the quasi-ordered set $S_i(G)$ and $H_i(G)$ are isomorphic.

__Proof__. Let G be a group. Define $f : H_i(G) \to S_i(G)$ as follows:given $\alpha \in H_i(G)$ choose $(X,\cdot) \in \alpha$ and $x \in X$ and then let $f(\alpha) = [G_x]$. First, we shall show that f is well-defined. Let $(X,\cdot) \sim (Y,*)$. Then $\exists$ an automorphism $\psi : G \to G$ and 1-1 onto $\phi : X \to Y$ such that $\phi(g.x) = \psi(g) * \phi(x)$ $\forall x \in X$ $g \in G$. Let $x \in X$. We shall show that $G_x \sim G_{\phi(x)}$. We have that ψ is an automorphism and we shall show $\psi(G_x) = G_{\phi(x)}$. Let $a \in G_x$ so $\phi(x) = \phi(a.x) = \psi(a) * \phi(x)$ so $\psi(a) \in G_{\phi(x)}$. Therefore $\psi(G_x) \subseteq G_{\phi(x)}$. Let $b \in G_{\phi(x)}$ so $b * \phi(x) = \phi(x)$. Because ψ is onto, $\exists a \in G$ such that $b = \psi(a)$. Then $\phi(x) = b*\phi(x) = \psi(a) * \phi(x) = \phi(a.x)$. Since ϕ is 1-1, $x = a.x$ ie. $a \in G_x$. So $b = \psi(a) \in \psi(G_x)$. Hence $G_{\phi(x)} \subseteq \psi(G_x)$. Therefore $G_{\phi(x)} = \psi(G_x)$. Then $G_x \sim G_{\phi(x)}$. Because $G_{\phi(x)} \sim G_y$ $\forall y \in G$, $G_x \sim G_y$ $\forall y \in G$. Hence f is well - defined.

Next we shall show that f is 1-1. Let $\alpha,\beta \in H_1(G)$ be such that $f(\alpha) = f(\beta)$. Choose $(X,\cdot) \in \alpha$, $(Y,*) \in \beta$, $x \in X$ and $y \in Y$. Then $G_x \sim G_y$. So $\exists$ an automorphism $\psi:G \to G$ such that $\psi(G_x) = G_y$. For each $u \in X$, $\exists\, h \in G$ such that $u = h.x$ so $\psi(h) * y \in Y$. Define $\phi:X \to Y$ by $\phi(u) = \psi(h) * y$. Because $\psi(G_x) \subseteq G_y$, ϕ is well-defined. Since $G_y \subseteq \psi(G_x)$, ϕ is 1-1. Next we shall show that ψ is onto. Let $v \in Y$ then $\exists\, h' \in G$ such that $v = h' * y$ so $\exists h \in G$ such that $h' = \psi(h)$ therefore $h.x \in X$ and $\phi(h.x) = \psi(h) * y = h' * y = v$. Hence ϕ is onto. Let $g \in G$, $u \in X$ then $\exists\, h \in G$ such that $u = h.x$ then $\phi(g.u) = \phi(g.h.x) = \psi(gh) * y = (\psi(g).\psi(h)) * y = \psi(g) * (\psi(h) * y) = \psi(g) * \phi(h.x) = \psi(g) * \phi(u)$. Hence $(X,\cdot) \sim (Y,*)$. Thus f is 1-1.

Next we shall show that f is onto. It is similar to a part of the proof of Theorem 2.4.16.

Next we shall show that f is isotone. Let $\alpha,\beta \in H_i(G)$ be such that $\alpha \leq \beta$. then $\exists (X,\cdot) \in \alpha$, $(Y,*) \in \beta$ an onto map $\phi:X \to Y$ and an automorphism $\psi:G \to G$ such that $\phi(g.u) = \psi(g) * \phi(u)$ $\forall g \in G$, $u \in X$. Similar to the above proof, $\psi(G_x) \subseteq G_{\phi(x)}$. Because $\psi(G_x) \sim G_x$, $[G_x] \leq [G_{\phi(x)}]$ ie. $f(\alpha) \leq f(\beta)$. Hence f is isotone.

Lastly we shall show that f^{-1} is isotone. Let $\alpha,\beta \in S_i(G)$ be such that $\alpha \leq \beta$ then $\exists\, H_1 \in \alpha$, $H_2 \in \beta$ such that $H_1 \subseteq H_2$. We want to show that $[(G/H_1,\cdot)] \subseteq [(G/H_2,*)]$. Define $\phi:G/H_1 \to G/H_2$ as follows: given $\gamma \in G/H_1$ choose $a \in \gamma$ and then let $\phi(\gamma) = [a]_2$. Then ϕ is an onto map such that $\phi(g.\gamma) = id_G(g) * \phi(\gamma)$ $\forall g \in G$, $\gamma \in G/H_1$. Hence $f^{-1}(\alpha) \leq f^{-1}(\beta)$. Thus f^{-1} is isotone. Therefore $H_i(G)$ is isomorphic to $S_i(G)$. #

Corollary 2.4.23 For each group G, the quasi-ordered sets $L_i(G)$ and $H_i(G)$ are isomorphic.

Now we shall define covariant functors from $\mathcal{G}_i$ to $\mathcal{Q}$.

1) Let G, G' be in Ob $\mathcal{G}_i$ and $\phi: G \to G'$ a group-isomorphism. Then $S_i(G)$, $S_i(G')$ are in Ob $\mathcal{Q}$. Define $S_i(\phi): S_i(G) \to S_i(G')$ as follows: given $\alpha \in S_i(G)$ choose $H \in \alpha$ and let $(S_i(\phi))(\alpha) = [\phi(H)]$. The proof that S_i is a covariant functor is similar to the proof that N^* is a covariant functor in Section 2.3.

2) Let G, G' be in Ob $\mathcal{G}_i$ and $\phi: G \to G'$ a group-isomorphism. Then $L_i(G), L_i(G')$ are in Ob $\mathcal{Q}$. Define $L_i(\phi): L_i(G) \to L_i(G')$ as follows: given $\alpha \in L_i(G)$ choose $\rho \in \alpha$ and let $(L_i(\phi))(\alpha) = [(\phi \times \phi)(\rho_1)]$. The proof that L_i is a covariant functor is similar to the proof that C^* is a covariant functor in Section 2.3.

3) Let G, G' be in Ob $\mathcal{G}_i$ and $\phi: G \to G'$ a group-isomorphism. Then $H_i(G)$, $H_i(G')$ are in Ob $\mathcal{Q}$. Define $H_i(\phi): H_i(G) \to H_i(G')$ as follows: given $\alpha \in H_i(G)$ choose $(X, \cdot) \in \alpha$ and $x \in X$ then let $(H_i(\phi))(\alpha) = [(G'/\phi(G_x), \cdot)]$. First we shall show that $H_i(\phi)$ is well-defined. Let $(X, \cdot) \sim (Y, *)$. Choose $x \in X$, $y \in Y$. Then $G_x \sim G_y$ so $\phi(G_x) \sim \phi(G_y)$. Then $(G'/\phi(G_x), \cdot) \sim (G'/\phi(G_y), *)$. Hence $H_i(\phi)$ is well-defined. Next we shall show that $H_i(\phi)$ is isotone. Let $\alpha, \beta \in H_i(G)$ be such that $\alpha \leqslant \beta$. Then $\exists (X, \cdot) \in \alpha$, $(Y, *) \in \beta$, an onto $\psi_1: X \to Y$ and an automorphism $\psi_2: G \to G$ such that $\psi_1(g.u) = \psi_2(g) * \psi_1(u)$. Choose $x \in X$ so $\psi_1(x) \in Y$. Let $y = \psi_1(x)$. Then $\psi_2(G_x) \subsetneq G_y$. Let $\psi_2' = \phi \circ \psi_2 \circ \phi^{-1}$. So ψ_2' is an automorphism. Define $\psi_1': G'/\phi(G_x) \to G'/\phi(G_y)$

as follows: given $\gamma \in G/\phi(G_x)$ choose $a \in \gamma$ and then let $\psi_1'(\gamma) = [\psi_2'(a)]_y$. First we shall show that ψ_1' is well-defined. Let $a,b \in G'$ be such that $a^{-1}b \in \phi(G_x)$ then $\phi^{-1}(a^{-1}b) \in G_x$ so $\psi_2(\phi^{-1}(a^{-1}b) \in \psi_2(G_x) \subseteq G_y$. Hence $\psi_2'(a^{-1}b) = (\phi o \psi_2 o \phi^{-1})(a^{-1}b) \in \phi(G_y)$. Therefore $(\psi_2'(a))^{-1}.\psi_2'(b) \in \phi(G_y)$. ie. ψ_1 is well-defined. Clearly ψ_1' is onto and $\psi_1'(g'.\gamma) = \psi_2'(g') * \psi_1'(\gamma)$ $\forall \gamma \in G'/\phi(G_x)$, $g' \in G$. Hence $(H_i(\phi))(\alpha) \leqslant (H_i(\phi))(\beta)$. Therefore $H_i(\phi)$ is isotone. Next we shall show that H_i is a covariant functor from $\mathcal{S}_i$ to $\mathcal{Q}$. Clearly $H_i(id_G) = id_{H_i(G)}$ $\forall$ G in Ob $\mathcal{S}_i$. Let $\phi:G \to G'$ and $\phi':G' \to G''$ be group-isomorphisms. Then $\phi' o \phi:G \to G''$. Let $\alpha \in H_i(G)$, choose $(X,\cdot) \in \alpha$ and choose $x \in X$ then $(H_i(\phi')oH_i(\phi))(\alpha) = (H_i(\phi'))\ [(G'/\phi(G_x),\cdot)] = [(G''/\phi'(G_{[1]}),\cdot)] = [(G''/\phi' o \phi(G_x),\cdot)] = (H_i(\phi' o \phi))(\alpha)$. Hence $H_i(\phi')oH_i(\phi) = H_i(\phi' o \phi)$. Therefore H_i is a covariant functor from $\mathcal{S}_i$ to $\mathcal{Q}$.

Now we shall show that S_i, L_i, H_i are naturally equivalent. The proof that S_i and L_i are naturally equivalent is similar to the proof that N^* and C^* are naturally equivalent in Section 2.3. The proof that H_i and S_i are naturally equivalent is similar to the proof that H_o and S_o are naturally equivalent in this section.

<u>Remark</u>: Definitions, theorems and our investigations in Section 2.2 are true for group-spaces.