CHAPTER I

INTRODUCTION

We shall give some definitions and some theorems used in this thesis.

<u>Definition 1.1</u> A <u>category</u> $\mathcal{C}$ consists of a class of things called <u>objects</u> and denoted by Ob $(\mathcal{C})$ and for each ordered pair of objects (X,Y) in Ob $(\mathcal{C})$ a set is assigned called <u>the set of morphisms</u> and denoted by Mor (X,Y) (Mor (X,Y) may be empty) and $\exists$ a map, called <u>composition</u> and denoted by o, from Mor $(X,Y) \times$ Mor $(Y,Z) \to$ Mor (X,Z) such that

i) if $f \in$ Mor (X,Y), $g \in$ Mor (Y,Z) and $h \in$ Mor (Z,W) then $ho(gof) = (hog)of$,

ii) for each Y in Ob $(\mathcal{C})$ there exists a special morphism denoted by $id_Y \in$ Mor (Y,Y) such that if $f \in$ Mor (X,Y) then $id_Y of = f$ and if $g \in$ Mor (Y,X) then $goid_Y = g$. id_Y is called the <u>identity</u> on Y.

<u>Remarks</u>: 1) If $f \in$ Mor (X,Y) we shall sometimes write this as $f:X \to Y$ or $X \xrightarrow{f} Y$.

2) id_Y is the unique element of $Mor(Y,Y)$ satisfying the property ii)

<u>Examples</u> 1) The category $\mathcal{S}$ of sets and set maps; let Ob $(\mathcal{S})$ = the class of all sets, if X, Y are sets then let $Mor(X,Y)$ be the set of all

set maps $f:X \to Y$. If $f:X \to Y$ and $g:Y \to Z$ then $g \circ f$ is the composition of set maps.

2) The category $\mathcal{G}$ of groups and group-homomorphisms; let $Ob(\mathcal{G})$ = the class of all groups, if G_1, G_2 are groups then let Mor $(G_1 G_2)$ be the set of all group-homomorphisms $f:G_1 \to G_2$. If $f:G_1 \to G_2$ and $g:G_2 \to G_3$. Then $g \circ f$ is the composition of group homomorphisms.

3) The category $\mathcal{R}$ of rings and onto ring homomorphisms; let $Ob(\mathcal{R})$ = the class of all rings, if R_1, R_2 are rings then let Mor (R_1, R_2) be the set of all onto ring homomorphisms $f:R_1 \to R_2$. Mor (R_1, R_2) may be empty. If $f:R_1 \to R_2$ and $g:R_2 \to R_3$ then $g \circ f$ is the composition of ring homomorphism.

Definition 1.2 Let X be a nonempty set and $x \in X$. We say that (X,x) is a pointed set. Let (X,x) and (Y,y) be pointed sets. Then a pointed set map from (X,x) to (Y,y) is a set map $\phi:X \to Y$ such that $\phi(x) = y$.

4) The category $\mathcal{P}$ of pointed sets and pointed set maps ; let $Ob(\mathcal{P})$ = the class of all pointed sets, if (X,x), (Y,y) are pointed sets then let $Mor((X,x), (Y,y))$ be the set of all pointed set maps $f:(X,x) \to (Y,y)$. If $f:(X,x) \to (Y,y)$ and $g:(Y,y) \to (Z,z)$ then $g \circ f$ is the composition of pointed set maps.

Definition 1.3 Let $\mathcal{C}$ be a category, X,Y in Ob $(\mathcal{C})$, $f \in$ Mor (X,Y) and $g \in$ Mor (Y,X). Then g is called a left inverse of f if $g \circ f = id_X$. In this case, f is called a right inverse of g. Also if $f \circ g = id_Y$ then g is called a two-sided inverse of f. If f has a two-sided inverse then f is called an isomorphism.

Remarks: 1) Let $f \in Mor(X,Y)$ have a left inverse g and a right inverse h then $g = h$. So f has two-sided inverse.

2) If $f \in Mor(X,Y)$ has a two-sided inverse then the two-sided inverse is unique.

Definition 1.4 Let $\mathscr{C}$ be a category and X,Y in $Ob(\mathscr{C})$. Then X is said to be equivalent to Y iff there exists an isomorphism $f \in Mor(X,Y)$.

Definition 1.5 Let $\mathscr{C}$, $\mathscr{D}$ be categories. A covariant functor F from $\mathscr{C}$ to $\mathscr{D}$ is a correspondence that takes objects in $\mathscr{C}$ to objects in $\mathscr{D}$ and morphisms $f:X \to Y$ in $\mathscr{C}$ to morphisms $F(f):F(X) \to F(Y)$ in $\mathscr{D}$ such that

i) $F(id_X) = id_{F(X)}$ for all X in $Ob(\mathscr{C})$,

ii) $F(g \circ f) = F(g) \circ F(f)$ whenever $g \circ f$ is defined.

Definition 1.6 Let $\mathscr{C}$, $\mathscr{D}$ be categories. A contravariant functor F from $\mathscr{C}$ to $\mathscr{D}$ is a correspondence that takes objects in $\mathscr{C}$ to objects in $\mathscr{D}$ and morphisms $f:X \to Y$ in $\mathscr{C}$ to morphisms $F(f):F(Y) \to F(X)$ in $\mathscr{D}$ such that

i) $F(id_X) = id_{F(X)}$ for all X in $Ob(\mathscr{C})$,

ii) $F(g \circ f) = F(f) \circ F(g)$ whenever $g \circ f$ is defined.

Examples of covariant functors and contravariant functors are given in Chapter II and Chapter III.

Definition 1.7 Let $\mathscr{C}$, $\mathscr{D}$ be categories and F_1, F_2 functors from

$\mathscr{C}$ to $\mathscr{D}$ of the same variance. A <u>natural transformation</u> ϕ from F_1 to F_2 is a correspondence which assigns to each object X in $\mathscr{C}$ a morphism $\phi_X \in \mathrm{Mor}_{\mathscr{D}}((F_1(X), F_2(X))$ such that for each $f \in \mathrm{Mor}_{\mathscr{C}}(X,Y)$ one of the following diagrams is commutative:

$$\begin{array}{ccc} F_1(X) & \xrightarrow{\phi_X} & F_2(X) \\ {\scriptstyle F_1(f)}\downarrow & & \downarrow{\scriptstyle F_2(f)} \\ F_1(Y) & \xrightarrow{\phi_Y} & F_2(Y) \end{array} \qquad \begin{array}{ccc} F_1(X) & \xrightarrow{\phi_X} & F_2(X) \\ {\scriptstyle F_1(f)}\uparrow & & \uparrow{\scriptstyle F_2(f)} \\ F_1(Y) & \xrightarrow{\phi_Y} & F_2(Y) \end{array}$$

In addition, if for each X in Ob ($\mathscr{C}$) ϕ_X is an isomorphism then we say that ϕ is a <u>natural equivalence</u> and F_1 is <u>naturally equivalent</u> to F_2.

Examples of natural equivalences are given in chapter II and chapter III of this thesis.

<u>Definition 1.8</u> Let $\mathscr{A}$, $\mathscr{B}$ be categories. Then $\mathscr{B}$ is said to be a <u>subcategory</u> of $\mathscr{A}$ iff if X is in Ob ($\mathscr{B}$) then X is in Ob($\mathscr{A}$) and if X,Y are in Ob($\mathscr{B}$) then $\mathrm{Mor}_{\mathscr{B}}(X,Y) \subsetneq \mathrm{Mor}_{\mathscr{A}}(X,Y)$ and if $g \in \mathrm{Mor}_{\mathscr{B}}(X,Y)$, $f \in \mathrm{Mor}_{\mathscr{B}}(Y,Z)$ then $f o_{\mathscr{B}}\, g = f o_{\mathscr{A}}\, g$.

<u>Example</u> Let $\mathscr{G}$ be the category of groups and group-homomorphisms. Then the following categories are subcategories of $\mathscr{G}$:

i) the category of groups and onto group homomorphisms,

ii) the category of abelian groups and group homomorphisms.

We shall now define a quasi-ordered set as given in [1].

<u>Definition 1.9</u> A <u>quasi ordered set</u> is a pair $(Q,\leqslant)$ where Q is a set and $\leqslant$ is a binary relation on Q which satisfies for all $x,y,z \in Q$ the following conditions:

1) For all $x, x \leqslant x$. (Reflexive)

2) If $x \leqslant y$ and $y \leqslant z$ then $x \leqslant z$. (Transitive)

<u>Example</u> Fix a vector space V of dimension > 1. Let $Q = \mathcal{P}(V)$. For $A,B \in \mathcal{P}(V)$ say that $A \leqslant B$ iff the linear subspace of V generated by A $\subsetneq$ the linear subspace of V generated by B. Clearly $(Q,\leqslant)$ is a quasi ordered set.

<u>Definition 1.10</u> A <u>partially ordered set (poset)</u> is a pair $(P,\leqslant)$ where P is a set and $\leqslant$ is a binary relation on P which satisfies for all $x,y,z \in P$ the following conditions:

1) For all x, $x \in X$. (Reflexive)

2) If $x \leqslant y$ and $y \leqslant x$ then $x = y$. (antisymmetric)

3) If $x \leqslant y$ and $y \leqslant z$ then $x \leqslant z$. (Transitive)

We shall now define an isotone map as given in [1].

<u>Definition 1.11</u> An <u>isotone map</u> from a quasi-ordered set Q_1 to a quasi-ordered set Q_2 is a map $\phi: Q_1 \to Q_2$ such that $x \leqslant y$ implies that $\phi(x) \leqslant \phi(y)$ for all $x,y \in Q_1$.

<u>Remark</u> Every poset is a quasi-ordered set so the above definition is used for posets also.

Definition 1.12 An upper bound of a subset X of a poset $(P,\leq)$ is an element a in P such that $x \leq a$ for all x in X. The least upper bound of X is an upper bound a of X such that if b is an upper bound of X then $a \leq b$. l.u.b.X denotes the least upper bound of X. The notions of lower bound of X and greatest lower bound (g.l.b.X) of X are defined dually.

Definition 1.13 A lattice is a poset any two of whose elements have a g.l.b. and a l.u.b.

Examples 1) Let $\mathcal{P}(X)$ be the set of all subsets of a set X and $\subseteq$ set inclusion. Then clearly $(\mathcal{P}(X), \subseteq)$ is a poset. Let $A,B \in \mathcal{P}(X)$ then $A \cap B = $ g.l.b.$\{A,B\}$ and $A \cup B = $ l.u.b.$\{A,B\}$. Hence $\mathcal{P}(X)$ is a lattice.

2) Let $\mathcal{S}(G)$ be the set of all subgroups of a group G and $\subseteq$ set inclusion. Then clearly $(\mathcal{S}(G), \subseteq)$ is a poset. Let $H_1, H_2 \in \mathcal{S}(G)$ then $H_1 \cap H_2 = $ g.l.b $\{H_1,H_2\}$ and the subgroup of G generated by $H_1 \cup H_2 = $ l.u.b. $\{H_1,H_2\}$. Hence $\mathcal{S}(G)$ is a lattice.

3) Let $\mathcal{N}(G)$ be the set of all normal subgroups of a group G and $\subseteq$ set inclusion. Then clearly $(\mathcal{N}(G), \subseteq)$ is a poset. Let $N_1, N_2 \in \mathcal{N}(G)$ then $N_1 \cap N_2 = $ g.l.b.$\{N_1,N_2\}$ and $N_1.N_2 = \{n_1.n_2 | n_1 \in N_1, n_2 \in N_2\}$ $=$ l.u.b.$\{N_1,N_2\}$. Hence $\mathcal{N}(G)$ is a lattice.

4) Let $\mathcal{I}(R)$ be the set of all ideals in a ring R and $\subseteq$ set inclusion. Then clearly $(\mathcal{I}(R), \subseteq)$ is a poset. Let $I_1, I_2 \in \mathcal{I}(R)$ then $I_1 \cap I_2 = $ g.l.b.$\{I_1,I_2\}$ and $I_1 + I_2 = \{i_1 + i_2 | i_1 \in I_1, i_2 \in I_2\} = $ l.u.b. $\{I_1,I_2\}$. Hence $\mathcal{I}(R)$ is a lattice.

Definition 1.14 Let $\phi: X \to Y$ be a set map. Then $\phi \times \phi: X \times X \to Y \times Y$ will denote the map given by $(\phi \times \phi)(x,y) = (\phi(x),\phi(y))$ for all $x,y \in X$

Definition 1.15 An algebraic system is a n + 1 tuple $(X,f_1,\ldots,f_n)$ where X is a nonempty set, $n \in \mathbb{N}$ and f_i is a map from $X \times X$ to X for all $i \in \{1,2,\ldots,n\}$.

Examples Semigroups, groups, semirings, rings and skew fields (division rings) are algebraic systems.

Definition 1.16 Let $(X,f_1,\ldots,f_n)$ and $(Y,g_1,\ldots,g_m)$ be algebraic systems. Say that $(X,f_1,\ldots,f_n)$ and $(Y,g_1,\ldots,g_m)$ are of the same type iff m = n.

Examples Two semigroups are of the same type, two groups are of the same type, semigroups and groups are of the same type, semirings and rings are of the same type.

Definition 1.17 Let $(X,f_1,\ldots,f_n)$ and $(Y,g_1,\ldots,g_n)$ be algebraic systems of the same type. An operation preserving map from $(X,f_1,\ldots,f_n)$ to $(Y,g_1,\ldots,g_n)$ is a set map $\phi: X \to Y$ such that for each $i \in \{1,2,\ldots,n\}$, $a,b \in X$ $\phi(f_i(a,b)) = g_i(\phi(a),\phi(b))$.

Examples Semigroup-homomorphisms, semiring homomorphisms are operation preserving maps.

Definition 1.18 Let $\mathcal{A}$ be a category. Then $\mathcal{A}$ is said to be a category of algebraic systems iff

1) all the objects of $\mathcal{A}$ are algebraic systems of the same type and

2) every morphism in $\mathcal{A}$ is an operation preserving map.

<u>Remark</u>: In this thesis we shall only study categories of algebraic systems and the following categories:

i) the category $\mathcal{L}$ of lattices and isotone maps,

ii) the category $\mathcal{P}$ of posets and isotone maps,

iii) the category $\mathcal{Q}$ of quasi-ordered sets and isotone maps.

<u>Examples</u> The following categories are categories of algebraic systems:

i) the category $\mathcal{S}_g$ of semigroups and semigroup homomorphisms,

ii) the category $\mathcal{R}_1$ of rings with multiplicative identity 1 and 1-1 ring homomorphisms,

iii) the category $\mathcal{F}$ of skew fields and 1-1 ring homomorphisms.

<u>Definition 1.19</u> Let $(X,f_1,\ldots,f_n)$ be an algebraic system. An <u>operation preserving relation</u> on $(X,f_1,\ldots,f_n)$ is an equivalence relation ρ on X such that $x \rho y$ implies that $f_i(a,x) \rho f_i(a,y)$ and $f_i(x,a) \rho f_i(y,a)$ for all $x,y,a \in X$, for all $i \in \{1,2,\ldots, n\}$.

Let $(X,f_1,\ldots,f_n)$ be an algebraic system, ρ an <u>operation preserving relation</u> on $(X,f_1\ldots,f_n)$ and X/ρ the set of equivalence classes of X. For each $i \in \{1,2,\ldots,n\}$ define $f_i':X/\rho \times X/\rho \to X/\rho$ as follows: given $\alpha,\beta \in X/\rho$ choose $a \in \alpha$, $b \in \beta$ and then let $f_i'(\alpha,\beta) = [f_i(a,b)]$. Clearly f_i' is well defined for all i. So we see that $(X/\rho,f_1',\ldots,f_n')$ is an algebraic system of the some type as $(X,f_1,\ldots,f_n)$ such that the projection map $\pi:X \to X/\rho$ is an operation preserving map.

Remarks: 1) For each object $(S,\cdot)$ in Ob $(\mathcal{S}_g)$ let ρ be an operation preserving relation on $(S,\cdot)$ then $(S/\rho,\cdot)$ as defined above is an object in $\mathcal{S}_g$.

2) Let ρ be an operation preserving relation on an object $(R,+,\cdot)$ in Ob $(\mathcal{R}_1)$. If $(0,1) \in \rho$ then $(0,x) \in \rho$ for all $x \in R$ and hence $(x,y) \in \rho$ for all $x,y \in R$ i.e. $\rho = R \times R$, so we shall see that R/ρ has only one element, ie. $(R/\rho,+,\cdot)$ is not an object in $\mathcal{R}_1$.

Definition 1.20 Let $\mathcal{A}$ be a category of algebraic systems. A congruence ρ on an object $(X,f_1,\ldots,f_n)$ in Ob $(\mathcal{A})$ is an operation preserving relation on $(X,f_1,\ldots,f_n)$ such that the algebraic system $(X/\rho,f_1',\ldots,f_n')$ constructed above is an object in Ob $(\mathcal{A})$.

Examples 1) An operation preserving relation on an object $(S,\cdot)$ of Ob $(\mathcal{S}_g)$ is a congruence on $(S,\cdot)$.

2) ρ is an operation preserving relation on an object $(R,+,\cdot)$ of $\mathcal{R}_1$ such that $(0,1) \notin \rho$ iff ρ is a congruence on $(R,+,\cdot)$.

3) ρ is an operation preserving relation on an object $(F,+,\cdot)$ of $\mathcal{F}$ such that $(0,1) \notin \rho$ iff ρ is a congruence on $(F,+,\cdot)$.

Remark: We have seen that in some categories an operation preserving relation is a congruence and in some categories an operation preserving relation is not a congruence.

<u>Definition 1.21</u> Let $\mathcal{A}$ be a category of algebraic systems. Say that $\mathcal{A}$ <u>has the congruence functor</u> iff

1) for each object $(X,f_1,\ldots,f_n)$ in Ob $(\mathcal{A})$ the set $C_{(X,f_1,\ldots,f_n)}$ of all congruences on $(X,f_1,\ldots,f_n)$ is a lattice with respect to set inclusion,

2) if $\phi:(X,f_1,\ldots,f_n) \to (Y,g_1,\ldots,g_n)$ is a morphism in $\mathcal{A}$ and $A \in C_{(Y,g_1,\ldots,g_n)}$ then $(\phi \times \phi)^{-1}(A) \in C_{(X,f_1,\ldots,f_n)}$.

1) and 2) implies that there exists a contravariant functor C from $\mathcal{A}$ to $\mathcal{L}$ such that C takes $(X,f_1,\ldots,f_n)$ to $C_{(X,f_1,\ldots,f_n)}$ and morphism $\phi:(X,f_1,\ldots,f_n) \to (Y,g_1,\ldots,g_n)$ to the isotone map $(\phi \times \phi)^{-1}: C_{(Y,g_1,\ldots,g_n)} \to C_{(X,f_1,\ldots,f_n)}$ we shall call C the <u>congruence functor</u> of $\mathcal{A}$.

In Chapter II and Chapter III we shall see examples of categories of algebraic systems having the congruence functor. In this thesis we are interested in categories of algebraic systems which have the congruence functor and $\mathcal{Q}$, $\mathcal{P}$, $\mathcal{L}$.

<u>Definition 1.22</u> Let $\mathcal{A}$ be a category of algebraic systems having the congruence functor. Let C denote the congruence functor of $\mathcal{A}$. Say that $\mathcal{A}$ <u>has a congruence set</u> iff for each object $(X,f_1,\ldots,f_n)$ in $\mathcal{A}$ we can choose a set $B_{(X,f_1,\ldots,f_n)}$ of subsets of X such that

1) $B_{(X,f_1,\ldots,f_n)}$ is a lattice with respect to set inclusion,

2) if $\phi:(X,f_1,\ldots,f_n) \to (Y,g_1,\ldots,g_n)$ is a morphism in $\mathcal{A}$

and $A \in B_{(Y,g_1,\ldots,g_n)}$ then $\phi^{-1}(A) \in B_{(X,f_1,\ldots,f_n)}$,

3) the contravariant functor B from $\mathcal{A}$ to $\mathcal{L}$ taking $(X,f_1,\ldots,f_n)$ to $B_{(X,f_1,\ldots,f_n)}$ and morphism $\phi:(X,f_1,\ldots,f_n) \to (Y,g_1,\ldots,g_n)$ to the isotone map $\phi^{-1}: B_{(Y,g_1,\ldots,g_n)} \to B_{(X,f_1,\ldots,f_n)}$ is naturally equivalent to C.

We shall call B a <u>congruence set functor</u> of $\mathcal{A}$. If $A \in B_{(X,f_1,\ldots,f_n)}$ then we shall call A a <u>congruence set</u> in $(X,f_1,\ldots,f_n)$ with respect to the congruence set functor B.

In Chapter II and Chapter III we shall see examples of categories of algebraic systems having a congruence set functor and a congruence set.

We shall show that $\mathcal{F}$ has at least two congruence set functors.

<u>Proposition</u> 1.23 If ρ is a congruence on an object $(F,+,\cdot)$ in $\mathcal{F}$ then $\rho = \Delta$.

<u>Proof</u>. Assume that ρ is a congruence on $(F,+,\cdot)$. Suppose that $\rho \neq \Delta$ so $\exists\,(a,b) \in \rho$ such that $a \neq b$. Then $(a - b, 0) \in \rho$. Because $a - b \neq 0$, $(1, 0) = ((a - b)^{-1}.(a - b),(a - b)^{-1}.0) \in \rho$ which is a contradiction. Hence $\rho = \Delta$.

We define contravariant functors C, B, B' from $\mathcal{F}$ to $\mathcal{L}$ as follows:

1) For each object $(F,+,\cdot)$ in $\mathcal{F}$, let $C(F,+,\cdot) = \{\Delta\}$ and for each morphism $\phi:(F,+,\cdot) \to (F',+',\cdot')$ let $C(\phi)(\Delta') = \Delta$.

2) For each object $(F,+,\cdot)$ in $\mathcal{F}$, let $B(F,+,\cdot) = \{\{0\}\}$ and for each morphism $\phi:(F,+,\cdot) \to (F',+',\cdot')$ let $B(\phi)(\{0'\}) = \{0\}$.

3) For each object $(F,+,\cdot)$ in $\mathcal{F}$, let $B'(F,+,\cdot) = \{\{1\}\}$ and for each morphism $\phi:(F,+,\cdot) \to (F',+',\cdot')$ let $B'(\phi)(\{1'\}) = \{1\}$.

Hence C, B and B' are naturally equivalent contravariant functors. Then C is the congruence functor of $\mathcal{F}$, B and B' are congruence set functors of $\mathcal{F}$, $\{0\}$ is a congruence set with respect to B, $\{1\}$ is a congruence set with respect to B'. Hence $\mathcal{F}$ has at least two congruence set functors.

Our aim in this thesis is to study congruences and partial congruences in the category of semigroups and semigroup homomorphisms and the category of semirings and semiring homomorphisms. We shall see that certain special subcategories of these categories have a congruence set. Besides the well-known congruence sets on the category of groups and the category of rings already mentioned, we have discovered that the category of P.R.D.'s and P.R.D homomorphisms and the category of semifields and semifield homomorphisms both have a congruence set. We shall apply the concepts of congruence sets to prove new theorems about algebraic systems.

In addition, we have found that a generalization of a ring, called a skew ring, also has a congruence set.

Our notation used for sets of numbers are:

$\mathbb{N} = \{1,2,3,.....\}$;

$\mathbb{N}_0 = \{0,1,2,.....\}$;

$\mathbb{Z}$ = the set of all integers;

$\mathbb{Q}$ = the set of all rational numbers;

$\mathbb{Q}^+ = \{ x \in \mathbb{Q} \mid x > 0 \}$;

$\mathbb{Q}_0^+ = \{ x \in \mathbb{Q} \mid x \geqslant 0 \}$;

$\mathbb{R}$ = the set of all real numbers ;

$\mathbb{R}^+ = \{ x \in \mathbb{R} \mid x > 0 \}$;

$\mathbb{R}_0^+ = \{ x \in \mathbb{R} \mid x \geqslant 0 \}$;

$\mathbb{Q}_\infty^+ = \mathbb{Q}^+ \cup \{\infty\}$;

$\mathbb{R}_\infty^+ = \mathbb{R}^+ \cup \{\infty\}$.