CHAPTER I

CONGRUENCE-FREE CONGRUENCES

General properties of congruence-free congruences on any semigroups are presented in this chapter. Moreover, some remarks on congruence-free Rees congruences are given.

Let S be a semigroup and ρ be a congruence on S. Suppose δ is a congruence on S which contains ρ. Let $\bar{\delta}$ be the relation on the semigroup S/ρ defined by

$$\bar{\delta} = \{(a\rho, b\rho) \mid (a, b) \in \delta\}.$$

Then $\bar{\delta}$ is a congruence on the semigroup S/ρ.

Conversely, assume that $\bar{\sigma}$ is a congruence on the semigroup S/ρ. Then the relation σ on S given by

$$\sigma = \{(a, b) \mid (a\rho, b\rho) \in \bar{\sigma}\}$$

is clearly a congruence on S which contains ρ.

Let δ_1 and δ_2 be congruences on S which contain ρ such that $\bar{\delta}_1 = \bar{\delta}_2$ where $\bar{\delta}_1$ and $\bar{\delta}_2$ are defined from δ_1 and δ_2, respectively, as above. To show $\delta_1 = \delta_2$, let $(a, b) \in \delta_1$. Then $(a\rho, b\rho) \in \bar{\delta}_1$. But $\bar{\delta}_1 = \bar{\delta}_2$, so $(a\rho, b\rho) \in \bar{\delta}_2$. Hence $(a, b) \in \delta_2$. Therefore $\delta_1 \subseteq \delta_2$. Similarly $\delta_2 \subseteq \delta_1$. Hence $\delta_1 = \delta_2$.

From the above proof, we obtain the following : Let ρ be a congruence on a semigroup S. For each congruence δ on S which contains ρ, let $\bar{\delta}$ denote the congruence on the semigroup S/ρ defined as follow: $(a\rho, b\rho) \in \bar{\delta}$ if and only if $(a, b) \in \delta$ $(a, b \in S)$. Then the association

δ with $\bar{\delta}$ sets up an inclusion preserving one-to-one correspondence of the set of all congruences on S which contain ρ onto the set of all congruences on the semigroup S/ρ.

Hence, nice behaviors of congruence-free congruences on a semigroup are obtained directly as follows :

1.1 Proposition. Let ρ be a congruence on a semigroup S. Then ρ is a congruence-free congruence on S if and only if either ρ is the universal congruence or ρ is a maximal nonuniversal congruence on S.

1.2 Proposition. Let ρ be a congruence on a semigroup S. Then ρ is congruence-free if and only if every chain $\mathcal{C}$ of congruence-free congruences on S such that $\rho \in \mathcal{C}$ contains at most two elements.

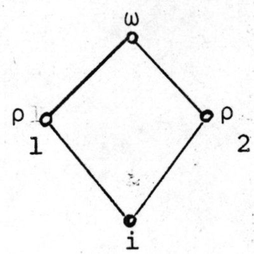
By Proposition 1.1, we then have

1.3 Corollary. Let ρ be a nonuniversal congruence on a semigroup S. Then ρ is congruence-free if and only if ρ is a maximal nonuniversal congruence on S.

We shall have many occasions to refer back to this result in our study of congruence-free symmetric inverse semigroups.

From proposition 1.1, the definitions of maximal nonuniversal congruences and of nonuniversal congruence-free congruences on a semigroup are the same. But "(nonuniversal) congruence-free congruences" are preferred to use because many known results of congruence-free semigroups are referred in this research.

Let S be a semigroup, let ω and i denote the universal congruence and the identity congruence on S, respectively. Let $\mathcal{C}$ be the collection of congruence-free congruences on S and the identity congruence on S. Partially order $\mathcal{C}$ by inclusion. If ρ_1 and ρ_2 are two distinct nonuniversal congruence-free congruences on S, then on $\mathcal{C}$, we have the following Hasse diagram :



Then every nonuniversal congruence-free congruence on a semigroup S is a minimal congruence-free congruence on S. Hence every semigroup has a minimal congruence-free congruence. However, a semigroup need not have a minimum congruence-free congruence. From the above diagram, it is shown that any semigroup having two distinct nonuniversal congruence-free congruences does not have the minimum congruence-free congruence.

<u>Example</u>. Let S = {0, a, b}. Define an operation * on S by

$$x*y = \begin{cases} x & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

Then S is a semigroup. Let $\rho_1 = \{(a, a), (0, 0), (0, b), (b, 0), (b, b)\}$ and $\rho_2 = \{(b, b), (0, 0), (0, a), (a, 0), (a, a)\}$. It is clearly seen that i, ρ_1, ρ_2 and ω are all the congruences on S, and ρ_1, ρ_2, ω are all the congruence-free congruences on S. Since $\rho_1 \nsubseteq \rho_2$ and $\rho_2 \nsubseteq \rho_1$, the minimum congruence-free congruence on S does not exist.

From the previous diagram, a characterization of a semigroup which has a minimum congruence-free congruence is given as follows :

1.4 Proposition. A semigroup S has a minimum congruence-free congruence if and only if S has at most one nonuniversal congruence-free congruence.

Let ρ be a nonuniversal congruence on a semigroup S. If ρ has exactly two classes, then ρ is congruence-free. However, the converse is not true, and we can see its counter examples in Chapter II.

An arbitrary intersection of congruences on a semigroup S is a congruence on S. The intersection of two distinct nonuniversal congruence-free congruences on a semigroup S is the identity congruence on S. Therefore, the intersection of congruence-free congruences on a semigroup S need not be congruence-free. It can be easily seen from the example of page 9-10.

Let S be a semigroup. For an ideal A of S, let ρ_A denote the Rees congruence on S induced by the ideal A, that is, $(x, y) \in \rho_A$ if and only if either $x, y \in A$ or $x = y$. For each ideal A of S, the quotient semigroup S/ρ_A is a semigroup with zero, and for any $a \in A$, $a\rho_A = A$ which is the zero of the semigroup S/ρ_A.

Any congruence-free semigroup S without zero is clearly a simple semigroup.

Let S be a semigroup with zero 0. Suppose S is congruence-free. Then {0} and S are the only ideals of S. If S is a zero semigroup, then every subset of S containing 0 is an ideal of S, so $|S| \leq 2$. Assume that S is not a zero semigroup. Then $S^2 \neq \{0\}$, so S is a 0-simple semigroup.

Therefore, if a semigroup S with zero is congruence-free, then either S is 0-simple or a zero semigroup of order less than 3.

However, a simple semigroup and a 0-simple semigroup are not necessarily congruence-free. Let S be a left zero semigroup with $|S| > 2$. Then S is simple. Let $a \in S$ and $\rho = \{(a, a)\} \cup \{(x, y) \in S \times S \mid x \neq a, y \neq a\}$. Then ρ is a nonuniversal congruence on S. Since $|S| > 2$, $\rho \neq i$. Hence S is not congruence-free. Next, let S^0 denote the semigroup obtained by joining the zero 0 to S. Then S^0 is clearly 0-simple. Let $\delta = \{(0, 0)\} \cup (S \times S)$. Then δ is a congruence on the semigroup S^0, $\delta \neq \omega$ and $\delta \neq i$ on S^0. Therefore S^0 is not congruence-free.

Let S be a semigroup and A be an ideal of S. For each ideal B of S such that $A \subseteq B$, let $\bar{B} = \{b\rho_A \mid b \in B\}$. The map $B \mapsto \bar{B}$ gives an inclusion preserving one-to-one correspondence of the set of all ideals of S which contain A onto the set of all ideals of the Rees semigroup S/ρ_A.

1.5 <u>Proposition</u>. Let A be an ideal of a semigroup S. If the Rees congruence ρ_A on S is congruence-free, then either A = S or A is a maximal ideal of S.

Proof : Suppose that $A \neq S$. To show that A is a maximal ideal of S, let K be an ideal of S such that $A \subsetneq K$. Then $\bar{K} = \{k\rho_A \mid k \in K\}$ is an ideal of the semigroup S/ρ_A and $|\bar{K}| > 1$. Because ρ_A is a congruence-free congruence on S, S/ρ_A is 0-simple or a zero semigroup of order less than 3. But $\bar{K}$ is an ideal of the semigroup S/ρ_A and $|\bar{K}| > 1$, so $\bar{K} = S/\rho_A$. Hence $K = S$. #

The converse of Proposition 1.5 is not true. A counter example is given as follows :

Example. Let G be a nontrivial group and let $S = G^0$. Then S is a group with zero. Let $A = \{0\}$. Clearly, A is a maximal ideal of S. Let $\rho = \{(0, 0)\} \cup (G \times G)$. Then ρ is a congruence on S, $\rho \neq i$ and $\rho \neq \omega$. Therefore S is not congruence-free. Since S/ρ_A is isomorphic to S, ρ_A is not a congruence-free Rees congruence on S. #